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On a length preserving curve flow

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Abstract In this paper, we consider a new length preserving curve flow for closed convex curves in the plane. We show that the flow exists globally, the area of the region bounded by the evolving curve is increasing, and the evolving curve converges to the circle in C^∞ topology as $t \to \infty$.

Keywords Non-local flow · Length preserving · Curve flow

Mathematics Subject Classification (2000) 35K15 · 35K55 · 53A04

1 Introduction

In this paper, we study a nature flow for closed convex curves in the plane. This flow preserves the length of evolving curves and then it is a non-local curve flow. We shall obtain the entropy estimate and integral estimates for the evolution flow to get a global flow. Then we show that it converges to a circle at $t \to \infty$ in C^∞ sense. We remark that our method is similar to the one used in [7], where the authors have studied the curve shortening flow which shrinks to a point at finite time. Since our flow has different nature, we must give some detail. Curve shortening flow has been studied extensively

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in the last few decades (see [3,4,7–9,13,16] for background and more references). It can be showed that the convexity of curves along the curve shortening flow is preserved and the curves become more and more circular before they collapse to a point. Other flows for curves have also been proposed. One may see Andrews's papers (see for example, [1]) and Tsai's papers [18] for more flows for curves. As showed by Gage [6], some non-local flows for convex curves are also very interesting. In a very recent paper [17], Pan and Yang have considered a very interesting length preserving curve flow for convex curves in the plane of the form

$$\frac{\partial}{\partial t}\gamma(t) = \left(\frac{L}{2\pi} - k^{-1}\right) N,$$

where L, N, and k are the length, unit normal vector, and the curvature of the curve $\gamma(t)$ respectively. They have proved that the convex planar curve flow will become more and more circular and converges to circle in the C^∞ sense. Apparently it is interesting to study planar curve flows which preserve some geometry quantity, such as the area of the region bounded by the curve. For an area-preserving planar curve flow, one may see [14]. In [10], the author has studied a higher dimensional volume-preserving flow for hyper-surfaces.

The main result of this paper is the following theorem.

Theorem 1.1 *Suppose $\gamma(u, 0)$ is a convex curve in the plane R^2. Then there is a unique maximal curve flow $\gamma(t) := \gamma(u, t)$ of convex curves satisfying the following evolving equation*

$$\frac{\partial}{\partial t}\gamma(t) = (k - \alpha(t))N, \tag{1.1}$$

where k is the curvature of the evolving curve $\gamma(t)$ and

$$\alpha(t) = \frac{1}{2\pi}\int k^2 ds := \frac{1}{2\pi}\int_{\gamma(t)} k^2 ds.$$

The flow exists globally and is length preserving. Furthermore, the flow $\gamma(t)$ converges in C^∞ to the circle of fixed length L as $t \to \infty$.

We point out that the local existence and uniqueness of the flow (1.1) follows in the similar way as in Theorem 3.4 in [11] or using the trick of supporting function of convex curve. By now, this part is standard and we omit the detail. The uniqueness follows also from lemma 32.14 in the book of Kriegl and Michor [12]. We remark that for $\alpha(t) = \frac{2\pi}{L}$ in (1.1), where L is the length of the curve $\gamma(t)$, the evolution equation (1.1) is an area-preserving flow, which has been studied by Gage in [6]. In below, we shall denote $\int_{\gamma(t)}$ by $\int$ for the evolving curve $\gamma(t)$ and use C to denote various uniform positive constants.

The paper is organized as follows. In section 2, we introduce necessary formulae for the flow (1.1). In section 3, we obtain key estimates about the curvature of the evolving curve flow (1.1). We show that the evolving curve is a convex curve and the flow does not blow up in finite time. We also obtain the theorem 1.1 in the C^0 case. In the last section, we show the C^∞ convergence of the flow.

2 Preparation

First of all, we derive basic formulae for our curve flow (1.1) for closed convex planar curves. We denote the evolving curve by $\gamma(t) := \gamma(u, t)$. We let T and N be the unit tangent vector and the (inward pointing) unit normal vectors to the evolving curve.

Lemma 2.1 *Let* $w = |\gamma_u|$. *Then we have*

$$w_t = -k(k - \alpha(t))w,$$

and

$$\frac{\partial}{\partial t}\frac{\partial}{\partial s} - \frac{\partial}{\partial s}\frac{\partial}{\partial t} = k(k-\alpha)\frac{\partial}{\partial s}.$$

Proof Note that

$$w^2 = |\gamma_u|^2.$$

Then we have

$$w w_t = <\gamma_u, \gamma_{tu}> = <\gamma_u, ((k-\alpha)N)_u> = w^2(k-\alpha)<T, N_s>.$$

Using

$$N_s = -kT,$$

we have

$$w_t = -k(k-\alpha)w.$$

Then,

$$\begin{aligned}\frac{\partial}{\partial t}\frac{\partial}{\partial s} - \frac{\partial}{\partial s}\frac{\partial}{\partial t} &= \frac{\partial}{\partial t}\left(\frac{1}{w}\right)\frac{\partial}{\partial u} \\ &= k(k-\alpha)\partial_s.\end{aligned}$$

□

Recall that $ds = wdu$. Then we have

$$(ds)_t = w_t du = -k(k-\alpha)ds.$$

We shall use this formula later.

