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Mutually permutable products and conjugacy classes

A. Ballester-Bolinches · John Cossey · Yangming Li

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Abstract The question of how certain arithmetical conditions on the lengths of the conjugacy classes of a finite group G influence the group structure has been studied by several authors with many results available. The purpose of this paper is to analyse the restrictions imposed by the lengths of the conjugacy classes of some elements of the factors of a finite group $G = G_1 G_2 \cdots G_r$, which is the product of the pairwise mutually permutable subgroups $G_1, G_2, \ldots, G_r$, on its structure. Some earlier results appear as corollaries of our main theorems.

Keywords Finite groups · Mutually permutable products · Conjugacy classes.

Mathematics Subject Classification 20D10 · 20D20 · 20D40 · 20E45

Dedicated to Professor Hermann Heineken on his 75th birthday.

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A. Ballester-Bolinches (✉)
Departament d'Àlgebra, Universitat de València, Dr. Moliner, 50, 46100 Burjassot, València, Spain
e-mail: Adolfo.Ballester@uv.es

J. Cossey
Department of Mathematics, School of Mathematical Science, The Australian National University, Canberra 0200, Australia
e-mail: John.Cossey@anu.edu.au

Y. Li
Department of Mathematics, Guangdong University of Education, Guangzhou 510310, People's Republic of China
e-mail: liyangming@gdei.edu.cn

1 Introduction and statement of results

This paper deals with several problems concerning finite factorised groups, that is, groups $G = G_1 G_2 \cdots G_r$ which are the product of pairwise permutable subgroups $G_1, G_2, \ldots, G_r$. Arguably, the study of factorised groups is a productive and interesting area of research in finite group theory, with the structural impact of the factors one of the central questions. During the past two decades, finite factorised groups whose factors are connected by means of certain permutability properties, namely mutually and totally permutable products, have gained more and more popularity (see Ref. [1]). Recall that two subgroups A and B of a group G are called mutually permutable if A permutes with every subgroup of B and B permutes with every subgroup of A. A group $G = G_1 G_2 \cdots G_r$ which is the product of its pairwise permutable subgroups $G_1, G_2, \ldots, G_r$ is said to be the mutually permutable product of $G_1, G_2, \ldots, G_r$ if G_i and G_j are mutually permutable subgroups of G for all $i, j \in \{1, 2, \ldots, r\}$.

Our paper features some results which give information about the structural restrictions on a mutually permutable factorised finite group in which the lengths of conjugacy classes of some elements of its factors have certain arithmetical properties.

In the sequel all groups considered are finite.

The terminology here is as follows: for an element x of a group G, we denote the conjugacy class of x in G by x^G. We denote by $|x^G|$ the length of the conjugacy class x^G, that is, the number of elements of x^G. If p is a prime, we say that an element x of the group G is *p-regular* if its order is not divisible by p; we say that x is *p-singular* if its order is divisible by p. As usual, a *p-element* of G is a p-singular element of prime power order.

Our first main result shows that the structure of a mutually permutable product in which the length of the conjugacy classes of the elements of the factors are not divisible by a prime p is quite restricted.

Theorem 1.1 *Let the group $G = G_1 G_2 \cdots G_r$ be the product of the pairwise mutually permutable subgroups $G_1, G_2, \ldots, G_r$, and let p be a prime. Then:*

1. *No conjugacy class length $|x^G|$, where x is a p-regular element of prime power order in $\bigcup_{i=1}^r G_i$, is divisible by p if and only if $G = O_p(G) \times O_{p'}(G)$.*
2. *$|x^G|$ is not divisible by p for every element $x \in \bigcup_{i=1}^r G_i$ if and only if $G = O_p(G) \times O_{p'}(G)$ with $O_p(G)$ abelian.*

In the particular case when $G = G_1 = \cdots = G_r$, we have:

Corollary 1.2 [4] *Let p be a prime integer. Then $|x^G|$ is not divisible by p for every element x of G if and only if $G = O_p(G) \times O_{p'}(G)$ with $O_p(G)$ abelian.*

The case when $r = 2$, when $G = G_1 G_2$ is just the product of the mutually permutable subgroups G_1 and G_2 is of special interest, since sometimes the problems reduce from an arbitrary number r of factors to the case $r = 2$. Our second main result shows that the order of the Sylow p-subgroups of the chief factors of a mutually permutable product of two factors in which the conjugacy class lengths of the p-regular elements of the factors are not divisible by p^2 is not divisible by p^2 either.

