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Global existence and blow-up phenomena for a periodic 2-component Camassa–Holm equation

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Abstract We first establish local well-posedness for a periodic 2-component Camassa–Holm equation. We then present two global existence results for strong solutions to the equation. We finally obtain several blow-up results and the blow-up rate of strong solutions to the equation.

Keywords A periodic 2-component Camassa–Holm equation · Global existence · Blow-up · Blow-up rate

Mathematics Subject Classification (2000) 35G40 · 35L05

1 Introduction

In the paper we consider the Cauchy problem of the following periodic 2-component Camassa–Holm equation

$$\begin{cases} y_t + y_x u + 2yu_x + \sigma\rho\rho_x = 0, & t > 0,\ x \in \mathbb{R}, \\ \rho_t + (\rho u)_x = 0, & t > 0, x \in \mathbb{R}, \\ u(0,x) = u_0(x), & x \in \mathbb{R}, \\ \rho(0,x) = \rho_0(x), & x \in \mathbb{R}, \\ u(t,x) = u(t,x+1), & t \geq 0,\ x \in \mathbb{R}, \\ \rho(t,x) = \rho(t,x+1), & t \geq 0,\ x \in \mathbb{R}, \end{cases} \tag{1.1}$$

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where $y = u - u_{xx}$, $\sigma = \pm 1$. (The Camassa–Holm equation can be obtained via the obvious reduction $\rho \equiv 0$.)

The 2-component generalization of Camassa–Holm equation (1.1) was recently derived by Constantin and Ivanov [17] in the context of shallow water theory. $u(t,x)$ describes the horizontal velocity of the fluid and $\rho(t,x)$ is in connection with the horizontal deviation of the surface from equilibrium, all measured in dimensionless units [17].

Equation (1.1) with $\sigma = -1$ corresponds to the situation in which the gravity acceleration points upwards. For $\sigma = -1$ in Eq. (1.1) was introduced by Chen et al. in [2,7,24] and Falqui in [24]. Similar to the Camassa–Holm equation, Eq. (1.1) can be identified with the first negative flow of the AKNS hierarchy and possesses the interesting peakon and multi-kink solutions, cf. [7]. Moreover Eq. (1.1) is connected with the time dependent Schrödinger spectral problem [1,7]. Popowicz has been observed that Eq. (1.1) is related to the bosonic sector of an $N = 2$ supersymmetric extension of the classical Camassa–Holm equation [34]. There are many further works to study its mathematical properties, cf. [7,17,23,31].

With $\rho \equiv 0$ in Eq. (1.1), we find the Camassa–Holm equation, which models the wave motion on shallow water, $u(t,x)$ representing the fluid's free surface above a flat bottom (or equivalently the fluid velocity at time $t \geq 0$ in the spatial x direction) [6,22,32]. Many interesting phenomena like solitons [3,21], bi-Hamiltonian structure [8,25], integrability [6,10] and wave breaking [9,13–15,19,33,35,38] are found in the Camassa–Holm equation. And there is a geometric interpretation of Eq. (1.1) in terms of geodesic flow on the diffeomorphism group of the circle [18]. There are numerous papers to study the Camassa–Holm equation on its mathematical issues, such as local well-posedness [11,14,33,35], global existence of strong solutions modeling permanent waves [14,16,19], the existence and uniqueness of global weak solutions with initial data $u_0 \in H^1(\mathbb{R})$ [4,5,20,37], and the behavior of compactly supported solutions [12,27].

For $\rho \neq 0$, the Cauchy problem of Eq. (1.1) on the line (nonperiodic case) with $\sigma = -1$ and with $\sigma = 1$ has been discussed in [23] and [17,26], respectively. In [23], Escher et al. establish the local well-posedness and present the precise blow-up scenarios and several blow-up results of strong solutions to Eq. (1.1) with $\sigma = -1$ on the line. In [17], Constantin and Ivanov investigate the global existence and blow-up phenomena of strong solutions of Eq. (1.1) with $\sigma = 1$ on the line. Later, Guan and Yin obtain a new global existence result for strong solutions to Eq. (1.1) with $\sigma = 1$ and get several blow-up results [26] which improve the recent results in [17]. Henry studies the infinite propagation speed for Eq. (1.1) with $\sigma = 1$ in [28]. The blow-up phenomena of Eq. (1.1) with $\sigma = -1$ on the circle have been studied in [30]. However, Eq. (1.1) with $\sigma = 1$ on the circle (periodic case) has not been studied yet. The aim of this paper is to present two global existence results for strong solutions to Eq. (1.1) with $\sigma = 1$, and to show that it has solutions which blow up in finite time, provided their initial data satisfy certain conditions.

The paper is organized as follows. In Sect. 2, we briefly give some needed results including the local well-posedness of Eq. (1.1), the precise blow-up scenarios and some useful lemmas to study global existence and blow-up phenomena. In Sect. 3, we address the global existence of Eq. (1.1) by introducing a continuous family of diffeomorphisms of the line and using an important conservation law. In Sect. 4, we give several blow-up criteria and the precise blow-up rate, which exhibit that Eq. (1.1) has blow-up solutions modeling wave breaking.

2 Preliminaries

In the section, we briefly give the needed results to pursue our goal. We first present the local well-posedness for the Cauchy problem of Eq. (1.1) in $H^s(\mathbb{S}) \times H^{s-1}(\mathbb{S})$, $s \geq 2$, with $\mathbb{S} = \mathbb{R}/\mathbb{Z}$ (the circle of unit length) by applying Kato's theory.

Let us introduce some notations. Let X and Y be Hilbert spaces such that Y is continuously and densely embedded in X and let $Q : Y \to X$ be a topological isomorphism. $L(Y, X)$ denotes the space of all bounded linear operators from Y to X ($L(X)$, if $X = Y$.). $\|\cdot\|_X$ denotes the norm of Banach space X. $G(X, 1, \beta)$ denotes the set of all linear operators A in X, such that $-A$ generates a C_0-semigroup $T(t)$ on X and that $\|T(t)\|_{L(X)} \leq e^{t\beta}$ for all $t \geq 0$.