Lemma 2.2

$$\frac{\partial}{\partial t}T = \partial_s k N, \quad \frac{\partial}{\partial t}N = -\partial_s k T.$$

Proof By a direct computation, we have

$$\begin{aligned}\frac{\partial}{\partial t}T &= \frac{\partial}{\partial t}\frac{\partial}{\partial s}\gamma = \frac{\partial}{\partial s}\frac{\partial}{\partial t}\gamma + k(k-\alpha)\partial_s\gamma \\ &= \partial_s((k-\alpha)N) + k(k-\alpha)T \\ &= \partial_s k N.\end{aligned}$$

Note that

$$0 = \frac{\partial}{\partial t}<T, N> = <\frac{\partial}{\partial t}T, N> + <T, \frac{\partial}{\partial t}N> = \partial_s k + <T, \frac{\partial}{\partial t}N>.$$

Then we have

$$\frac{\partial}{\partial t}N = -\partial_s k T.$$

□

We denote the angle between the tangent of the evolving curve and the X-axis by θ. Then we have

$$cos\theta = <T, X>$$

and the curvature of the curve is given by

$$k = \frac{\partial\theta}{\partial s}.$$

For convex curves we can use the angle θ of the tangent line as a parameter [2]. We may write the curvature $k = k(\theta)$ in terms of this parameter.

Lemma 2.3

$$\frac{\partial\theta}{\partial t} = \partial_s k.$$

Proof Note that

$$-sin\theta\frac{\partial\theta}{\partial t} = <\frac{\partial T}{\partial t}, \quad X> = \partial_s k<N, X>.$$

Since $< N, X >= \cos\left(\theta + \frac{\pi}{2}\right) = -\sin\theta$. The lemma follows immediately. □

We can derive the important evolution equation for the curvature.

Lemma 2.4

$$\frac{\partial}{\partial t}k = \partial_s^2 k + k^2(k - \alpha).$$

Proof Just do the computation.

$$\frac{\partial}{\partial t}\frac{\partial}{\partial s}\theta = \frac{\partial}{\partial s}\frac{\partial}{\partial t}\theta + k(k - \alpha)\partial_s\theta = \partial_s^2 k + k^2(k - \alpha).$$

□

Note that $\int kds = 2\pi$ along the flow. Hence, we get

Lemma 2.5

$$\frac{\partial}{\partial t}\int kds = 0.$$

We remark that another proof of above fact is below through using lemma 2.4.

Proof

$$\frac{\partial}{\partial t}\int kds = \int \frac{\partial}{\partial t}kds + k\frac{\partial}{\partial t}ds = \int \partial_s^2 k + k^2(k - \alpha) - k^2(k - \alpha)ds = 0.$$

□

Using lemma 2.5, we derive

Lemma 2.6 *The length L of the evolving curve is preserved under the flow.*

Proof

$$\begin{aligned}\frac{\partial}{\partial t}L &= \int \frac{<\gamma_u, \gamma_{tu}>}{|\gamma_u|}du = \int <T, \gamma_{ts}>ds \\ &= \int <T, ((k - \alpha)N)_s>ds = -\int k(k - \alpha)ds \\ &= -\int k^2 ds + \alpha\int kds = 2\pi\alpha - \int k^2 ds = 0.\end{aligned}$$

□

Another important fact for us is the following.

Lemma 2.7 *The area $A(t)$ of the domain bounded by the curve $\gamma(t)$ is increasing. That is,*

$$\frac{d}{dt}A(t) = \alpha L - 2\pi \geq 0$$

and the equality occurs only when k is a constant, i.e. the curve is a circle.

Proof Since

$$-2A(t) = \int <\gamma, N> ds,$$

we have

$$\begin{aligned}
-2\frac{d}{dt}A(t) &= \int \frac{d}{dt}<\gamma, N> ds + \int <\gamma, N> \frac{d}{dt} ds \\
&= \int <(k-\alpha)N, N> ds + \int <\gamma, -\partial_s(k-\alpha)T> ds \\
&\quad + \int <\gamma, N>(-k(k-\alpha)) ds \\
&= \int (k-\alpha) ds - \int \partial_s(k-\alpha) <\gamma, T> ds - \int <\gamma, N> k(k-\alpha) ds \\
&= \int (k-\alpha) ds + \int (k-\alpha)(<T, T> + k<\gamma, N>) ds \\
&\quad - \int <\gamma, N> k(k-\alpha) ds \\
&= 2\int (k-\alpha) ds = 2(2\pi - L\alpha).
\end{aligned}$$

Note that

$$2\pi\alpha = \int k^2 ds \geq \frac{1}{L}\left(\int k ds\right)^2 = \frac{4\pi^2}{L}.$$

We remark that from the Cauchy-Schwartz inequality, the equality occurs only when k is a constant, i.e. the curve is a circle. Then we have

$$2\pi - \alpha L \leq 0,$$

which implies the result wanted. □

We now consider the growth of $\alpha = \alpha(t)$.

Lemma 2.8

$$\partial_t \alpha = -\frac{1}{\pi}\int (k_s)^2 ds + \frac{1}{2\pi}\int k^3(k-\alpha) ds.$$

Proof Do the computation.

$$
\begin{aligned}
\partial_t \alpha &= \frac{1}{\pi}\int k k_t ds - \frac{1}{2\pi}\int k^3(k-\alpha)ds \\
&= \frac{1}{\pi}\int k(k_{ss} + k^2(k-\alpha))ds - \frac{1}{2\pi}\int k^3(k-\alpha)ds \\
&= -\frac{1}{\pi}\int (k_s)^2 ds + \frac{1}{2\pi}\int k^3(k-\alpha)ds.
\end{aligned}
$$

The lemma follows immediately. □

3 Long time existence

In this section, we derive key estimates of the curvature k of the evolving curve $\gamma(t)$.

We firstly work with the general curve flow of convex curves with

$$\frac{\partial}{\partial t}\gamma = (k-\alpha)N + \eta T \tag{3.1}$$

where the function η will be given later.

Similar to results in the section above, we can compute the following basic formulae of the flow (3.1).