Theorem 1.3 *Let the group $G = AB$ be the mutually permutable product of the subgroups A and B. Suppose that for every p-regular element $x \in A \cup B$, $|x^G|$ is not divisible by p^2. Then the order of a Sylow p-subgroup of every chief factor of G is at most p. In particular, if G is p-soluble, we have that G is p-supersoluble.*

We do not know whether or not the above result holds for an arbitrary number of factors.

As an immediate deduction we have the

Corollary 1.4 [8] *Let G be a group and let p be a prime. Suppose that, for every p-regular element x of G, $|x^G|$ is not divisible by p^2. Then the order of a Sylow p-subgroup of every chief factor of G is at most p.*

If the condition of Theorem 1.3 holds for every prime, we get supersolubility.

Corollary 1.5 *Let the group $G = AB$ be the mutually permutable product of the subgroups A and B. Suppose that for every prime p and every p-regular element $x \in A \cup B$, $|x^G|$ is not divisible by p^2. Then G is supersoluble.*

Theorem 10 of Ref. [7] was the starting point of our investigation. It can be regarded as a particular case of the above corollary.

Corollary 1.6 [7] *Let the group $G = AB$ be a product of two subgroups A and B which are permutable in G. Suppose that for every prime p and every element $x \in A \cup B$, $|x^G|$ is not divisible by p^2. Then G is supersoluble.*

2 Proofs

Our first result, whose proof is straightforward, implies that the assumptions about conjugacy classes of our main results are inherited by normal subgroups and quotient groups. We shall use these properties frequently without further reference.

Lemma 2.1 *Let N be a normal subgroup of a group G and let p be a prime. Then:*

1. *$|x^N|$ divides $|x^G|$ for any $x \in N$.*
2. *$|(yN)^{G/N}|$ divides $|y^G|$ for any $y \in G$.*
3. *If xN is a p-element of G/N, then there is a p-element x_1 of G such that $xN = x_1N$.*

We shall also use the following lemma frequently and without further comment. It says that if G is a mutually permutable product and N is a normal subgroup of G, then G/N is also a mutually permutable product.

Lemma 2.2 [1, 4.1.11] *Let the group $G = G_1G_2 \cdots G_r$ be the product of the pairwise mutually permutable subgroups $G_1, G_2, \ldots, G_r$. Then G/N is the product of the mutually permutable subgroups $G_1N/N, G_2N/N, \ldots, G_rN/N$.*

Our next result will be applied to the consideration of groups with no conjugacy class length divisible by a given prime.

Lemma 2.3 [7, Theorem 5] *Let p be a prime and let G be a group such that $|x^G|$ is not divisible by p for any p-regular element x of prime power order in G. Then $G = O_p(G) \times O_{p'}(G)$.*

Lemma 2.4 *Let p be a prime and Q a p'-group acting faithfully on an elementary abelian p-group N with $|[x, N]| = p$ for all $1 \neq x \in Q$. Then Q is cyclic.*

Proof Suppose the result is not true and let Q be chosen as small as possible and then N also chosen as small as possible. Note that N cannot be cyclic. The hypotheses of the lemma remain true for subgroups of Q and so we must have $Q = \langle a, b\rangle$. Also since $N = [N, Q] \times C_N(Q)$ we have $C_N(Q) = 1$ by the minimality of N. Since $|[a, N]| = |[b, N]| = p$, $C_N(a)$ and $C_N(b)$ are maximal in N. Also, $C_N(a) \cap C_N(b) = C_N(Q) = 1$ and so if $C_N(a) = C_N(b)$ we would have $|N| = p$, a contradiction. Hence $N = C_N(a) \times C_N(b)$ and $[a, C_N(b)] = C_N(b)$ and $[b, C_N(a)] = C_N(a)$.