Let $G(x) := \frac{\cosh(x-[x]-1/2)}{2\sinh(1/2)}$, $x \in \mathbb{R}$. Then $(1-\partial_x^2)^{-1} f = G * f$ for all $f \in L^2(\mathbb{S})$ and $G * y = u$. Here, we denote by $*$ the convolution. By a direct calculation, one can rewrite Eq. (1.1) with $\sigma = 1$ as follows:

$$\begin{cases} u_t + uu_x + \partial_x G * (u^2 + \frac{1}{2}u_x^2 + \frac{1}{2}\rho^2) = 0, & t > 0,\ x \in \mathbb{R}, \\ \rho_t + u\rho_x + u_x\rho = 0, & t > 0,\ x \in \mathbb{R}, \\ u(0,x) = u_0(x), & x \in \mathbb{R}, \\ \rho(0,x) = \rho_0(x), & x \in \mathbb{R}, \\ u(t,x) = u(t,x+1), & t \geq 0,\ x \in \mathbb{R}, \\ \rho(t,x) = \rho(t,x+1), & t \geq 0,\ x \in \mathbb{R}. \end{cases} \tag{2.1}$$

Or the equivalent form:

$$\begin{cases} u_t + uu_x = -\partial_x(1-\partial_x^2)^{-1}(u^2 + \frac{1}{2}u_x^2 + \frac{1}{2}\rho^2), & t > 0,\ x \in \mathbb{R}, \\ \rho_t + u\rho_x + u_x\rho = 0, & t > 0,\ x \in \mathbb{R}, \\ u(0,x) = u_0(x), & x \in \mathbb{R}, \\ \rho(0,x) = \rho_0(x), & x \in \mathbb{R}, \\ u(t,x) = u(t,x+1), & t \geq 0,\ x \in \mathbb{R}, \\ \rho(t,x) = \rho(t,x+1), & t \geq 0,\ x \in \mathbb{R}. \end{cases} \tag{2.2}$$

We now have the following local well-posedness result.

Theorem 2.1 *Given* $z_0 = \begin{pmatrix} u_0 \\ \rho_0 \end{pmatrix} \in H^s(\mathbb{S}) \times H^{s-1}(\mathbb{S}),\ s \geq 2$, *there exists a maximal* $T = T(z_0) > 0$, *and a unique solution* $z = \begin{pmatrix} u \\ \rho \end{pmatrix}$ *to Eq.* (2.1) *such that*

$$z = z(., z_0) \in C([0,T); H^s(\mathbb{S}) \times H^{s-1}(\mathbb{S})) \cap C^1([0,T); H^{s-1}(\mathbb{S}) \times H^{s-2}(\mathbb{S})).$$

Moreover, the solution depends continuously on the initial data, i.e. the mapping $z_0 \to z(., z_0) : H^s(\mathbb{S}) \times H^{s-1}(\mathbb{S}) \to C([0,T); H^s(\mathbb{S}) \times H^{s-1}(\mathbb{S})) \cap C^1([0,T); H^{s-1}(\mathbb{S}) \times H^{s-2}(\mathbb{S}))$ *is continuous.*

The proof of Theorem 2.1 is similar to that of Theorem 2.2 in [23], we omit it here.

By the local well-posedness in Theorem 2.1 and the energy method, one can get the following precise blow-up scenario of strong solutions to Eq. (2.1).

Theorem 2.2 [23] *Let* $z_0 = \begin{pmatrix} u_0 \\ \rho_0 \end{pmatrix} \in H^s(\mathbb{S}) \times H^{s-1}(\mathbb{S}), s > \frac{5}{2}$, *and let* T *be the maximal existence time of the solution* $z = \begin{pmatrix} u \\ \rho \end{pmatrix}$ *to Eq.* (2.1) *with the initial data* z_0. *Then the corresponding solution blows up in finite time if and only if*

$$\lim_{t\to T} \inf_{x\in\mathbb{S}} \{u_x(t,x)\} = -\infty \quad or \quad \lim_{t\to T} \sup\{\|\rho_x(t,\cdot)\|_{L^\infty(\mathbb{S})}\} = +\infty.$$

The proof of Theorem 2.2 is similar to that of Theorem 3.2 in [23], we omit it here.

For initial data $z_0 = \begin{pmatrix} u_0 \\ \rho_0 \end{pmatrix} \in H^2(\mathbb{S}) \times H^1(\mathbb{S})$, we have the following precise blow-up scenario.

Theorem 2.3 [23] *Let* $z_0 = \begin{pmatrix} u_0 \\ \rho_0 \end{pmatrix} \in H^2(\mathbb{S}) \times H^1(\mathbb{S})$, *and let* T *be the maximal existence time of the solution* $z = \begin{pmatrix} u \\ \rho \end{pmatrix}$ *to Eq.* (2.1) *with the initial data* z_0. *Then the corresponding solution blows up in finite time if and only if*

$$\lim_{t\to T} \inf_{x\in\mathbb{S}} \{u_x(t,x)\} = -\infty.$$

The proof of the theorem is similar to the proof of Theorem 3.3 in [23], we omit it here.

Remark 2.1 If $\rho \equiv 0$, then Theorems 2.2–2.3 cover the corresponding results for the Camassa–Holm equation in [33,35].

Given initial data $z_0 = \begin{pmatrix} u_0 \\ \rho_0 \end{pmatrix} \in H^s(\mathbb{S}) \times H^{s-1}(\mathbb{S}), s \geq 2$, Theorem 2.1 ensures the existence and uniqueness of strong solutions to Eq. (2.1).

Consider the following initial value problem

$$\begin{cases} q_t = u(t,q), & t \in [0,T),\ x \in \mathbb{R}, \\ q(0,x) = x, & x \in \mathbb{R}, \end{cases} \tag{2.3}$$

where u denotes the first component of the solution z to Eq. (2.1) with the initial data z_0. Since $u(t,.) \in H^2(\mathbb{S}) \subset C^m(\mathbb{S})$ with $0 \leq m \leq \frac{3}{2}$, it follows that $u \in C^1([0,T) \times \mathbb{R}, \mathbb{R})$. Applying the classical results in the theory of ordinary differential equations, one can obtain the following results of q which is the key in the proof of global existence of solutions to Eq. (2.1) in Theorem 3.2.

Lemma 2.1 [17,23,26] *Let* $z_0 = \begin{pmatrix} u_0 \\ \rho_0 \end{pmatrix} \in H^s(\mathbb{S}) \times H^{s-1}(\mathbb{S})$, $s \geq 2$, *and let* $T > 0$ *be the maximal existence time of corresponding solution* $z = \begin{pmatrix} u \\ \rho \end{pmatrix}$ *to Eq.* (2.1) *with the initial data* z_0. *Then Eq.* (2.3) *has a unique solution* $q \in C^1([0, T) \times \mathbb{R}, \mathbb{R})$. *Moreover, the map* $q(t, \cdot)$ *is an increasing diffeomorphism of* $\mathbb{R}$ *with*

$$q_x(t, x) = \exp\left(\int_0^t u_x(s, q(s, x))ds\right) > 0, \quad (t, x) \in [0, T) \times \mathbb{R}.$$

Lemma 2.2 [23,26] *Let* $z_0 = \begin{pmatrix} u_0 \\ \rho_0 \end{pmatrix} \in H^s(\mathbb{S}) \times H^{s-1}(\mathbb{S})$, $s \geq 2$, *and let* $T > 0$ *be the maximal existence time of corresponding solution* $z = \begin{pmatrix} u \\ \rho \end{pmatrix}$ *to Eq.* (2.1) *with the initial data* z_0. *Then we have*

$$\rho(t, q(t, x))q_x(t, x) = \rho_0(x), \quad (t, x) \in [0, T) \times \mathbb{R}. \tag{2.4}$$

Moreover if there exists $x_0 \in \mathbb{S}$ *such that* $\rho_0(x_0) = 0$, *then* $\rho(t, q(t, x_0)) = 0$ *for all* $t \in [0, T)$.