Lemma 3.1 *Commutator:*

$$\partial_t \partial_s - \partial_s \partial_t = k(k-\alpha)\partial_s - \eta_s \partial_s.$$

The growth of tangent:

$$\partial_t T = (\partial_s(k-\alpha) + k\eta)N.$$

The change of angle:

$$\partial_t \theta = \partial_s k + k\eta.$$

Length invariant:

$$\partial_t L = 0.$$

Area growth:

$$\partial_t A = \int (\alpha - k)ds.$$

The computation is omitted.

We now take η such that $\partial_t \theta = 0$, i.e. $\eta = -\partial_\theta k$. Then by changing the space variable we can transform away the tangential component, without changing the shape of the curves (see also the proof of Theorem 4.1.4 in [7]).

Remark 3.1 Along the flow (3.1), the length is also preserved, and the variation of area is the same as the case when $\eta = 0$.

The evolution of the curvature of the evolving curve is given below.

Lemma 3.2

$$\frac{\partial k}{\partial t} = k^2 \partial_\theta^2 k + k^2 (k - \alpha).$$

Proof

$$\begin{aligned}\partial_t k &= \partial_s \partial_t \theta + k(k-\alpha)\partial_s \theta - \partial_s \eta \partial_s \theta \\ &= \partial_s(\partial_s k + k\eta) + k(k-\alpha)\partial_s \theta - \partial_s \eta k \\ &= \partial_s^2 k + \eta \partial_s k + k^2(k-\alpha).\end{aligned}$$

Since $\frac{ds}{d\theta} = k$, we have $\partial_s = k\partial_\theta$. Substituting $\partial_s = k\partial_\theta$ and $\eta = -\partial_\theta k$ into the above equality, we have

$$\begin{aligned}\partial_t k &= k\partial_\theta(k\partial_\theta k) + k\partial_\theta k(-\partial_\theta k) + k^2(k-\alpha) \\ &= k^2\partial_\theta^2 k + k(\partial_\theta k)^2 - k(\partial_\theta k)^2 + k^2(k-\alpha) \\ &= k^2 \partial_\theta^2 k + k^2(k-\alpha).\end{aligned}$$

Then the proof of the lemma is completed. □

One of the main results in this section is below.

Theorem 3.1 *Convexity is preserved along the flow* (3.1). *In fact, for any finite* $T \in (0, \infty)$ *such that the curve flow exists on* $[0, T]$*, we have that* $k(t)$ *is uniformly bounded from below by a positive constant on the interval* $[0, T]$.

Proof Fix any finite $T \in (0, \infty)$ such that the curve flow exists on $[0, T]$. By a direct computation, we have, for $t \in (0, T]$,

$$\partial_t \left(\frac{1}{k} - \frac{A}{L} - \frac{2\pi t}{L}\right) = k^2 \partial_\theta^2 \left(\frac{1}{k} - \frac{A}{L} - \frac{2\pi t}{L}\right) - 2k^3 \left(\partial_\theta \left(\frac{1}{k} - \frac{A}{L} - \frac{2\pi t}{L}\right)\right)^2 - k. \tag{3.2}$$

For any $t < T_0$, where $T_0 \in (0, T]$ is the first time such that $k(T_0) = 0$ (which implies that $\frac{1}{k} - \frac{A}{L} - \frac{2\pi t}{L}$ blows up at T_0), we have $k(t) > 0$. Take any $\epsilon > 0$ small enough.

Suppose the maximum of $\frac{1}{k} - \frac{A}{L} - \frac{2\pi t}{L}$ attains at $(x_0, t_0) \in \gamma \times [0, T_0 - \epsilon]$. Assume that $t_0 > 0$. Then we have, at (x_0, t_0), $-k < 0$,

$$\frac{\partial}{\partial t}\left(\frac{1}{k} - \frac{A}{L} - \frac{2\pi t}{L}\right)|_{(x_0,t_0)} \geq 0,$$
$$k^2 \partial_\theta^2 \left(\frac{1}{k} - \frac{A}{L} - \frac{2\pi t}{L}\right)|_{x_0,t_0} \leq 0,$$

and

$$2k^3 \left(\partial_\theta \left(\frac{1}{k} - \frac{A}{L} - \frac{2\pi t}{L}\right)\right)^2 |_{x_0,t_0} = 0.$$

All these relations imply a contradiction with (3.2). Then, $\frac{1}{k} - \frac{A}{L} - \frac{2\pi t}{L}$ attains its maximum at $t_0 = 0$, which implies that

$$\frac{1}{k} - \frac{A}{L} - \frac{2\pi t}{L} \leq \max_\theta \left[\frac{1}{k(0,\theta)} - \frac{A(0)}{L}\right].$$

This implies that $k(t) \geq \frac{1}{C_1 + C_2 t} := c(t)$, where $C_1 > 0$ and $C_2 > 0$ are uniform constants independent of $\epsilon > 0$. This implies that T_0 does not exist. Hence, the convexity is preserved along the flow on $[0, T]$ and $k(t)$ is uniformly bounded from below by a positive constant on the interval $[0, T]$. □

Following [7], we now do the entropy estimate.

Theorem 3.2 *$\int \log k(\theta, t) d\theta$ is non increasing along the flow and there is a uniform bound for $\int \log k(\theta, t) d\theta$ along the flow.*

Proof

$$\frac{\partial}{\partial t} \int \log k(\theta, t) d\theta = \int k \partial_\theta^2 k + k(k - \alpha) d\theta$$
$$= \int_0^{2\pi} -(\partial_\theta k)^2 + (k - \alpha)^2 d\theta + \alpha \int_0^{2\pi} (k - \alpha) d\theta.$$

By definition, we have $\int_0^{2\pi} (k - \alpha) d\theta = 0$. Using the Wirtinger inequality, we have

$$\frac{\partial}{\partial t} \int_0^{2\pi} \log k(\theta, t) d\theta \leq 0.$$

Therefore,

$$\int_0^{2\pi} \log k(\theta, t) d\theta \leq \int_0^{2\pi} \log k(\theta, 0) d\theta.$$

□

Based on the entropy estimate above we can derive the following result.