If $n \in C_N(a)$ then $n^{[a,b]} = n^{a^{-1}b^{-1}ab} = n^{b^{-1}ab} = (n^{b^{-1}})^b = n$ (since $n^{b^{-1}} \in C_N(a)$). Similarly $[a, b]$ centralises $C_N(b)$ and hence N, a contradiction. Thus Q is abelian. In this case we have $C_N(ab) = 1$ and this contradiction completes the proof. □

Proof of Theorem 1.1 We first give a proof for Statement 1. It is clear that only the necessity of the condition is in doubt. Assume that $G = G_1G_2 \cdots G_r$ is the product of the pairwise mutually permutable subgroups $G_1, G_2, \ldots, G_r$ and $|x^G|$ is not divisible by p for any p-regular element of prime power order $x \in \bigcup_{i=1}^r G_i$. Suppose that the theorem is not true, and let the group G provide a counterexample of least possible order. If M is a non-trivial normal subgroup of G, then G/M satisfies the hypotheses of the theorem. Thus, by the minimality of G, we have $G/M = (PM/M) \times (Q/M)$ with P a Sylow p-subgroup of G and $Q/M = \mathrm{O}_{p'}(G/M)$. Since the class of all groups which are the direct product of a p-group and a p'-group is a formation, we see easily that G has a unique minimal normal subgroup, say N. Assume that N is not abelian. We have by [1, 4.3.8] that there exists $j \in \{1, 2, \ldots, r\}$ such that $N \leq G_j$. Applying Lemma 2.3, N has a normal Sylow p-subgroup, which must be trivial because N is minimal normal in G. Thus the order of N is not divisible by p, and hence every p-regular element of PN is contained in N. By virtue of Lemma 2.3, P is a normal Sylow p-subgroup of PN. It follows immediately that P is normal in G, a contradiction which implies that N is an abelian subgroup of G. If N is a p-group then $Q = NT$, $N \cap T = 1$. Since T is a p' group, it follows that N centralises every element of T and then since T centralises P/N and N it centralises P ([5, I, 1.5]). Thus $G = P \times T$, a contradiction. Suppose now that N is a p'-group. Applying [1, 4.1.45], P is prefactorised in G, that is, $P = (P \cap G_1)(P \cap G_2) \cdots (P \cap G_r)$. Moreover, by [1, 4.1.22] (see also Ref. [2] for the case of 2 factors), every normal subgroup of G is prefactorised in G. Thus $U = PN = (U \cap G_1)(U \cap G_2) \cdots (U \cap G_r)$, and $N = (N \cap G_1)(N \cap G_2) \cdots (N \cap G_r)$. Moreover, U is the pairwise mutually permutable product of the subgroups $U \cap G_1, U \cap G_2, \cdots, U \cap G_r$. Suppose that U is a proper normal subgroup of G. Since U satisfies the hypotheses of the theorem, the minimality of G ensures that $U = P \times N$. We now have P centralises both Q/N and N and hence P centralises Q (by [5, I, 1.5]). Thus $G = P \times Q$, a contradiction. This contradiction yields $U = PN = G$. Let $n_i \in (N \cap G_i)$, $i \in \{1, 2, \ldots, r\}$. Then

$C_G(n_i)$ contains a Sylow p-subgroup of G. Since N is abelian, we have $C_G(n_i) = G$. It follows that $N \leq Z(G)$ and so $G = P \times N$, a final contradiction.

It is clear that Statement 2 is a direct consequence of Statement 1. This completes the proof of the theorem. □