We then give several useful conservation laws of strong solutions to Eq. (2.1).

Lemma 2.3 *Let* $z_0 = \begin{pmatrix} u_0 \\ \rho_0 \end{pmatrix} \in H^s(\mathbb{S}) \times H^{s-1}(\mathbb{S})$, $s \geq 2$, *and let* T *be the maximal existence time of the solution* $z = \begin{pmatrix} u \\ \rho \end{pmatrix}$ *to Eq.* (2.1) *with the initial data* z_0. *Then for all* $t \in [0, T)$, *we have*

$$\int_{\mathbb{S}} u(t, x)dx = \int_{\mathbb{S}} u_0(x)dx,$$
$$\int_{\mathbb{S}} \rho(t, x)dx = \int_{\mathbb{S}} \rho_0(x)dx.$$

Proof Integrating the first equation in (2.1) by parts, in view of the periodicity of u and G, we get

$$\frac{d}{dt}\int_{\mathbb{S}} udx = -\int_{\mathbb{S}} uu_x dx - \int_{\mathbb{S}} \partial_x G * \left(u^2 + \frac{1}{2}u_x^2 + \frac{1}{2}\rho^2\right) dx = 0.$$

On the other hand, integrating the second equation in (2.1) by parts, in view of the periodicity of u and ρ, we get

$$\frac{d}{dt}\int_{\mathbb{S}} \rho dx = -\int_{\mathbb{S}} (u\rho)_x dx = 0.$$

This completes the proof of the lemma. □

Lemma 2.4 *Let* $z_0 = \begin{pmatrix} u_0 \\ \rho_0 \end{pmatrix} \in H^s(\mathbb{S}) \times H^{s-1}(\mathbb{S})$, $s \geq 2$, *and let* T *be the maximal existence time of the solution* $z = \begin{pmatrix} u \\ \rho \end{pmatrix}$ *to Eq.* (2.1) *with the initial data* z_0. *Then for all* $t \in [0, T)$, *we have*

$$\int_{\mathbb{S}} \Big(u^2(t,x) + u_x^2(t,x) + \rho^2(t,x)\Big)\, dx = \int_{\mathbb{S}} \Big(u_0^2(x) + u_{0x}^2(t,x) + \rho_0^2(x)\Big)\, dx.$$

Proof Multiplying the first equation in (2.1) by u and integrating by parts, we have

$$\frac{d}{dt}\int_{\mathbb{S}} \Big(u^2(t,x) + u_x^2(t,x)\Big)\, dx = \int_{\mathbb{S}} u_x(t,x)\rho^2(t,x)dx.$$

Multiplying the second equation in (2.1) by ρ and integrating by parts, we get

$$\frac{d}{dt}\int_{\mathbb{S}} \rho^2(t,x)dx = -\int_{\mathbb{S}} u_x(t,x)\rho^2(t,x)dx.$$

Adding the above two equalities, we obtain

$$\frac{d}{dt}\int_{\mathbb{S}} \Big(u^2(t,x) + u_x^2(t,x) + \rho^2(t,x)\Big)\, dx = 0.$$

This completes the proof of the lemma. □

Lemma 2.5 [13] *Let* $T > 0$ *and* $v \in C^1([0,T); H^2(\mathbb{R}))$. *Then for every* $t \in [0,T)$, *there exists at least one point* $\xi(t) \in \mathbb{R}$ *with*

$$m(t) := \inf_{x\in\mathbb{R}} [v_x(x,t)] = v_x(t, \xi(t)).$$

The function $m(t)$ *is almost everywhere differentiable on* $(0,t)$ *with*

$$\frac{dm(t)}{dt} = v_{tx}(t, \xi(t)), \quad a.e. \text{ on } \quad (0,t).$$

Lemma 2.6 [36,38] (i) *For every* $f \in H^1(\mathbb{S})$, *we have*

$$\max_{x\in[0,1]} f^2(x) \leq \frac{e+1}{2(e-1)}\|f\|^2_{H^1(\mathbb{S})},$$

where the constant $\frac{e+1}{2(e-1)}$ *is sharp.*

(ii) *For every* $f \in H^3(\mathbb{S})$, *we have*

$$\max_{x\in[0,1]} f^2(x) \leq c\|f\|^2_{H^1(\mathbb{S})},$$

with the best possible constant c *lying within the range* $(1, \frac{13}{12}]$. *Moreover, the best constant* c *is* $\frac{e+1}{2(e-1)}$.

By the conservation law stated in Lemma 2.4 and Lemma 2.6 (i), we have the following corollary.

Corollary 2.1 *Let* $z_0 = \begin{pmatrix} u_0 \\ \rho_0 \end{pmatrix} \in H^s(\mathbb{S}) \times H^{s-1}(\mathbb{S})$, $s \geq 2$ *be given and assume that* T *is the maximal existence time of the corresponding solution* $z = \begin{pmatrix} u \\ \rho \end{pmatrix}$ *to Eq.* (2.1) *with the initial data* z_0. *Then for all* $t \in [0, T)$, *we have*

$$\|u(t,\cdot)\|^2_{L^\infty(\mathbb{S})} \leq \frac{e+1}{2(e-1)}\|u(t,\cdot)\|^2_{H^1(\mathbb{S})} \leq \frac{e+1}{2(e-1)}(\|u_0\|^2_{H^1(\mathbb{S})} + \|\rho_0\|^2_{L^2(\mathbb{S})}).$$

Lemma 2.7 [29] *If* $f \in H^3(\mathbb{S})$ *is such that* $\int_{\mathbb{S}} f(x)dx = \frac{a_0}{2}$, *then for every* $\epsilon > 0$, *we have*

$$\max_{x\in[0,1]} f^2(x) \leq \frac{\epsilon+2}{24}\int_{\mathbb{S}} f_x^2 dx + \frac{\epsilon+2}{4\epsilon}a_0^2.$$

Moreover,

$$\max_{x\in[0,1]} f^2(x) \leq \frac{\epsilon+2}{24}\|f\|^2_{H^1(\mathbb{S})} + \frac{\epsilon+2}{4\epsilon}a_0^2.$$

3 Global existence

In the section, we give two global existence results for strong solutions to Eq. (2.1).

Theorem 3.1 [23] *Let* $z_0 = \begin{pmatrix} u_0 \\ \rho_0 \end{pmatrix} \in H^s(\mathbb{S}) \times H^{s-1}(\mathbb{S})$, $s \geq 2$ *be given and assume that* T *is the maximal existence time of the corresponding solution* $z = \begin{pmatrix} u \\ \rho \end{pmatrix}$ *to Eq.* (2.1) *with the initial data* z_0. *If there exits* $M > 0$ *such that*

$$\|u_x(t,\cdot)\|_{L^\infty(\mathbb{S})} + \|\rho(t,\cdot)\|_{L^\infty(\mathbb{S})} + \|\rho_x(t,\cdot)\|_{L^\infty(\mathbb{S})} \leq M, \ \ t \in [0, T),$$

then the $H^s(\mathbb{S}) \times H^{s-1}(\mathbb{S})$*-norm of* $z(t,\cdot)$ *does not blow up on* $[0, T)$.