Theorem 3.3 *Assume that the curve flow exists on* $[0, T)$. *Then for any* $\delta > 0$, *we can find a constant* $C(T) > 0$ *such that* $k(\theta, t) \leq C(T)$ *except on intervals of length less than or equal to* δ.

Proof If $k \geq C(T)$ on $a \leq \theta \leq b$ and $b - a \geq \delta$, then

$$\int_0^{2\pi} \log k(\theta, t) d\theta \geq \delta \log C(T) + (2\pi - \delta) \log k_{min}(t)$$
$$\geq \delta \log C(T) + (2\pi - \delta) \log c(T),$$

where $c(T)$ is the lower bound of k on $[0, T)$. Using the fact that $\int_0^{2\pi} \log k(\theta, t) d\theta$ is non-increasing, we know that $C(T)$ is bounded above. □

Along the flow we have the following inverse Poincare type inequality.

Lemma 3.3 *We have*

$$\int \left(\frac{\partial k}{\partial \theta}\right)^2 d\theta \leq \int k^2 + D$$

for some uniform constant D which depends only on the initial curve $\gamma(0)$.

Proof Compute,

$$\begin{aligned}
&\frac{\partial}{\partial t} \int (k - \alpha)^2 - \left(\frac{\partial k}{\partial \theta}\right)^2 d\theta \\
&= 2 \int (k - \alpha)(\partial_t k - \partial_t \alpha) - 2 \frac{\partial k}{\partial \theta} \frac{\partial^2 k}{\partial \theta \partial t} d\theta \\
&= 2 \int (k - \alpha + \partial_\theta^2 k) \partial_t k d\theta - 2 \int (k - \alpha) \partial_t \alpha d\theta \\
&= 2 \int (k - \alpha + \partial_\theta^2 k)^2 k^2 d\theta + 2 \partial_t \alpha (2\pi \alpha - \int k d\theta).
\end{aligned}$$

Note that

$$\int k d\theta = \int k^2 ds = 2\pi \alpha.$$

Then we have

$$\frac{\partial}{\partial t}\int (k-\alpha)^2 - \left(\frac{\partial k}{\partial \theta}\right)^2 d\theta = 2\int \left(k-\alpha+\partial_\theta^2 k\right)^2 k^2 d\theta.$$

Integrating the above inequality, we have

$$\int (k(t)-\alpha(t))^2 - \left(\frac{\partial k(t)}{\partial \theta}\right)^2 d\theta \geq \int (k(0)-\alpha(0))^2 - \left(\frac{\partial k(0)}{\partial \theta}\right)^2 d\theta = -D.$$

By this we have

$$\int \left(\frac{\partial k(t)}{\partial \theta}^2\right) d\theta \leq \int (k(t)-\alpha(t))^2 + D \leq \int k^2 d\theta + D.$$

This completes the proof.

Theorem 3.4 *If* $\int_0^{2\pi} \log k(\theta, t) d\theta$ *is uniformly bounded on* $[0, T)$, *then* $k(\theta, t)$ *is uniformly bounded on* $S^1 \times [0, T)$.

Proof For any given δ, by theorem 3.3, we have $k \leq C(T)$ except on intervals $[a, b]$ of length less than δ. On such an interval

$$\begin{aligned} k(\phi) &= k(a) + \int_a^\phi \frac{\partial k}{\partial \theta} d\theta \leq C(T) + \sqrt{\delta}\left(\int \left(\frac{\partial k}{\partial \theta}\right)^2 d\theta\right)^{1/2} \\ &\leq C(T) + \sqrt{\delta}\left(\int k^2 d\theta + D\right)^{1/2}. \end{aligned}$$

Assume that k attains its maximum at ϕ. Then we have

$$k_{max} \leq C(T) + \sqrt{\delta}\left(2\pi k_{max}^2 + D\right)^{1/2}.$$

By choosing δ small, we have

$$k_{max}^2 \leq \frac{2C^2(T) + 2\delta D}{1 - 4\pi\delta} \leq 4C^2(T).$$

□

Lemma 3.4 *If k is bounded, then $\frac{\partial k}{\partial \theta}$ is bounded.*

Proof

$$\partial_t \partial_\theta k = k^2 \partial_\theta^3 k + 2k\partial_\theta k \partial_\theta^2 k + 3k^2 \partial_\theta k - 2\alpha k \partial_\theta k.$$

Since k is bounded, α is bounded. Then $\partial_\theta k$ grows at most exponentially. □

In the following, we will use $k^{'}, k^{''}$, etc, to denote the derivatives of $\gamma(t)$ with respect to the variable θ.

Lemma 3.5 *If k and $k^{'}$ are bounded, then $\int_0^{2\pi}(k^{''})^4 d\theta$ is bounded.*

Proof By the Holder inequality, we have

$$\begin{aligned}
\frac{\partial}{\partial t}\int_0^{2\pi}(k^{''})^4 d\theta &= 4\int_0^{2\pi}(k^{''})^3(k^2k^{''}+k^2(k-\alpha))^{''}d\theta \\
&= -12\int_0^{2\pi}(k^{''})^2(k^{'''})(k^2k^{'''}+2kk^{'}k^{''}+3k^2k^{'}-2\alpha kk^{'})d\theta \\
&= -12\int_0^{2\pi}k^2(k^{''})^2(k^{'''})^2+2kk^{'}(k^{''})^3k^{'''}+3k^2k^{'}(k^{''})^2k^{'''}d\theta \\
&\quad +24\alpha\int_0^{2\pi}kk^{'}(k^{''})^2k^{'''}d\theta \\
&\le C_1\int_0^{2\pi}(k^{''})^4(k^{'})^2 d\theta + C_2\int_0^{2\pi}k^2(k^{'})^2(k^{''})^2 d\theta \\
&\quad +24\alpha C_3\int_0^{2\pi}(k^{'})^2(k^{''})^2 d\theta.
\end{aligned}$$

