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On Miquel's theorem and inversions in normed planes

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Abstract Asplund and Grünbaum proved that Miquel's six-circles theorem holds in strictly convex, smooth normed planes if the considered circles have equal radii. We extend this result in two directions. First we prove that Miquel's theorem for circles of equal radii (more precisely, a generalized version of it) is true in every normed plane, without the assumptions of strict convexity and smoothness, and give also some properties of the circle configuration related to this theorem. Second we clarify the situation if the circles of the corresponding configuration do not necessarily have equal radii.

Keywords Finite-dimensional Banach spaces · Inversion · Minkowski planes · Miquel's theorem · Möbius plane · Normed planes

Mathematics Subject Classification (2000) 46B20 · 51B10 · 52A10 · 52A21 · 52C05

1 Introduction

In [1] it is proved that the known (*six-circles*) *theorem of Miquel* (cf. [12, p. 424]) holds in an arbitrary strictly convex, smooth normed plane if the considered circles have equal radii. In [10] the same statement is extended to all strictly convex normed planes, without the assumption of smoothness. In the present paper we extend

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Miquel's theorem for circles of equal radii to *all* normed planes. We also give some further properties of the corresponding configuration of circles. In Sect. 4, we prove that if Miquel's theorem (without the assumption of equal radii) holds in a strictly convex, smooth normed plane, then this plane is Euclidean. For that reason we introduce the notion of *inversion* in such a plane. Note that our definition of inversion is different to that given by Stiles [16], and that these two definitions only coincide in the Euclidean case. Essential for our considerations in Sect. 4 is the fact that any strictly convex, smooth normed plane can be viewed as a *Möbius plane*; see [15] (note that the notion of "Möbius plane" is used in the sense of Benz' terminology, cf. [6]). A brief overview to this approach is given in Sect. 2.

2 Strictly convex, smooth normed planes as Möbius planes

Let $\mathbb{M}$ be a two-dimensional real vector space equipped with norm $\|\cdot\|$. Usually $\mathbb{M}$ is called a *normed* (or *Minkowski*) *plane*. The elements of $\mathbb{M}$ will be called either *vectors* or *points*, depending on the type of consideration. In the sequel, the line through two points $p_1, p_2 \in (\mathbb{M}, \|\cdot\|)$ is denoted by $\langle p_1, p_2\rangle$, the line segment with endpoints p_1, p_2 by $[p_1, p_2]$, and the ray emanating from p_1 and passing through p_2 by $[p_1, p_2\rangle$. A normed plane $(\mathbb{M}, \|\cdot\|)$ is said to be *strictly convex* if its *unit circle* $\mathcal{C} := \{x \in \mathbb{M} : \|x\| = 1\}$ is a strictly convex curve, i.e., does not contain a non-degenerate line segment. If $\mathcal{C}$ has a unique supporting line at each point, then $(\mathbb{M}, \|\cdot\|)$ is called a *smooth plane*. Every homothetic copy of the unit circle $\mathcal{C}$ is said to be a (*Minkowskian*) *circle* in $(\mathbb{M}, \|\cdot\|)$. Referring to incidence properties of circles and points in normed planes, the following facts are known; see, e.g., [11, Proposition 14, Proposition 41, and Lemma 13].

Theorem 2.1 *The following properties hold in a normed plane* $(\mathbb{M}, \|\cdot\|)$.

(i) *If* $(\mathbb{M}, \|\cdot\|)$ *is strictly convex, then through any three non-collinear points passes at most one circle.*
(ii) *If* $(\mathbb{M}, \|\cdot\|)$ *is smooth, then through any three non-collinear points passes at least one circle.*
(iii) *If* $(\mathbb{M}, \|\cdot\|)$ *is strictly convex, then two circle have at most two intersecting points.*