The proof of the theorem is similar to that of Theorem 3.1 *in* [23]*, so we omit it.*

By Lemmas 2.1–2.2 and Lemma 2.4, we obtain a new global existence of strong solutions of Eq. (2.1).

Theorem 3.2 *Let* $z_0 = \begin{pmatrix} u_0 \\ \rho_0 \end{pmatrix} \in H^2(\mathbb{S}) \times H^1(\mathbb{S})$ *be given. If* $\rho_0(x) \neq 0$ *for all* $x \in \mathbb{S}$*, then the corresponding strong solution* $z = \begin{pmatrix} u \\ \rho \end{pmatrix}$ *to Eq.* (2.1) *with the initial data* z_0 *exists globally in time.*

Proof Assumed that T is the maximal existence time of the corresponding solution z to Eq. (2.1) with the initial data z_0. In view of Theorem 2.3, it suffices to prove that there exits $M > 0$, such that $\inf_{x\in\mathbb{S}} u_x(t,x) \geq -M$ for all $t \in [0,T)$.

By Lemmas 2.1–2.2, we know that $\rho(0,x)$ has the same sign with $\rho(t, q(t,x))$. Since $\rho(0,x) \neq 0$ for all $x \in \mathbb{S}$, it follows that $\rho(t,q(t,x)) \neq 0$ for all $(t,x) \in [0,T) \times \mathbb{S}$.

By Lemma 2.1, we have that the map $q(t,\cdot)$ is an increasing diffeomorphism of $\mathbb{R}$. By the periodicity of u_x and the property of $q(t,\cdot)$, we have $\inf_{x\in\mathbb{R}} u_x(t,q(t,x)) = \inf_{x\in\mathbb{R}} u_x(t,x) = \inf_{x\in\mathbb{S}} u_x(t,x)$. Set $m(t,x) = u_x(t,q(t,x))$.

Next, we consider the function introduced in [17],

$$w(t,x) = \rho(0,x)\rho(t,q(t,x)) + \frac{\rho(0,x)}{\rho(t,q(t,x))}(1 + m^2(t,x)).$$

By Sobolev imbedding theorem, we have

$$0 < w(0,x) \leq \|\rho_0\|^2_{L^\infty(\mathbb{S})} + \|u_0\|^2_{H^1(\mathbb{S})} + 1 \leq \|z_0\|_{H^s(\mathbb{S})\times H^{s-1}(\mathbb{S})} + 1.$$

By the definition of $m(t,x)$ and the first equation in (2.1), we have

$$\frac{\partial m}{\partial t} = (u_{tx} + uu_{xx})(t, q(t,x)). \tag{3.1}$$

By Eq. (2.3) and the second equation in (2.1), we obtain

$$\frac{\partial \rho(t,q(t,x))}{\partial t} = -\rho(t,q(t,x))m(t,x). \tag{3.2}$$

Differentiating the first equation in (2.1) with respect to x, we get

$$u_{tx} = -\frac{1}{2}u_x^2 - uu_{xx} + u^2 + \frac{1}{2}\rho^2 - G * \left(u^2 + \frac{1}{2}u_x^2 + \frac{1}{2}\rho^2\right). \tag{3.3}$$

Substituting $(t, q(t,x))$ into (3.3), we obtain

$$\begin{aligned}\frac{\partial m}{\partial t} = &-\frac{1}{2}m^2(t) + u^2(t,q(t,x)) + \frac{1}{2}\rho^2(t,q(t,x)) \\ &-G * \left(u^2 + \frac{1}{2}u_x^2 + \frac{1}{2}\rho^2\right)(t,q(t,x)).\end{aligned} \tag{3.4}$$

Differentiating $w(t,x)$ with respect to t and using (3.2) and (3.4), we have

$$\begin{aligned}
\frac{dw}{dt} &= \frac{2\rho(0,x)}{\rho(t,q(t,x))} m(t,x)\left[u^2 - G*\left(u^2+\frac{1}{2}u_x^2+\frac{1}{2}\rho^2\right)+\frac{1}{2}\right]\\
&\le \frac{\rho(0,x)}{\rho(t,q(t,x))}(1+m^2(t,x))\left[u^2 - G*\left(u^2+\frac{1}{2}u_x^2+\frac{1}{2}\rho^2\right)+\frac{1}{2}\right]\\
&\le \left|u^2 - G*\left(u^2+\frac{1}{2}u_x^2+\frac{1}{2}\rho^2\right)+\frac{1}{2}\right| w(t,x)\\
&\le \left(\frac{e+1}{2(e-1)}\|u\|^2_{H^1(\mathbb{S})}+\|G\|_{L^\infty(\mathbb{S})}\|u^2+\frac{1}{2}u_x^2+\frac{1}{2}\rho^2\|_{L^1(\mathbb{S})}+\frac{1}{2}\right) w(t,x)\\
&\le \left(\frac{e+1}{2(e-1)}E_0+\frac{\cosh(1/2)}{2\sinh(1/2)}E_0+\frac{1}{2}\right) w(t,x),
\end{aligned}$$

where $E_0 = \|u_0\|^2_{H^1(\mathbb{S})}+\|\rho_0\|^2_{L^2(\mathbb{S})}$. Here we use Young's inequality, Corollary 2.1 and the fact that $\frac{1}{2\sinh(1/2)} \le G(x) \le \frac{\cosh(1/2)}{2\sinh(1/2)}$.

By Gronwall's inequality, we have

$$w(t,x) \le w(0,x)e^{Kt} \le (\|z_0\|_{H^s(\mathbb{S})\times H^{s-1}(\mathbb{S})}+1)e^{KT},$$

where $K = \left(\frac{e+1}{2(e-1)}E_0+\frac{\cosh(1/2)}{2\sinh(1/2)}E_0+\frac{1}{2}\right)$.

On the other hand, we get

$$w(t,x) \ge 2\sqrt{\rho^2(0,x)(1+m^2)} \ge 2a|m(t,x)|,$$

where $a = \inf_{x\in\mathbb{S}}|\rho_0(x)| > 0$.

Thus, we deduce that

$$m(t,x) \ge -\frac{1}{2a}w(t,x) \ge -\frac{1}{2a}((\|z_0\|_{H^s(\mathbb{S})\times H^{s-1}(\mathbb{S})}+1)e^{KT}) := -M.$$

This completes the proof of the theorem. □

From the proof of Theorem 3.2, we have the following corollary immediately.