By the bound of $k, k^{'}$, we see that $\int_0^{2\pi}(k^{''})^4 d\theta$ grows at most exponentially. □

Lemma 3.6 *If $k, k^{'}$, and $\int_0^{2\pi}(k^{''})^4 d\theta$ are bounded, then so is $\int_0^{2\pi}(k^{'''})^2 d\theta$.*

Proof Note that

$$\begin{aligned}
&\frac{\partial}{\partial\theta}\int_0^{2\pi}(k^{'''})^2 d\theta \\
&= -2\int_0^{2\pi}k^{''''}(k^2k^{''}+k^3-\alpha k^2)^{''}d\theta \\
&= -2\int_0^{2\pi}k^2(k^{''''})^2+4kk^{'}k^{'''}k^{''''}+2k(k^{''})^2k^{''''} \\
&\quad +2(k^{'})^2k^{''}k^{''''}+3k^2k^{''}k^{''''}+6k(k^{'})^2k^{''''} \\
&\quad -2\alpha(k^{'})^2k^{''''}-2\alpha kk^{''}k^{''''}d\theta
\end{aligned}$$

$$\leq C_1 \int (k')^2 (k''')^2 d\theta + C_2 \int (k'')^4 d\theta + C_3 \int \frac{(k')^4}{k^2} (k'')^2 d\theta$$
$$+ C_4 \int k^2 (k'')^2 d\theta + C_5 \int (k')^4 d\theta + C_6 \int \frac{(k')^4}{k^2} d\theta + C_7 \int (k'')^2 d\theta.$$

By the bounds of $k,\ k',\ ||k''||_4$, we see that $\int_0^{2\pi} (k''')^2 d\theta$ grows at most exponentially. □

Corollary 3.1 *Under the same hypothesis as above, k'' is bounded.*

Proof Recall the well-known fact that for any one dimensional smooth function f, we always have that

$$\max |f|^2 \leq C \int |f'|^2 + f^2.$$

We apply this to k'' to get the desired estimate. □

Lemma 3.7 *If k, k', and k'' are uniformly bounded, then so are k''' and all the higher derivatives of k.*

Proof We compute

$$\begin{aligned} \frac{\partial}{\partial t} k''' &= (k^2 k'' + k^3 - k^2 \alpha)''' \\ &= k^2 k^v + 6kk' k^{iv} + (8kk'' + 6k'^2 + 3k^2 - 2\alpha k) k''' \\ &\quad + (6k'(k'')^2 + 18kk'k'' + 6(k')^3 - 6\alpha k' k''). \end{aligned}$$

Since k, k', k'', α are bounded, k''' grows at most exponentially. Similarly, we can show that for any $n \geq 3$, the derivative $k^{(n)}$ is bounded on any finite intervals. □

Then we have

Theorem 3.5 *The curve flow does not blow up in any finite time.*

Proof By the above analysis, the curvature of the evolving curve does not blow up in any finite time. This implies that the curve flow does not blow up in finite time and it exists globally. □

We now recall the following inequality from [5].

Theorem 3.6 *For any closed, convex C^2 curve γ in the plane, we have*

$$\pi \frac{L}{A} \leq \int_0^L k^2 ds,$$

where L, A and k are the length of the curve, the area it encloses, and its curvature respectively.

Theorem 3.7 *If the convex curve $\gamma(t)$ evolves according to (3.1), then the isoperimetric deficit $L^2 - 4\pi A$ is decreasing during the evolution process (1.1) and converges to zero as the time t goes to infinity. Furthermore, the evolving curve converges to a circle in the Hausdorff sense.*

Proof Since the length of the curve is preserved, we have

$$\frac{d}{dt}(L^2 - 4\pi A) = -4\pi \frac{d}{dt}A(t) = -4\pi(L\alpha - 2\pi) \leq 0.$$

From the above theorem 3.6, we have

$$\frac{d}{dt}(L^2 - 4\pi A) \leq -4\pi \left(\frac{L^2}{2A} - 2\pi\right) = -\frac{2\pi}{A}(L^2 - 4\pi A).$$

Note that for any closed plane curve, we have

$$\frac{L^2}{4\pi} \geq A.$$

Then we have

$$\frac{d}{dt}(L^2 - 4\pi A) \leq -\frac{8\pi^2}{L^2}(L^2 - 4\pi A).$$

Hence,

$$L^2 - 4\pi A(t) \leq Cexp\left(-\frac{8\pi^2}{L^2}t\right),$$

and as $t \to \infty$, we have

$$L^2 - 4\pi A \to 0.$$

By the Bonnesen inequality (see [15]), $\frac{L^2}{A} - 4\pi \geq \frac{\pi^2}{A}(r_{out} - r_{in})^2$, we have $r_{out} - r_{in} \to 0$, as $t \to \infty$. Then the evolving curve converges to a circle in the Hausdorff sense. □

4 C^∞ convergence

In this section, we shall complete the proof of Theorem 1.1. We shall use the argument in Sect. 5 in the work [7] to prove the C^∞ convergence of the curve flow. The difference between our work with [7] is that Gage and Hamilton have used the normalized curvature for the curve-shortening flow in the plane, which satisfies a similar evolution equation as our curvature flow.

Define, for $w \in (0, \pi]$,

$$k_w^* = sup\{b | k(\theta) > b \; on \; some \; interval \; of \; length \; w\}.$$

Using similar argument as in Lemma 5.1 in [7] we have the following geometric estimate.