Let a formal point at infinity ∞ be added to the plane $\mathbb{M}$, and let all lines of $\mathbb{M}$ pass through ∞. Let $\mathfrak{P} = \mathbb{M} \cup \{\infty\}$ and $\mathfrak{C}$ be the set of all circles and lines of $(\mathbb{M}, \|\cdot\|)$. We call the elements of $\mathfrak{P}$ *generalized points*, and the elements of $\mathfrak{C}$ *generalized circles*. We say that two generalized circles from $\mathfrak{C}$ *touch each other* if they have exactly one common generalized point. If two generalized circles intersect in more than one generalized point, we say that they *intersect properly*. Note that two intersecting lines, treated as elements of $\mathfrak{C}$, intersect properly, and that two parallel lines touch each other. In [15] it is proved that the geometric structure $(\mathfrak{P}, \mathfrak{C})$ induced by a strictly convex, smooth normed plane satisfies the following axioms:

M1. Any three different generalized points of $\mathfrak{P}$ are contained in precisely one generalized circle.

M2. For any generalized point $p \in \mathfrak{P}$ on a generalized circle $P \in \mathfrak{C}$ and any generalized point $q \notin P$ there exists exactly one generalized circle Q through q with $Q \cap P = \{p\}$.

M3. There are four generalized points not on a generalized circle, and each generalized circle contains at least three generalized points.

It is clear that **M2** is really essential. Axiom **M1** follows from Theorem 2.1, and **M3** is trivial. According to Benz' terminology, a geometric structure $(\mathfrak{P}, \mathfrak{C})$ with $\mathfrak{P}$ a set and $\mathfrak{C}$ a set of subsets of $\mathfrak{P}$ satisfying **M1**, **M2**, and **M3** is called a *Möbius plane*. Therefore each strictly convex, smooth normed plane appears to be also a Möbius plane. Here we give some notions referring to Möbius planes that are necessary for our considerations. For more facts from the geometry of such planes we refer to [4], [7], and the survey [6]. Let $\Sigma = (\mathfrak{P}, \mathfrak{C})$ be a Möbius plane. The elements of $\mathfrak{P}$ are called *generalized points*, and the elements of $\mathfrak{C}$ are the *generalized circles* of Σ. A set of generalized circles having a unique common generalized point form a *parabolic bundle*. If the following statement (F) holds in a Möbius plane Σ, then Σ is called an (F)-*plane*:[1]

(F) Every generalized circle that touches three different generalized circles of a parabolic bundle belongs to the same bundle.

It is easy to check that every strictly convex, smooth normed plane is an (F)-plane. If the set of the generalized points of a Möbius plane $(\mathfrak{P}, \mathfrak{C})$ is homeomorphic to a 2-sphere $\mathcal{S}^2$ in $\mathbb{R}^3$ and every generalized circle of $\mathfrak{C}$ is homeomorphic to a 1-sphere $\mathcal{S}^1$, then $(\mathfrak{P}, \mathfrak{C})$ is said to be a *spherical Möbius plane*. Due to the natural homeomorphisms between $\mathbb{M} \cup \infty$ and $\mathcal{S}^2$ and between each Jordan curve and $\mathcal{S}^1$, every strictly convex, normed plane is also a spherical Möbius plane. For a comprehensive study of spherical planes we refer to [17] and [18]. All the mentioned facts referring to strictly convex, normed planes considered as Möbius planes we summarize in the following

Theorem 2.2 *Every strictly convex, smooth normed plane is a spherical Möbius plane and, in addition, an* (F)*-plane.*

Let $(\mathfrak{P}, \mathfrak{C})$ be a Möbius plane and $p_1, p_2, p_3, p_4 \in \mathfrak{P}$. The generalized points p_1, p_2, p_3, p_4 are said to be *concyclic* if there exists a generalized circle $C \in \mathfrak{C}$ containing them. Let $p_1, \dots, p_8$ be eight generalized points. To every generalized point p_i, $i = 1, \dots, 8$, we assign a vertex of a cube. Consider the six quadruples of generalized points that correspond to the vertices of each facet of the cube, e.g.,

$$\begin{aligned}&(p_1, p_2, p_3, p_4), (p_1, p_2, p_5, p_6), (p_2, p_3, p_7, p_6),\\&\quad(p_3, p_4, p_8, p_7), (p_1, p_4, p_8, p_5), (p_5, p_6, p_7, p_8).\end{aligned} \tag{1}$$