Corollary 3.1 *Let* $z_0 = \begin{pmatrix} u_0 \\ \rho_0 \end{pmatrix} \in H^s(\mathbb{S}) \times H^{s-1}(\mathbb{S})$, $s > \frac{5}{2}$, *and assume that* T *is the maximal existence time of the corresponding solution* $z = \begin{pmatrix} u \\ \rho \end{pmatrix}$ *to Eq.* (2.1) *with the initial data* z_0. *If* $\rho_0(x) \neq 0$ *for all* $x \in \mathbb{S}$, *then* $u_x(t,x)$ *has a lower bound for all* $(t,x) \in [0,T) \times \mathbb{S}$, *i.e. the corresponding solution blows up in finite time if and only if*

$$\limsup_{t\to T}\{\|\rho_x(t,\cdot)\|_{L^\infty(\mathbb{S})}\} = +\infty.$$

4 Blow-up phenomena

In the section we investigate the blow-up phenomena of strong solutions to Eq. (2.1). We now present the first blow-up result.

Theorem 4.1 *Let* $z_0 = \begin{pmatrix} u_0 \\ \rho_0 \end{pmatrix} \in H^s(\mathbb{S}) \times H^{s-1}(\mathbb{S})$, $s > \frac{5}{2}$, *and let* T *be the maximal existence time of solution* $z = \begin{pmatrix} u \\ \rho \end{pmatrix}$ *to Eq.* (2.1) *with the initial data* z_0*. Assume that* $E_0 := \|u_0\|^2_{H^1(\mathbb{S})} + \|\rho_0\|^2_{L^2(\mathbb{S})} \neq 0$ *and* $\int_{\mathbb{S}} \rho_0(x)dx = 0$*. If there exists some* $x_0 \in \mathbb{S}$ *and* $K_0 = K_0(E_0) > 0$ *such that*

$$\int_{\mathbb{S}} u_{0x}^3 dx < -K_0,$$

then the corresponding solution to Eq. (2.1) *blows up in finite time.*

Proof Let z be the solution to Eq. (2.1) with the initial data $z_0 \in H^s(\mathbb{S}) \times H^{s-1}(\mathbb{S})$, $s > \frac{5}{2}$, and let $T > 0$ be the maximal time of existence of the solution z with the initial data z_0. If the statement is not true, then by Theorem 2.2 it follows that there exists $M > 0$, such that $u_x(t,x) > -M$ for any $(t,x) \in [0,T) \times \mathbb{S}$, and $\|\rho_x(t,\cdot)\|_{L^\infty(\mathbb{S})} \le M$ for all $t \in [0,T)$.

Applying $u_x^2 \partial_x$ to both side of the first equation in (2.2) and integrating by parts, we get

$$\begin{aligned}\frac{d}{dt}\int_{\mathbb{S}} u_x^3 dx + \frac{1}{2}\int_{\mathbb{S}} u_x^4 dx &= 3\int_{\mathbb{S}} u_x^2 (u^2 + \frac{1}{2}\rho^2) dx \\ &\quad -3\int_{\mathbb{S}} u_x^2 G * (u^2 + \frac{1}{2}u_x^2 + \frac{1}{2}\rho^2) dx. \qquad (4.1)\end{aligned}$$

Note that

$$\left|\int_{\mathbb{S}} u_x^3 dx\right| \le \left(\int_{\mathbb{S}} u_x^4 dx\right)^{\frac{1}{2}} \left(\int_{\mathbb{S}} u_x^2 dx\right)^{\frac{1}{2}}.$$

By Lemma 2.4, we have $\|u_0\|^2_{H^1(\mathbb{S})} \le E_0$. Thus we get

$$\int_{\mathbb{S}} |u_x|^4 dx \ge \frac{1}{\|u\|^2_{H^1(\mathbb{S})}} \left(\int_{\mathbb{S}} u_x^3 dx\right)^2 \ge \frac{1}{E_0}\left(\int_{\mathbb{S}} u_x^3 dx\right)^2.$$

By the above inequality and (4.1), we obtain

$$\frac{d}{dt}\int_{\mathbb{S}} u_x^3 dx + \frac{\left(\int_{\mathbb{S}} u_x^3 dx\right)^2}{2E_0} \le 3\int_{\mathbb{S}} u_x^2\left(u^2+\frac{1}{2}\rho^2\right)dx$$
$$-3\int_{\mathbb{S}} u_x^2 G*\left(u^2+\frac{1}{2}u_x^2+\frac{1}{2}\rho^2\right)dx$$
$$\le \frac{3}{2}\int_{\mathbb{S}} u_x^2u^2dx + \frac{3}{2}\int_{\mathbb{S}} u_x^2\rho^2dx, \tag{4.2}$$

where we use the relations $\int_{\mathbb{S}} u_x^2 G*\rho^2 dx \ge 0$ and $G*(u^2+\frac{1}{2}u_x^2) \ge \frac{1}{2}u^2$.

Using Young's inequality and Corollary 2.1, we have

$$\left|\int_{\mathbb{S}} u_x^2u^2\,dx\right| \le \|u\|^2_{L^\infty(\mathbb{S})}\left|\int_{\mathbb{S}} u_x^2 dx\right| \le \frac{e+1}{2(e-1)}E_0. \tag{4.3}$$

By the assumption $\int_{\mathbb{S}} \rho_0(x)dx = 0$ and Lemma 2.3, we have

$$\int_{\mathbb{S}} \rho(t,x)dx = \int_{\mathbb{S}} \rho_0(x)dx = 0.$$

It then follows that for any $t \in [0,T)$, $\rho(t,\cdot)$ has a zero point η_t. Thus we have

$$\rho(t,x) = \int_{\eta_t}^{x} \rho_x(t,s)ds, \quad x\in[\eta_t,\eta_t+1],$$

which implies that

$$|\rho(t,x)| = \left|\int_{\eta_t}^{x} \rho_x(t,s)ds\right| \le M,$$
$$\left|\int_{\mathbb{S}} u_x^2\rho^2\,dx\right| \le M^2\int_{\mathbb{S}} u_x^2dx \le M^2E_0. \tag{4.4}$$

By (4.2)–(4.4), we obtain

$$\frac{d}{dt}\int_{\mathbb{S}} u_x^3dx \le -\frac{1}{2E_0}\left(\int_{\mathbb{S}} u_x^3dx\right)^2 + \frac{3}{2}\int_{\mathbb{S}} u_x^2u^2dx + \frac{3}{2}\int_{\mathbb{S}} u_x^2\rho^2\,dx$$
$$\le -\frac{1}{2E_0}\left(\int_{\mathbb{S}} u_x^3dx\right)^2 + \frac{3(e+1)}{4(e-1)}E_0 + \frac{3}{2}M^2E_0.$$

Set $m(t)=\int_{\mathbb{S}} u_x^3dx$ and $K=\left(\frac{3(e+1)}{4(e-1)}E_0+\frac{3}{2}M^2E_0\right)^{\frac{1}{2}}$. Note that if $m(0)<-\sqrt{2E_0}K$, then $m(t)<-\sqrt{2E_0}K$. Therefore, we can solve the above inequality to obtain

$$\frac{m(0)+\sqrt{2E_0}K}{m(0)-\sqrt{2E_0}K}e^{\sqrt{2/E_0}Kt}-1\leq\frac{2\sqrt{2E_0}K}{m(t)-\sqrt{2E_0}K}\leq 0.$$

Due to $0<\frac{m(0)+\sqrt{2E_0}K}{m(0)-\sqrt{2E_0}K}<1$, then there exists T_1 satisfying

$$0<T_1<\frac{1}{\sqrt{2/E_0}K}\ln\Big(\frac{m(0)-\sqrt{2E_0}K}{m(0)+\sqrt{2E_0}K}\Big),$$

such that $\lim_{t\uparrow T_1}m(t)=-\infty$. This contradicts the assumption $u_x(t,x)>-M$ for all $(t,x)\in[0,T)\times\mathbb{S}$. Let $K_0=\sqrt{2E_0}K$. Applying Theorem 2.2, we deduce that the solution z blows up in finite time. This completes the proof of the theorem. □

Next, we give the second blow-up result.