Lemma 4.1

$$k_w^*(t) r_{in}(t) \leq \frac{1}{1 - K(w)(\frac{r_{out}}{r_{in}} - 1)},$$

where r_{in} and r_{out} are the radii of the largest inscribed circle and the smallest circumscribed circle of the curve defined by the curvature function $k(\cdot, t)$ respectively. The function K is defined by

$$K(w) = \frac{2cos(\frac{w}{2})}{1 - cos(\frac{w}{2})}.$$

Remark 4.1 Since the proof is almost same as in Lemma 5.1 in [7], we omit the detail. Here we note that the function $K(w)$ is a positive decreasing function of w with $K(0) = \infty$ and $K(\pi) = 0$.

From theorem 3.7 in the above section, we know that the curve flow $\gamma(t)$ converges to a circle in the Hausdorff sense as $t \to \infty$, i.e.

$$\pi (r_{out} - r_{in})^2 \leq L^2 - 4\pi A \to 0.$$

We also have $\pi r_{out}^2 \geq A(t) \geq A(0)$. Hence, for any sufficient large time T_1 we have $r_{in}(t) \geq \sqrt{\frac{A(0)}{2\pi}}$ for $t \geq T_1$.

We firstly fix a small w. Then there is a sufficient large time $T_2 \geq T_1$ such that

$$K(w) \left(\frac{r_{out}}{r_{in}} - 1 \right) \leq 1/2$$

for $t \geq T_2$. By theorem 4.1, we have $k_w^*(t) r_{in}(t) \leq 2$, i.e., $k_w^*(t) \leq 2\sqrt{\frac{2\pi}{A(0)}}$ for $t \geq T_2$.

Then, we have

Theorem 4.1 *The curvature $k(t)$ is uniformly bounded along the flow.*

Proof We just need to consider the curvature k of the evolving curve $\gamma(t)$ for $t \geq T_2$. First we fix a small $w < (\frac{1}{4\pi})^2$. Assume that $[a, b]$ is an interval such that $k \geq k_w^*$.

By this, we have $|b-a| \le w$ and $k(a) = k_w^*$. For any $\phi \in [a, b]$, we have

$$k(\phi) = k(a) + \int_a^{\phi} \frac{\partial k}{\partial \theta} d\theta \le k_w^* + \sqrt{w}\left(\int \left(\frac{\partial k}{\partial \theta}\right)^2 d\theta\right)^{1/2}$$
$$\le k_w^* + \sqrt{w}\left(\int k^2 d\theta + D\right)^{1/2}.$$

Let k_{max} denote the maximum value of k. Then

$$k_{max} \le k_w^* + \sqrt{w}\left(2\pi k_{max}^2 + D\right)^{1/2} \le k_w^* + 2\pi\sqrt{w} k_{max} + \sqrt{w} D.$$

Combining the above results together, we know that k is bounded uniformly for $t \ge T_2$. □

Since $k(\theta, t)$ is uniformly bounded, $\int (\partial_\theta k)^2 d\theta$ is uniformly bounded. By this we know that $k(\cdot, t)$ is equi-continuous. Then for any sequence $k(\theta, t_i)$, we can choose a sequence $k(\theta, t_{i_n})$ converging uniformly to $k(\theta, \infty)$. Note that the evolving curve converges to the circle in the Hausdorff sense. Then $k(\theta, \infty) = const$. Since the sequence $k(\theta, t_i)$ is arbitrary and it has a subsequence which converges to the same curve with $k(\theta, \infty) = const$, we know that $k(\theta, t)$ converges to $k(\theta, \infty) = const$ uniformly.

Using an argument similar to section 5 in [7], we shall give various energy bounds for the curvature.

Lemma 4.2 $||k'||_4$ *are bounded by constants independent of* t.

Proof Compute,

$$\frac{\partial}{\partial t}\int (k')^4 d\theta = 4\int (k')^3 (k^2 k'' + k^2(k-\alpha))' d\theta$$
$$= -12\int_0^{2\pi} k^2 (k')^2 (k'')^2 - 12\int_0^{2\pi} (k')^2 k'' k^3 d\theta$$
$$-8\alpha \int_0^{2\pi} (k')^4 k d\theta$$
$$\le 3\int_0^{2\pi} k^4 (k')^2 d\theta - 8\alpha \int_0^{2\pi} (k')^4 d\theta.$$

Since k converges to a constant at $t = \infty$, $\alpha(t)$ converges to the constant at $t = \infty$. Using the Holder inequality, we have

$$\frac{\partial f}{\partial t} \le C_1 f^{1/2} - C_2 f$$

for $f = \int_0^{2\pi} (k^{'})^4 d\theta$, where $C_1 > 0$ and $C_2 > 0$ are constants independent of t. Using lemma 5.7.4 in [7], we know that $||k^{'}||_4(t)$ is uniformly bounded. □

Lemma 4.3 $||k^{''}||_2$ *is bounded by a constant which is independent of* t.