If five of the quadruples in (1) are concyclic, then this configuration is called a *Miquel configuration*. If *all six* quadruples in a Miquel configuration are concyclic, we say that in the respective Möbius plane $(\mathfrak{P}, \mathfrak{C})$ *Miquel's theorem* holds. The plane $(\mathfrak{P}, \mathfrak{C})$ is then said to be a *Miquelian Möbius plane*.

[1] Coming from the German word "Fährte".

Let Σ and Σ' be two Möbius planes. If there exists a one-to-one correspondence $\sigma : \Sigma \to \Sigma'$ mapping concyclic generalized points into concyclic generalized points, and non-concyclic generalized points into non-concyclic ones, then Σ and Σ' are called *isomorphic* and σ is said to be a *homography from* Σ *to* Σ'. Clearly, the isomorphism between Möbius planes is an equivalence relation. Note also that if a Möbius plane is an (F)-plane, then all planes isomorphic to it are (F)-planes, too.

We mention that a class of Möbius planes can be constructed in an algebraic way. Let $\mathbb{F}$ and $\mathbb{E} \supset \mathbb{F}$ be commutative fields and $[\mathbb{E} : \mathbb{F}] = 2$. One can consider the elements of $\mathbb{E} \cup \{\infty\}$, where ∞ is a formal symbol, as generalized points and define generalized circles (usually called *chains*) as sets

$$\left\{x \in \mathbb{E} \cup \{\infty\} | \frac{p-r}{q-r} : \frac{p-x}{q-x} \in \mathbb{F} \cup \infty\right\},$$

where p, q, and r are three pairwise different points from $\mathbb{E}$. Then the so-defined incidence structure is a Möbius plane (see [6, § 2]) and we denote it by $\mathrm{Mo}(\mathbb{F}, \mathbb{E})$. Note also that in the classical case $\mathbb{F} = \mathbb{R}$, $\mathbb{E} = \mathbb{C}$, and $\mathrm{Mo}(\mathbb{F}, \mathbb{E})$ is the inversive plane.

3 Miquel configurations of circles of equal radii

In Sect. 2 a Miquel configuration is defined for points of a Möbius plane. But such a configuration can also be defined for an arbitrary normed plane $(\mathbb{M}, \|\cdot\|)$. The following theorem is related to Miquel configurations of circles of equal radii in a strictly convex normed plane $(\mathbb{M}, \|\cdot\|)$; see [10, Theorem 4.2]. In fact, this was already proved by Asplund and Grünbaum [1], but under the assumption that, in addition, the plane $(\mathbb{M}, \|\cdot\|)$ is smooth.

Theorem 3.1 *In a strictly convex normed plane* $(\mathbb{M}, \|\cdot\|)$*, let there be given a Miquel configuration*

$$(p_1, p_2, p_3, p_4), (p_1, p_2, p_5, p_6), (p_2, p_3, p_7, p_6),$$
$$(p_3, p_4, p_8, p_7), (p_1, p_4, p_8, p_5), (p_5, p_6, p_7, p_8).$$

If the first five quadruples are concyclic such that the corresponding circles are different, but all have equal radii λ*, then the points* p_5, p_6, p_7, p_8 *lie on a circle of the same radius* λ*; see Fig. 1.*

In order to extend this theorem to an arbitrary normed plane we need some further facts. Let $(\mathbb{M}, \|\cdot\|)$ be a normed plane with unit circle $\mathcal{C}$. According to Theorem 2.4 in [2] (see also [11, Proposition 22]) the intersection $\mathcal{I}$ of two circles $p + \lambda\mathcal{C}$ and $q + \mu\mathcal{C}$ in $(\mathbb{M}, \|\cdot\|)$ can only have the following forms:

(i) $\mathcal{I} = \emptyset$;
(ii) $\mathcal{I} = p + \lambda\mathcal{C} = q + \mu\mathcal{C}$;
(iii) $\mathcal{I}$ consists of two closed, disjoint segments (one of them or both may be reduced to a point) lying on the opposite sides of the line G through p and q;

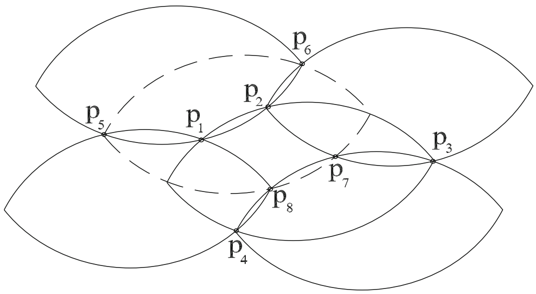

Fig. 1 Miquel's theorem in a strictly convex normed plane

(iv) $\mathcal{I}$ consists of two segments (one of them or both may be reduced to a point) with common point p_1 or p_2, where $\{p_1, p_2\} = G \cap (p + \lambda\mathcal{C})$.

Note that if $\lambda = \mu$ and $p \neq q$, (iii) or (iv) occur (see Fig. 2) if and only if $\|p-q\| \leq 2\lambda$. In these two cases we say that the circles $p + \lambda\mathcal{C}$ and $q + \lambda\mathcal{C}$ *intersect*. If the plane is strictly convex, then the intersection of $p + \lambda\mathcal{C}$ and $q + \lambda\mathcal{C}$ consists of exactly two points if and only if $\|p - q\| < 2\lambda$. If $\|p - q\| = 2\lambda$, then $(p + \lambda\mathcal{C}) \cap (q + \lambda\mathcal{C})$ consists of exactly one point.

In a normed plane, let the circles $p + \lambda\mathcal{C}$ and $q + \lambda\mathcal{C}$ intersect and their intersection consist of the segments $\mathcal{I}_1$ and $\mathcal{I}_2$ (possibly degenerate). If the point r_1 belongs to $\mathcal{I}_1$, we call the point

$$r_2 = p + q - r_1$$

the *conjugate of* r_1 *with respect to the circles* $p + \lambda\mathcal{C}$ *and* $q + \lambda\mathcal{C}$. If the intersection of both the circles consists only of one point (i.e., in (iv) the two segments are reduced to points that coincide), then this point is the conjugate of itself. Clearly, we have $r_2 \in \mathcal{I}_2$, and r_1 is the conjugate of r_2. If the plane is strictly convex and $(p + \lambda\mathcal{C}) \cap (q + \lambda\mathcal{C}) = \{r_1, r_2\}$, then r_1 and r_2 are conjugates of each other.

Let there be given a circle $x + \lambda\mathcal{C}$. Let the points $p_1, \ldots, p_4$ be placed on $x + \lambda\mathcal{C}$ in this order. Then the *monotonicity lemma* (see, e.g., [11, § 3.5]) implies that $\|p_i - p_{i+1}\| \leq 2\lambda$ for $i = 1, \ldots, 4$ and $p_5 = p_1$. Therefore the circles $p_i + \lambda\mathcal{C}$ and $p_{i+1} + \lambda\mathcal{C}$ intersect. Clearly the point x belongs to $(p_i + \lambda\mathcal{C}) \cap (p_{i+1} + \lambda\mathcal{C})$ for

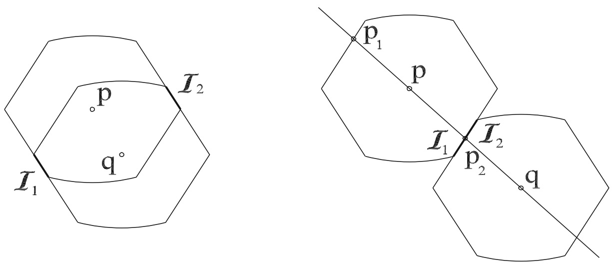

Fig. 2 The intersection of two circles having equal radii

every $i = 1, \dots, 4$. Let y_i be the conjugate point of x with respect to $p_i + \lambda\mathcal{C}$ and $p_{i+1} + \lambda\mathcal{C}$. We call the configuration of circles

$$y_1 + \lambda\mathcal{C}, \dots, y_4 + \lambda\mathcal{C}, x + \lambda\mathcal{C} \tag{2}$$

the *Miquelian configuration induced by the points* $p_1, \dots, p_4$.