Theorem 4.2 *Let* $z_0=\begin{pmatrix}u_0\\ \rho_0\end{pmatrix}\in H^s(\mathbb{S})\times H^{s-1}(\mathbb{S})$, $s\geq 2$, *and let T be the maximal existence time of solution* $z=\begin{pmatrix}u\\ \rho\end{pmatrix}$ *to Eq.* (2.1) *with the initial data z_0. If there is some $x_0\in\mathbb{S}$ such that $\rho_0(x_0)=0$ and*

$$u_0'(x_0)<-\left[\frac{e+1}{2(e-1)}(\|u_0\|^2_{H^1(\mathbb{S})}+\|\rho_0\|^2_{L^2(\mathbb{S})})\right]^{\frac{1}{2}},$$

then the corresponding solution to Eq. (2.1) *blows up in finite time.*

Proof Let z be the solution to Eq. (2.1) with the initial data $z_0\in H^s(\mathbb{S})\times H^{s-1}(\mathbb{S})$, $s\geq 2$, and let $T>0$ be the maximal time of existence of the solution z with the initial data z_0. Note that $\partial_x^2G*f=G*f-f$. Differentiating the first equation in (2.1) with respect to x, we get

$$u_{tx}=-\frac{1}{2}u_x^2-uu_{xx}+u^2+\frac{1}{2}\rho^2-G*\left(u^2+\frac{1}{2}u_x^2+\frac{1}{2}\rho^2\right). \tag{4.5}$$

Define $m(t)=u_x(t,q(t,x_0))$ and $h(t)=\rho(t,q(t,x_0))$. By Eq. (2.1) and Eq. (2.3), we have

$$\frac{dm}{dt}=(u_{tx}+u_{xx}q_t)(t,q(t,x_0))=(u_{tx}+uu_{xx})(t,q(t,x_0))$$

and

$$\frac{dh}{dt}=\rho_t+\rho_xq_t=-hm.$$

Substituting $(t, q(t, x_0))$ into Eq. (4.5), we obtain

$$m'(t) = -\frac{1}{2}m^2(t) + u^2(t, q(t, x_0)) + \frac{1}{2}h^2 - G * \left(u^2 + \frac{1}{2}u_x^2 + \frac{1}{2}\rho^2\right)(t, q(t, x_0))$$
$$\leq -\frac{1}{2}m^2(t) + \left(\frac{1}{2}u^2\right)(t, q(t, x_0)) + \frac{1}{2}h^2. \quad (4.6)$$

Here we use the relation $G * (u^2 + \frac{1}{2}u_x^2) \geq \frac{1}{2}u^2$, and $G * \rho^2 \geq 0$.

Recalling Corollary 2.1, we have

$$\|u\|_{L^\infty(\mathbb{S})} \leq \sqrt{\frac{e+1}{2(e-1)}} \left(\|u_0\|^2_{H^1(\mathbb{S})} + \|\rho_0\|^2_{L^2(\mathbb{S})}\right)^{\frac{1}{2}}.$$

Note that $h(0) = \rho(0, q(0, x_0)) = \rho_0(x_0) = 0$. By Lemma 2.2, we have $h(t) = 0$ for all $t \in [0, T)$. Thus, we deduce that

$$m'(t) \leq -\frac{1}{2}m^2(t) + \frac{e+1}{4(e-1)} \left(\|u_0\|^2_{H^1(\mathbb{S})} + \|\rho_0\|^2_{L^2(\mathbb{S})}\right).$$

Set $E_0 = \|u_0\|^2_{H^1(\mathbb{S})} + \|\rho_0\|^2_{L^2(\mathbb{S})}$ and $K = \left(\frac{e+1}{4(e-1)}E_0\right)^{\frac{1}{2}}$. Note that if $m(0) < -\sqrt{2}K$, then $m(t) < -\sqrt{2}K$, for all $t \in [0, T)$. Therefore, we can solve the above inequality to obtain

$$\frac{m(0) + \sqrt{2}K}{m(0) - \sqrt{2}K} e^{\sqrt{2}Kt} - 1 \leq \frac{2\sqrt{2}K}{m(t) - \sqrt{2}K} \leq 0.$$

Due to $0 < \frac{m(0)+\sqrt{2}K}{m(0)-\sqrt{2}K} < 1$, there exists T_1, and $0 < T_1 < \frac{1}{\sqrt{2}K} \ln(\frac{m(0)-\sqrt{2}K}{m(0)+\sqrt{2}K})$, such that $\lim_{t\uparrow T_1} m(t) = -\infty$. Applying Theorem 2.3, the solution z does not exist globally in time . ◻

Note that ρ_0 is periodic. If ρ_0 is odd, then $\int_{\mathbb{S}} \rho_0(x)dx = \int_{-\frac{1}{2}}^{\frac{1}{2}} \rho_0(x)dx = 0$. Since $\int_{\mathbb{S}} \rho_0(x)dx = 0$, this implies that there is some $x_0 \in \mathbb{S}$ such that $\rho_0(x_0) = 0$. By Theorem 4.2, we have the following corollary immediately.

Corollary 4.1 *Let* $z_0 = \begin{pmatrix} u_0 \\ \rho_0 \end{pmatrix} \in H^s(\mathbb{S}) \times H^{s-1}(\mathbb{S})$, $s \geq 2$, *and let* T *be the maximal existence time of solution* $z = \begin{pmatrix} u \\ \rho \end{pmatrix}$ *to Eq.* (2.1) *with the initial data* z_0. *If* $\int_{\mathbb{S}} \rho_0(x)dx = 0$ *or* ρ_0 *is odd, and*

$$u_0'(x_0) < -\left[\frac{e+1}{2(e-1)}(\|u_0\|^2_{H^1(\mathbb{S})} + \|\rho_0\|^2_{L^2(\mathbb{S})})\right]^{\frac{1}{2}},$$

then the corresponding solution to Eq. (2.1) *blows up in finite time.*

We now give the third blow-up result.