Proof Compute,

$$\begin{aligned}
&\partial_t \frac{1}{2}\int_0^{2\pi} (k^{''})^2 d\theta \\
&= \int_0^{2\pi} k^{''} (\partial_t k)^{''} d\theta \\
&= -\int_0^{2\pi} k^{'''} (k^2 k^{''})^{'} d\theta + \int_0^{2\pi} k^{''} ((k^3)^{''} - \alpha (k^2)^{''}) d\theta \\
&= -\int_0^{2\pi} k^2 (k^{'''})^2 d\theta - 2\int_0^{2\pi} k k^{'} k^{''} k^{'''} d\theta \\
&\quad -3\int_0^{2\pi} k^2 k^{'} k^{'''} d\theta - 2\alpha \int_0^{2\pi} k (k^{''})^2 d\theta - 4\alpha \int_0^{2\pi} k^{''} (k^{'})^2 d\theta.
\end{aligned}$$

By the Cauchy inequality, we have

$$\begin{aligned}
\partial_t \int_0^{2\pi} (k^{''})^2 d\theta \le\ & C_1 \int_0^{2\pi} (k^{'} k^{''})^2 d\theta + C_2 \int_0^{2\pi} k^2 (k^{'})^2 d\theta - 4\alpha \int_0^{2\pi} k (k^{''})^2 d\theta \\
& -4\alpha \int_0^{2\pi} k^{''} (k^{'})^2 d\theta.
\end{aligned}$$

We can control the first term by using the inequality in lemma 4.2. In fact,

$$\begin{aligned}
\partial_t \int_0^{2\pi} (k^{''})^2 d\theta \le\ & C_2 \int_0^{2\pi} k^2 (k^{'})^2 d\theta - 4\alpha \int_0^{2\pi} k (k^{''})^2 d\theta \\
& -4\alpha \int_0^{2\pi} k^{''} (k^{'})^2 d\theta - C_3 \partial_t \int_0^{2\pi} (k^{'})^4 d\theta
\end{aligned}$$

$$-C_4\int_0^{2\pi}(k')^2k''k^3d\theta - C_5\int_0^{2\pi}(k')^4d\theta$$
$$\leq C_6 - C_7\int_0^{2\pi}(k'')^2d\theta - C_3\partial_t\int_0^{2\pi}(k')^4d\theta.$$

We denote $\int_0^{2\pi}(k'')^2d\theta$ by $f(t)$. Then we have

$$\partial_t f \leq C_6 - C_7 f - C_3\partial_t\int_0^{2\pi}(k')^4d\theta.$$

Multiplying e^{c_7t} on both sides of the above inequality and integrating, we have

$$e^{c_7t}f(t)|_0^t \leq C_6\int_0^t e^{C_7t}dt - C_3\int_0^t e^{C_7t}\partial_t\int_0^{2\pi}(k')^4d\theta dt.$$

Then we have

$$e^{C_7t}f(t) \leq \frac{C_6}{C_7}e^{C_7t} + C_3C_7\int_0^t e^{C_7t}\int_0^{2\pi}(k')^4d\theta dt + c_3\int_0^{2\pi}(k')^4d\theta + C_8$$
$$\leq \frac{C_6}{C_7}e^{C_7t} + C_3Me^{C_7t} + C_3M + C_8,$$

where M is the bound of $\int_0^{2\pi}(k')^4d\theta$ which is independent of t. So $\int_0^{2\pi}(k'')^2d\theta$ is uniformly bounded.

Then using similar argument as in lemma 5.7.8 of [7], we have □

Lemma 4.4 $||k'||_\infty$ *converges to 0 as* $t\to\infty$.

Similar to lemma 5.7.9 in [7], we have

Lemma 4.5 *For any* $\beta\in(0,1)$ *we can choose* A *so that for* $t > A$,

$$\int(k'')^2d\theta \geq 4\beta\int(k')^2d\theta.$$

The proof of lemma 4.5 is omitted.

Lemma 4.6 *There is a constant* $C_1 > 0$ *such that* $||k'||_2 \leq C_1e^{-4C^2t}$, *where* C *is the constant such that* $k\to C$ *as* $t\to\infty$.

Proof Compute,

$$\partial_t \int (k^{'})^2 d\theta = 2\int k^{'}(k^2k^{''} + k^3 - \alpha k^2)^{'} d\theta$$
$$= -2\int k^2(k^{''})^2 d\theta + 6\int k^2(k^{'})^2 d\theta - 4\alpha\int k(k^{'})^2 d\theta.$$

Since $k \to C$, as $t \to \infty$, we have

$$\partial_t \int (k^{'})^2 d\theta \leq -2C^2\int (k^{''})^2 d\theta + 6C^2\int (k^{'})^2 d\theta - 4C^2\int (k^{'})^2 d\theta$$
$$\leq -8C^2\beta\int (k^{'})^2 d\theta + 6C^2\int (k^{'}) d\theta - 4C^2\int (k^{'})^2 d\theta$$
$$\leq -4C^2\int (k^{'})^2 d\theta.$$

So we have completed the proof. In below, we shall obtain good exponential decay bounds on the low order derivatives. □

Lemma 4.7 *For any* $\beta \in (0, 1)$, *there is a uniform constant* $C > 0$ *such that* $||k^{''}||_2 \leq Ce^{-2\beta t}$.

Proof By a direct computation, we have

$$\partial_t \int (k^{''})^2 d\theta$$
$$= 2\int k^{''}(k_t)^{''} d\theta = 2\int k^{''}(k^2k^{''} + k^2(k-\alpha))^{''} d\theta$$
$$= -2\int k^2(k^{'''})^2 d\theta - 4\int kk^{'}k^{''}k^{'''} d\theta - 6\int k^2k^{'}k^{'''} d\theta$$
$$-4\alpha\int (k^{'})^2k^{''} - 4\alpha\int k(k^{''})^2 d\theta$$
$$\leq -2\int k^2(k^{'''})^2 d\theta + 4\epsilon\int k^2(k^{'''})^2 d\theta + 1/\epsilon\int (k^{'})^2(k^{''})^2 d\theta$$
$$+6(\epsilon\int (kk^{'''})^2 + 1/4\epsilon\int k^2(k^{'})^2) - 4\alpha\int (k^{'})^2k^{''} d\theta$$
$$-4\alpha\int k(k^{''})^2 d\theta.$$

We choose $\epsilon > 0$ small such that

$$\partial_t \int (k^{''})^2 d\theta \leq C_1\int (k^{'})^2(k^{''})^2 d\theta + C_2\int k^2(k^{'})^2 d\theta$$
$$-4\alpha\int (k^{'})^2k^{''} d\theta - 4\alpha\int k(k^{''})^2 d\theta.$$