Remark 3.1 If (2) is the Miquelian configuration induced by the points $p_1, \dots, p_4$ which lie on $x + \lambda\mathcal{C}$, then we can speak about the conjugate point of p_{i+1} with respect to the circles $y_i + \lambda\mathcal{C}$ and $y_{i+1} + \lambda\mathcal{C}$, where $y_5 = y_1$. Indeed, since $y_i \in (p_i + \lambda\mathcal{C}) \cap (p_{i+1} + \lambda\mathcal{C})$, we have that $p_i, p_{i+1} \in y_i + \lambda\mathcal{C}$, which yields $p_{i+1} \in (y_i + \lambda\mathcal{C}) \cap (y_{i+1} + \lambda\mathcal{C})$.

Theorem 3.2 *In a normed plane* $(\mathbb{M}, \|\cdot\|)$ *with unit circle* $\mathcal{C}$*, let there be given a circle* $x + \lambda\mathcal{C}$*. Let the points* $p_1, \dots, p_4$ *be placed in this order on* $x + \lambda\mathcal{C}$*, and let*

$$y_1 + \lambda\mathcal{C}, \dots, y_4 + \lambda\mathcal{C}, x + \lambda\mathcal{C} \tag{3}$$

be the Miquelian configuration induced by the points $p_1, \dots, p_4$*. If* p'_{i+1} *is the conjugate point of* p_{i+1} *with respect to* $y_i + \lambda\mathcal{C}$ *and* $y_{i+1} + \lambda\mathcal{C}$*, then the points* p'_i *lie on a circle of radius* λ*, where* $i = 1, \dots, 4$, $p'_5 = p'_1$, $p_5 = p_1$*, and* $y_5 = y'_5$*; see Fig. 3.*

Proof Without loss of generality we assume that $x = 0$. Since the points $p_1, \dots, p_4$ induce the Miquelian configuration (3), we have

$$y_i = p_i + p_{i+1}, \quad i = 1, \dots, 4. \tag{4}$$

On the other hand, the points p_{i+1} and p'_{i+1} are conjugate with respect to $y_i + \lambda\mathcal{C}$ and $y_{i+1} + \lambda\mathcal{C}$. Therefore we get

$$p_{i+1} + p'_{i+1} = y_i + y_{i+1}, \quad i = 1, \dots, 4. \tag{5}$$

Thus, from (4) and (5) we obtain

$$p'_{i+1} = y_i + y_{i+1} - p_{i+1} = p_i + p_{i+1} + p_{i+2} = p_1 + \dots + p_4 - p_{i+3}, \tag{6}$$

where $p_6 = p_2$, $p_7 = p_3$. Hence $\|p'_{i+1} - (p_1 + \dots + p_4)\| = \|p_{i+3}\| = \lambda$. □