Theorem 4.3 *Let* $z_0 = \begin{pmatrix} u_0 \\ \rho_0 \end{pmatrix} \in H^s(\mathbb{S}) \times H^{s-1}(\mathbb{S})$, $s \geq 3$, *and let* T *be the maximal existence time of solution* $z = \begin{pmatrix} u \\ \rho \end{pmatrix}$ *to Eq.* (2.1) *with the initial data* z_0. *Assume that* $\int_{\mathbb{S}} u_0 dx = \frac{a_0}{2}$. *If there is some* $x_0 \in \mathbb{S}$ *such that* $\rho_0(x_0) = 0$ *and for any* $\epsilon > 0$

$$u_0'(x_0) < -\left(\frac{\epsilon+2}{24} E_0 + \frac{\epsilon+2}{4\epsilon} a_0^2\right)^{\frac{1}{2}},$$

where $E_0 = \|u_0\|^2_{H^1(\mathbb{S})} + \|\rho_0\|^2_{L^2(\mathbb{S})}$, *then the corresponding solution to Eq.* (2.1) *blows up in finite time.*

Proof By Lemma 2.3, we have $\int_{\mathbb{S}} u(t,x)dx = \int_{\mathbb{S}} u_0(x)dx = \frac{a_0}{2}$. Using Lemma 2.7, Corollary 2.1 and the above conservation law, we have

$$\|u\|_{L^\infty(\mathbb{S})} \leq \sqrt{\frac{\epsilon+2}{24}\left(\|u_0\|^2_{H^1(\mathbb{S})} + \|\rho_0\|^2_{L^2(\mathbb{S})}\right) + \frac{\epsilon+2}{4\epsilon} a_0^2}.$$

Let $m(t)$ and $h(t)$ be the same as those defined in Theorem 4.2. Using (4.6) and the above inequality, we have

$$m'(t) \leq -\frac{1}{2}m^2(t) + \left(\frac{\epsilon+2}{48}\left(\|u_0\|^2_{H^1(\mathbb{S})} + \|\rho_0\|^2_{L^2(\mathbb{S})}\right) + \frac{\epsilon+2}{8\epsilon} a_0^2\right).$$

Set $K = \left(\frac{\epsilon+2}{48}\left(\|u_0\|^2_{H^1(\mathbb{S})} + \|\rho_0\|^2_{L^2(\mathbb{S})}\right) + \frac{\epsilon+2}{8\epsilon} a_0^2\right)^{\frac{1}{2}}$. Following the same argument in Theorem 4.2, we deduce that the solution z blows up in finite time. □

Letting $a_0 = 0$ and $\epsilon \to 0$ in Theorem 4.3, we have the following corollary immediately.

Corollary 4.2 *Let* $z_0 = \begin{pmatrix} u_0 \\ \rho_0 \end{pmatrix} \in H^s(\mathbb{S}) \times H^{s-1}(\mathbb{S})$, $s \geq 3$, *and let* T *be the maximal existence time of solution* $z = \begin{pmatrix} u \\ \rho \end{pmatrix}$ *to Eq.* (2.1) *with the initial data* z_0. *Assume that* $\int_{\mathbb{S}} u_0 dx = 0$. *If there is some* $x_0 \in \mathbb{S}$ *such that* $\rho_0(x_0) = 0$ *and*

$$u_0'(x_0) < -\frac{\sqrt{3}}{6}\left(\|u_0\|^2_{H^1(\mathbb{S})} + \|\rho_0\|^2_{L^2(\mathbb{S})}\right)^{\frac{1}{2}},$$

then the corresponding solution to Eq. (2.1) *blows up in finite time.*

Remark 4.1 If u_0 is odd, then by the perodicity of u_0, we have $\int_{\mathbb{S}} u_0 dx = \int_{-\frac{1}{2}}^{\frac{1}{2}} u_0 dx = 0$. If ρ_0 is odd, we also have $\int_{\mathbb{S}} \rho_0 dx = 0$, which implies that there is some $x_0 \in \mathbb{S}$ such that $\rho_0(x_0) = 0$. Thus Corollary 4.2 is also true for u_0 and ρ_0 being odd.

Next, we give a blow-up result if u_0 is odd and ρ_0 is even.

Theorem 4.4 *Let* $z_0 = \begin{pmatrix} u_0 \\ \rho_0 \end{pmatrix} \in H^s(\mathbb{S}) \times H^{s-1}(\mathbb{S})$, $s \geq 2$, *and let* T *be the maximal existence time of solution* $z = \begin{pmatrix} u \\ \rho \end{pmatrix}$ *to Eq.* (2.1) *with the initial data* z_0. *Assume that* u_0 *is odd,* ρ_0 *is even,* $\rho_0(0) = 0$ *and* $u_0'(0) < 0$, *then the corresponding solution to Eq.* (2.1) *blows up in finite time and* $T < -2/u_0'(0)$.

Proof Let z be the solution to Eq. (2.1) with the initial data $z_0 \in H^s(\mathbb{S}) \times H^{s-1}(\mathbb{S})$, $s \geq 2$, and let $T > 0$ be the maximal time of existence of the solution z with the initial data z_0. Note that $\partial_x^2 G * f = G * f - f$. Differentiating the first equation in (2.1) with respect to x, we get

$$u_{tx} = -\frac{1}{2}u_x^2 - uu_{xx} + u^2 + \frac{1}{2}\rho^2 - G * \left(u^2 + \frac{1}{2}u_x^2 + \frac{1}{2}\rho^2\right). \tag{4.7}$$

Note that Eq. (2.1) is invariant under the transformation $(u, x) \to (-u, -x)$ and $(\rho, x) \to (\rho, -x)$. Thus, we deduce that if $u_0(x)$ is odd and $\rho_0(x)$ is even, then $u(t, x)$ is odd and $\rho(t, x)$ is even for any $t \in [0, T)$. By the oddness of $u(t, x)$, we have that $u(t, 0) = 0$. Define $m(t) = u_x(t, 0)$ and $h(t, x) = \rho(t, q(t, x))$. Note that $h(0, 0) = \rho(0, q(0, 0)) = \rho_0(0) = 0$. By Eq. (2.3) and the second equation in (2.1), we have

$$\frac{dh}{dt} = \rho_t + \rho_x q_t = -h(t, x)u_x(t, q(t, x)).$$

In view of Eq. (2.3), we deduce that if $u(t, x)$ is odd with respect to x, then $q(t, x)$ is also odd with respect to x. Then we have $q(t, 0) = 0$. By Lemma 2.2, we have $h(t, 0) = \rho(t, q(t, 0)) = \rho(t, 0) = 0$ for all $t \in [0, T)$.

Substituting $(t, 0)$ into Eq. (4.7), we obtain

$$\begin{aligned} m'(t) &= -\frac{1}{2}m^2(t) + u^2(t, 0) + \frac{1}{2}h^2(t, 0) - G * \left(u^2 + \frac{1}{2}u_x^2 + \frac{1}{2}\rho^2\right)(t, 0) \\ &\leq -\frac{1}{2}m^2(t). \end{aligned} \tag{4.8}$$

Here we use the relations $u(t, 0) = 0$, $h(t, 0) = 0$ and $G * (u^2 + \frac{1}{2}u_x^2 + \frac{1}{2}\rho^2) \geq 0$.