Since $||k^{'}||_\infty$ converges to 0, $k \to C, \alpha \to C$ as $t \to \infty$, we have

$$\partial_t \int (k^{''})^2 d\theta \le -2C^2 \int (k^{''})^2 d\theta + C_2 C e^{-C_3 t}.$$

We denote $\int (k^{''})^2 d\theta$ by f (and we may repeat the use of the same f to denote various quantities in different lemmas). Then we have

$$\partial_t f \le -2C^2 f + C_2 C e^{-C_3 t}.$$

Using lemma 5.7.5 of [7], we complete the proof. □

Lemma 4.8 *For any $\beta \in (0, 1)$, we can find a uniform constant C such that $||k^{''}||_4 \le Ce^{-\beta t}$.*

Proof Compute,

$$\begin{aligned}
&\partial_t \int (k^{''})^4 d\theta \\
&= 4 \int (k^{''})^3 (k^2 k^{''} + k^2 (k-\alpha))^{''} d\theta \\
&\quad = -12 \int k^2 (k^{''})^2 (k^{'''})^2 d\theta - 24 \int k k^{'} (k^{''})^3 k^{'''} d\theta \\
&\qquad -36 \int k^2 k^{'} (k^{''})^2 k^{'''} d\theta - 8\alpha \int (k^{'})^2 (k^{''})^3 d\theta - 8\alpha \int k (k^{''})^4 d\theta \\
&\quad \le -12 \int k^2 (k^{''})^2 (k^{'''})^2 d\theta + 24 \left(\epsilon \int k^2 (k^{''})^2 (k^{'''})^2 + 1/4\epsilon \int (k^{'})^2 (k^{''})^4 d\theta \right) \\
&\qquad +36 \left(\epsilon \int k^2 (k^{''})^2 (k^{'''})^2 + 1/4\epsilon \int k^2 (k^{'})^2 (k^{''})^2 d\theta \right) \\
&\qquad -8\alpha \int (k^{'})^2 (k^{''})^3 d\theta - 8\alpha \int k (k^{''})^4 d\theta \\
&\quad \le C_1 \int (k^{'})^2 (k^{''})^4 d\theta + C_2 \int k^2 (k^{'})^2 (k^{''})^2 d\theta \\
&\qquad -8\alpha \int (k^{'})^2 (k^{''})^3 d\theta - 8\alpha \int k (k^{''})^4 d\theta.
\end{aligned}$$

By the Young inequality, we have

$$\int (k^{'})^2 (k^{''})^3 d\theta \le \epsilon \int (k^{'})^{4/3} (k^{''})^4 d\theta + C(\epsilon) \int (k^{'})^4 d\theta,$$
$$\int k^2 (k^{'})^2 (k^{''})^2 d\theta \le \epsilon \int (k^{''})^4 d\theta + C(\epsilon) \int k^4 (k^{'})^4 d\theta.$$

Since $||k^{'}||_{\infty} \to 0$ as $t \to \infty$, we have

$$\partial_t \int (k^{''})^4 d\theta \leq -C_1 \int (k^{''})^4 d\theta + C_2 \int (k^{'})^4 d\theta.$$

We denote $\int (k^{''})^4 d\theta$ by f. Then we have

$$\partial_t f \leq -C_1 f + C_2 e^{-C_3 t}.$$

By a use of lemma 5.7.5 in [7], we can complete the proof. □

Lemma 4.9 *For any $\beta \in (0, 1)$, there is some uniform constant C such that $||k^{'''}||_2 \leq Ce^{-2\beta t}$.*

Proof Compute,

$$\begin{aligned}\partial_t \int (k^{'''})^2 d\theta = & -2 \int k^2 (k^{''''})^2 d\theta - 8 \int k k^{'} k^{'''} k^{''''} d\theta \\ & -4 \int (k^{'})^2 k^{''} k^{''''} d\theta - 4 \int k (k^{''})^2 k^{''''} d\theta - 6 \int k^2 k^{''} k^{''''} d\theta \\ & -12 \int k (k^{'})^2 k^{''''} d\theta - 4\alpha \int k (k^{'''})^2 d\theta - 12\alpha \int k^{'} k^{''} k^{'''} d\theta.\end{aligned}$$

By the Young inequality, we have

$$\begin{aligned}\partial_t \int (k^{'''})^2 d\theta \leq & -2 \int k^2 (k^{''''})^2 d\theta + \epsilon \int k^2 (k^{''''})^2 d\theta + C_1(\epsilon) \int (k^{'})^2 (k^{'''})^2 d\theta \\ & + C_2(\epsilon) \int \frac{(k^{'})^4 (k^{''})^2}{k^2} d\theta + C_3(\epsilon) \int (k^{''})^4 d\theta \\ & + C_4(\epsilon) \int (k^{'})^4 d\theta + C_5(\epsilon) \int k^2 (k^{''})^2 d\theta - 4\alpha \int k (k^{'''})^2 d\theta \\ & + 12\alpha \left(\epsilon \int (k^{'''})^2 d\theta + C_6(\epsilon) \int (k^{'})^2 (k^{''})^2 d\theta \right).\end{aligned}$$

By the above estimate of $k^{'}$ and $k^{''}$, for sufficiently large t, we have

$$\partial_t \int (k^{'''})^2 d\theta \leq -C_6 \int (k^{'''})^2 d\theta + C_7 e^{-\beta t}.$$

The result follows from lemma 5.7.5 in [7]. □

In fact, the method in [7] can be applied to our case. The high order estimate of k is similar to [7], so we omit the detail. By now, we have completed the proof of Theorem 1.1.

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