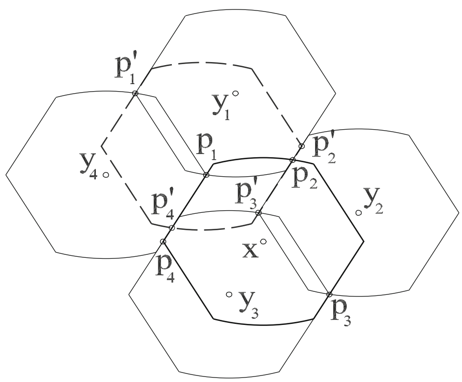

Fig. 3 Miquel's theorem in a normed plane that is not strictly convex

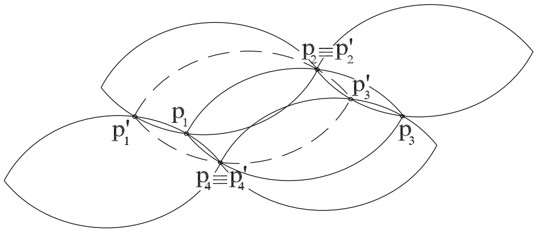

Fig. 4 A special case of Theorem 3.2

Remark 3.2 If the plane $(\mathbb{M}, \|\cdot\|)$ in Theorem 3.2 is strictly convex, then p_{i+1} and p'_{i+1} are the intersection points of $y_i + \lambda\mathcal{C}$ and $y_{i+1} + \lambda\mathcal{C}$. In such a case, Theorem 3.2 appears to be a reformulation of Theorem 3.1. However, restricted to the strictly convex case, Theorem 3.2 is more general than Theorem 3.1. If the points p_1 and p_3 are opposite with respect to $x + \lambda\mathcal{C}$, then from (4) we have $\|y_1 - y_2\| = 2\lambda$ and $\|y_3 - y_4\| = 2\lambda$ (under the assumption that $x = 0$). This means that $y_1 + \lambda\mathcal{C}$ and $y_2 + \lambda\mathcal{C}$, as well as $y_3 + \lambda\mathcal{C}$ and $y_4 + \lambda\mathcal{C}$, have exactly one common point. In other words, if two pairs of circles corresponding to the quadruples in Theorem 3.1 have only one point in common, Theorem 3.1 is also true; see Fig. 4.

In order to present further properties of Miquelian configurations, we need more facts from the geometry of triangles and quadrangles in normed planes. Let $(\mathbb{M}, \|\cdot\|)$ be a strictly convex normed plane with unit circle $\mathcal{C}$. Let p_1, p_2, p_3 be three points on a circle $x + \lambda\mathcal{C}$. The circle $\frac{1}{2}(p_1 + p_2 + p_3 - x) + \frac{1}{2}\mathcal{C}$ is called the *Feuerbach circle* of the triangle $p_1p_2p_3$. This circle passes through six "remarkable" points, namely the midpoints of the sides of $p_1p_2p_3$, and the midpoints of the segments $[p, p_i], i = 1, 2, 3$, where $p = p_1 + p_2 + p_3 - 2x$. This fact was proved by Asplund and Grünbaum in [1, Theorem 5] for planes that, in addition, are smooth, but it holds also for strictly convex normed planes without smoothness assumption; see [9]. Note that in the Euclidean case the point p coincides with the orthocenter of the triangle $p_1p_2p_3$, and that there the circle $\frac{1}{2}(p_1 + p_2 + p_3 - x) + \frac{1}{2}\mathcal{C}$ is known as *nine-point* (or Feuerbach) *circle of the triangle* $p_1p_2p_3$. Further on, we consider four points $p_1, \ldots, p_4$ placed on a circle $x + \lambda\mathcal{C}$. Then, according to Theorem 4.8 in [9], the Feuerbach circles of all four triangles derived from $\{p_1, \ldots, p_4\}$ have a common point $p = \frac{1}{2}(p_1 + p_2 + p_3 + p_4) - x$, and their centers lie on the circle $p + \frac{1}{2}\lambda\mathcal{C}$. This circle is called the *Feuerbach circle of the inscribed quadrangle* $p_1p_2p_3p_4$. If we consider the Miquelian configuration induced by the points $p_1, \ldots, p_4$, then also the points $p'_1, \ldots, p'_4$ determined by Theorem 3.2 lie on one circle. Therefore we can also speak about the *Feuerbach circle of the quadrangle* $p'_1p'_2p'_3p'_4$.