Note that $m(0) = u_0'(0) < 0$ and $m'(t) \leq 0$. We deduce that $m(t) < 0$ for all $t > 0$. Solving (4.8), we have

$$\frac{1}{m(0)} + \frac{1}{2}t \leq \frac{1}{m(t)} < 0.$$

The above inequality implies that $T < -\frac{2}{m(0)}$, and $u_x(t, 0)$ tends to negative infinity as t goes to T. This completes the proof of the theorem. ◻

Remark 4.2 If the condition $u_0'(0) < 0$ in Theorem 4.4 is replaced by the conditions $u_0'(0) \leq 0$ and $u_0 \neq 0$, then one can also deduce that the solution of Eq. (2.1) blows up in finite time.

Finally we give more insight into the blow-up rate for the wave breaking solutions to Eq. (2.1). In view of (2.4), if there exists $M > 0$ such that $u_x(t,x) \geq -M$ for all $(t,x) \in [0,T) \times \mathbb{S}$, then for all $t \in [0,T)$, we have

$$\|\rho(t,\cdot)\|_{L^\infty(\mathbb{S})} = \|\rho(t,\cdot)\|_{L^\infty(\mathbb{R})} \leq e^{MT}\|\rho_0(\cdot)\|_{L^\infty(\mathbb{R})} = e^{MT}\|\rho_0(\cdot)\|_{L^\infty(\mathbb{S})}.$$

This implies that if $\rho(t,x)$ becomes unbounded in finite time, then $u_x(t,x)$ must be unbounded from below in finite time. Thus we might assume that $\rho(t,x)$ is bounded for all $t \in [0,T)$ in the following theorem.

Theorem 4.5 *Let* $z = \begin{pmatrix} u \\ \rho \end{pmatrix}$ *be the solution to Eq.* (2.1) *with the initial data* $z_0 = \begin{pmatrix} u_0 \\ \rho_0 \end{pmatrix} \in H^2(\mathbb{S}) \times H^1(\mathbb{S})$ *and let* $T > 0$ *be the maximal time of existence of the solution* z*. Assume that there exists* $M_1 > 0$ *such that* $\|\rho(t,\cdot)\|_{L^\infty(\mathbb{S})} \leq M_1$ *for all* $t \in [0,T)$*. If* $T < \infty$*, we have*

$$\lim_{t\to\infty}(\inf_{x\in\mathbb{S}} u_x(t,x)(T-t)) = -2,$$

while the solution remains uniformly bounded.

Proof By Corollary 2.1, we get the uniform bound of u. Define $m(t) = u_x(t,\xi(t)) = \inf_{x\in\mathbb{S}} u_x(t,x)$. Note that $u_{xx}(t,\xi(t)) = 0$ for all $t \in [0,T)$. Substituing $(t,\xi(t))$ into Eq. (4.5), we have

$$\begin{aligned} m'(t) &= -\frac{1}{2}m^2(t) + \left(u^2 + \frac{1}{2}\rho^2 - G*(u^2 + \frac{1}{2}u_x^2 + \frac{1}{2}\rho^2)\right)(t,\xi(t)) \\ &\leq -\frac{1}{2}m^2(t) + \left(\frac{1}{2}u^2 + \frac{1}{2}\rho^2\right)(t,\xi(t)). \end{aligned} \tag{4.9}$$

Here we use the relation $G*(u^2+\frac{1}{2}u_x^2) \geq \frac{1}{2}u^2$, and $G*\rho^2 \geq 0$.

By Corollary 2.1 and the assumption $\|\rho(t,\cdot)\|_{L^\infty} \leq M_1$ for all $t \in [0,T)$, we have

$$|m'(t) + \frac{1}{2}m^2| \leq \frac{e+1}{4(e-1)}\left(\|u_0\|^2_{H^1(\mathbb{S})} + \|\rho_0\|^2_{L^2(\mathbb{S})}\right) + \frac{1}{2}M_1^2 := K.$$

It follows that

$$-K \leq m'(t) + \frac{1}{2}m^2 \leq K \quad a.e.\ on\ (0,T). \tag{4.10}$$

Choose $\epsilon \in (0, \frac{1}{2})$. Since $\liminf_{t\to T}\{\inf_{x\in\mathbb{S}} u_x(t,x)\} = -\infty$ by Theorem 2.3, there is some $t_0 \in (0, T)$ with $m(t_0) < 0$ and $m^2(t_0) > \frac{K}{\epsilon}$. Let us first prove that

$$m^2(t) > \frac{K}{\epsilon} \quad t \in [t_0, T). \tag{4.11}$$

Since m is locally Lipschitz (it belongs to $W^{1,\infty}_{loc}(\mathbb{R})$ by Lemma 2.5), there is some $\delta > 0$ such that

$$m^2(t) > \frac{K}{\epsilon} \quad t \in (t_0, t_0 + \delta).$$

Pick $\delta > 0$ maximal with this property. If $\delta < T - t_0$ we would have $m^2(t_0 + \delta) = \frac{K}{\epsilon}$ while

$$m'(t) \leq -\frac{1}{2}m^2 + K < -\frac{1}{2}m^2 + \epsilon m^2 \quad a.e.\ on\ (t_0, t_0 + \delta).$$

Note that m is locally Lipschitz and therefore absolutely continuous. Integrating the previous relation on $[t_0, t_0 + \delta]$ yields that

$$m(t_0 + \delta) \leq m(t_0) < 0.$$

It follows from the above inequality that

$$m^2(t_0 + \delta) \geq m^2(t_0) > \frac{K}{\epsilon}.$$

The obtained contradiction completes the proof of the relation (4.10).

By (4.9)–(4.10), we infer

$$\frac{1}{2} - \epsilon \leq -\frac{m'(t)}{m^2} \leq \frac{1}{2} + \epsilon \quad a.e. on\ (0, T). \tag{4.12}$$

Since m is locally Lipschtiz on $[0, T)$ and (4.10) holds, it is easy to check $\frac{1}{m}$ is locally Lipschtiz on (t_0, T). Differentiating the relation $m(t) \cdot \frac{1}{m(t)} = 1,\ t \in (t_0, T)$, we get

$$\frac{d}{dt}\frac{1}{m(t)} = -\frac{m'(t)}{m^2(t)} \quad a.e. on\ (0, T).$$

For $t \in (t_0, T)$, integrating (4.11) on (t, T) to get

$$\left(\frac{1}{2} - \epsilon\right)(T - t) \leq -\frac{1}{m(t)} \leq \left(\frac{1}{2} + \epsilon\right)(T - t), \quad t \in (t_0, T).$$

Since $m(t) < 0$ on $[t_0, T)$, it follows that

$$\frac{1}{\frac{1}{2}+\epsilon} \leq -m(t)(T-t) \leq \frac{1}{\frac{1}{2}+\epsilon}, \quad t \in (t_0, T).$$

By the arbitraryness of $\epsilon \in (0, \frac{1}{2})$, the statement of the theorem follows. □

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