Proof of Theorem 1.3 Suppose that the result is false and choose for G a counterexample of minimal order. Since the properties of G, as enunciated in the statement of the theorem, are inherited by quotients of G and the class of all groups whose chief factors have Sylow p-subgroups of order at most p is a formation, the minimality of G implies that G has a unique minimal normal subgroup N, the Sylow p-subgroups of N have order at least p^2, and the chief factors of G above N have Sylow p-subgroups of order at most p. Suppose that N is not soluble. Then $N = N_1 \times \cdots \times N_t$, is a direct product of pairwise isomorphic non-abelian simple groups N_i, $i \in \{1, 2, \ldots, t\}$. Applying [1, 4.3.8] (see also Ref. [2]), either $N \leq A$ or $N \leq B$. According to [4, Proposition 3], for each i, there exists an element $x_i \in N_i$ such that $|C_{N_i}(x_i)|$ is not divisible by p. Thus $C_{N_i}(x_i)$ is a p'-group for every $i \in \{1, 2, \ldots, t\}$. Let $x = x_1 \ldots x_t$. Clearly x is a p-regular element of G belonging to A or B, and so $|x^N|$ is not divisible by p^2. Moreover, $C_N(x) = C_{N_1}(x_1) \times \cdots \times C_{N_t}(x_t)$ is a p'-group. This implies that the Sylow p-subgroups of N have order at most p, contrary to assumption. Consequently, N must be soluble and so N is an elementary abelian p-group which is not central in G. Let $Z/C_G(N)$ be a minimal normal subgroup of $G/C_G(N)$. Then, by Ref. [5, A, 13.6], $Z/C_G(N)$ is not a p-group. In addition, the Sylow p-subgroups of $Z/C_G(N)$ have order at most p. We may assume, by Ref. [1, 4.3.11] (see also Ref. [3]), that $Z/C_G(N)$ is contained in $AC_G(N)/C_G(N)$. Suppose that $Z/C_G(N)$ is not abelian. If the Sylow p-subgroups of $Z/C_G(N)$ have order p, then it follows that $Z/C_G(N)$ is a non-abelian simple group. Applying Ref. [4, Proposition 3], there exists an element $x \in A$ such that $xC_G(N) \in Z/C_G(N)$, and $C_{Z/C_G(N)}(xC_G(N))$ is a p'-group. Since $xC_G(N)$ is a p-regular element of $AC_G(N)/C_G(N)$, we may suppose that x is a p-regular element of A. Now $|N/C_N(x)||(Z/C_G(N))/C_{Z/C_G(N)}(xC_G(N))|$ divides $|Z/C_Z(x)|$. Since $N \neq C_N(x)$ both $|N/C_N(x)|$ and $|(Z/C_G(N))/C_{Z/C_G(N)}(xC_G(N))|$ are divisible by p and hence $|Z/C_Z(x)|$ divisible by p^2, a contradiction. Thus $Z/C_G(N)$ is a p'-group. Since for every p-regular element $1 \neq xC_G(N) \in Z/C_G(N)$ we have $|N/C_N(x)| = p$ it now follows from Lemma 2.4 that $Z/C_G(N)$ is cyclic, say $Z/C_G(N) = \langle xC_G(N) \rangle$ with x p-regular. But then $N = [N, Z] \times C_N(Z)$, giving $[N, Z] = [N, x]$ is normal in G and has order p, a contradiction. This final contradiction establishes the theorem. □

Proof of Corollary 1.5 We must prove that G is supersoluble. Suppose $G = AB$ is chosen satisfying the hypothesis of the corollary but not supersoluble. Let N be a minimal normal subgroup of G. Then G/N is supersoluble. By [1, 4.3.3], $\{A \cap N, B \cap N\} \subseteq \{N, 1\}$. If $A \cap N = B \cap N = 1$, then [1, 4.3.9] implies that N is of prime order. Hence G is supersoluble, contrary to supposition. Therefore either $N \leq A$ or $N \leq B$. According Theorem 1.3, every chief factor of N has cyclic Sylow subgroups. Applying Ref. [6, IV, 2.9], N is soluble. Hence G is soluble and Theorem 1.3 yields the final contradiction. □

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References

1. Ballester-Bolinches, A., Esteban-Romero, R., Asaad, M.: Products of finite groups, vol. 53 of de Gruyter Expositions in Mathematics. Walter de Gruyter, Berlin (2010)
2. Beidleman, H., Heineken, J.C.: Mutually permutable subgroups and group classes. Arch. Math. (Basel) **85**, 18–30 (2005)
3. Beidleman, J.C., Heineken, H.: Group classes and mutually permutable products. J. Algebra **297**, 409–416 (2006)
4. Chillag, M., Herzog, D.: On the length of the conjugacy classes of finite groups. J. Algebra **131**, 110–125 (1990)
5. Doerk, K., Hawkes, T.: Finite Soluble Groups, vol. 4 of De Gruyter Expositions in Mathematics. Walter de Gruyter, Berlin (1992)
6. Huppert, B.: Endliche Gruppen I vol. 134 of Grund Math Wiss. Springer, Berlin (1967)
7. Liu, X., Wang, Y., Wei, H.: Notes on the length of conjugacy classes of finite groups. J. Pure Appl. Algebra **196**(1), 111–117 (2005)
8. Qian, G.H., Wang, Y.M.: A note on conjugacy class sizes in finite groups. Acta Math. Sinica (Chin. Ser.) **52**(1), 125–130 (2009)