Corollary 3.1 *In a strictly convex normed plane* $(\mathbb{M}, \|\cdot\|)$ *with the unit circle* $\mathcal{C}$*, let there be given four points* $p_1, \ldots, p_4$ *placed on* $\lambda\mathcal{C}$ *in this order. Let*

$$y_1 + \lambda\mathcal{C}, \ldots, y_4 + \lambda\mathcal{C}, \lambda\mathcal{C}$$

be the Miquelian configuration induced by $p_1, \ldots, p_4$*, and the points* $p'_1, \ldots, p'_4$ *be defined as in Theorem 3.2. Then*

(i) *the points in each of the quadruples* $\{y_1, y_2, y_3, y_4\}, \{p_1, p_2, p_3', p_4'\}$, *and* $\{p_1', p_2', p_3, p_4\}$ *form a parallelogram whose diagonals intersect in the point* $\frac{1}{2}(p_1 + p_2 + p_3 + p_4)$;
(ii) *the circles in each of the pairs* $\{y_1 + \lambda\mathcal{C}, y_3 + \lambda\mathcal{C}\}, \{y_2 + \lambda\mathcal{C}, y_4 + \lambda\mathcal{C}\}$, *and* $\{\lambda\mathcal{C}, p_1+p_2+p_3+p_4+\lambda\mathcal{C}\}$ *are symmetric with respect to* $\frac{1}{2}(p_1+p_2+p_3+p_4)$;
(iii) *the Feuerbach circles of the quadrangles* $p_1p_2p_3p_4$ *and* $p_1'p_2'p_3'p_4'$ *coincide.*

Proof Statement (i) follows directly from (5) and (6). Statement (i) implies (ii). Further on, the Feuerbach circle of $p_1p_2p_3p_4$ is $\frac{1}{2}(p_1+\cdots+p_4)+\frac{1}{2}\lambda\mathcal{C}$. From the proof of Theorem 3.2 we have that the points $p_1', \ldots, p_4'$ lie on the circle $p_1+p_2+p_3+p_4+\lambda\mathcal{C}$. Therefore the Feuerbach circle of $p_1'p_2'p_3'p_4'$ is $p' + \frac{1}{2}\lambda\mathcal{C}$, where

$$p' = \frac{1}{2}(p_1' + p_2' + p_3' + p_4') - (p_1 + p_2 + p_3 + p_4) = \frac{1}{2}(p_1 + p_2 + p_3 + p_4),$$

by (6). □

4 Normed planes in which Miquel's theorem holds

Let $(\mathbb{M}, \|\cdot\|)$ be a strictly convex, smooth normed plane, and consider this plane as a Möbius plane $\Sigma = (\mathfrak{P}, \mathfrak{C})$. A homography φ in Σ that is involutory and leaves the generalized points of a generalized circle C fixed such that no other generalized point is fixed is called *the inversion with respect to the generalized circle* C. It is clear that such a homography exists at least for the Euclidean case, and since we will show that this is the only norm with that property, uniqueness will follow, too.

Proposition 4.1 *In a strictly convex, smooth normed plane, let* φ *be an inversion with respect to the circle* C *with center* x. *Then* $\varphi(x) = \infty$.

Proof Assume that $\varphi(x) = x' \neq \infty$. Clearly, $x \neq x'$. Let $\langle x, x'\rangle \cap C = \{p_1, p_2\}$ and $\varphi(\infty) = y$. Then $y \neq x$ and $y \neq \infty$. We distinguish the following cases:

(1) $y \notin \langle x, x'\rangle$. Then the image of $\langle p_1, p_2\rangle$ is a circle passing through p_1, p_2, and x', a contradiction.
(2a) $y \in \langle x, x'\rangle$ and y is interior with respect to C. Consider the circle C_1 with center $\frac{x+y}{2}$ and radius $\frac{\|x-y\|}{2}$. Clearly, $C_1 \cap C = \emptyset$. On the other hand, $\varphi(C_1) = C_1'$ is a line through x'. If the point x' is interior with respect to C, then $C_1' \cap C \neq \emptyset$. This is not true, and thus we obtain that x' is exterior with respect to C. Let G be a supporting line of C through x', and let $G \cap C = \{z\}$. If $\varphi(G) = C_2$, then C_2 is the circle through x, y, z, and $C_2 \cap C = \{z\}$. Denote by x_2 the center of C_2. Since $(\mathbb{M}, \|\cdot\|)$ is strictly convex, the point x_2 lies on $\langle x, z\rangle$. Therefore x_2 is the midpoint of the segment $[x, z]$. If the line H supports C_2 at x, then H is parallel to G, by the fact that $(\mathbb{M}, \|\cdot\|)$ is smooth. Let $H \cap C = \{q_1, q_2\}$. If $\varphi(H) = C_3$, then C_3 is the circle through q_1, q_2, x'. Moreover, G supports C_3 at x' and $C_2 \cap C_3 = \emptyset$. If q_2 lies in the half-plane bounded by $\langle x, z\rangle$ and containing x', the circular arc of C_3 between q_1 and x' has to intersect C_2, a contradiction.

(2b) $y \in \langle x, x' \rangle$ and y is exterior with respect to C. Let C_1 be the circle with center $\frac{x+y}{2}$ and radius $\frac{\|x-y\|}{2}$, and let $C_1 \cap C = \{q_1, q_2\}$. According to [2, Theorem 2.4] (cited also in Sect. 3), the points q_1 and q_2 lie in the different half-planes with respect to $\langle x, y \rangle$. Note that the intersection point of $\langle q_1, q_2 \rangle$ and $\langle x, y \rangle$ is x', since $\varphi(C_1) = \langle q_1, q_2 \rangle$. Thus we get that x' is interior with respect to C. Consider the circle C_2 through x' and y with center $\frac{x'+y}{2}$. If G supports C_2 at x', the points x and y lie on opposite sides of G. Let $C_2 \cap C = \{t_1, t_2\}$. Then the segment $[t_1, t_2]$ has to intersect $\langle x, y \rangle$. Denote the respective intersection point by t. Since $\varphi(C_2) = \langle t_1, t_2 \rangle$, the point t has to coincide with x. Due to the fact that t and x lie in different half-planes with respect to G, again we get a contradiction. □

Remark 4.1 In [16], Stiles defined the inversion with respect to the unit circle $\mathcal{C}$ of a normed plane as a mapping φ of $\mathbb{M} \backslash \{0\}$ onto itself that maps a point $x \neq 0$ onto the point $\frac{1}{\|x\|^2} x$. He proved that if the inversive image of some line is a circle, then $(\mathbb{M}, \| \cdot \|)$ is Euclidean. Theorem 4.2 below shows that Stiles' definition of inversion and ours are only equivalent in the Euclidean case.

For the proof of Theorem 4.2, we also need the following characterization of the Euclidean plane, proved in [3].

Theorem 4.1 *A normed plane with unit circle $\mathcal{C}$ is Euclidean if and only if*

$$x, y \in \mathcal{C}, \ \inf\{\|\alpha x + (1-\alpha) y\| : 0 \leq \alpha \leq 1\} = \frac{1}{2} \quad \Rightarrow \quad x + y \in \mathcal{C}.$$

Theorem 4.2 *Let* $(\mathbb{M}, \| \cdot \|)$ *be a strictly convex, smooth normed plane, and let there exist the inversion φ with respect to some circle of* $(\mathbb{M}, \| \cdot \|)$. *Then the plane is Euclidean.*

Proof Without loss of generality we can assume that in $(\mathbb{M}, \| \cdot \|)$ there exists an inversion with respect to the unit circle of $(\mathbb{M}, \| \cdot \|)$. Let p_1, p_2 be points on $\mathcal{C}$ and x be a point on $[p_1, p_2]$ such that the segment $[p_1, p_2]$ supports $\frac{1}{2}\mathcal{C}$ at x. Let H be the supporting line of $\mathcal{C}$ at $2x$; see Fig. 5. The smoothness of $(\mathbb{M}, \| \cdot \|)$ implies that H is parallel to $\langle p_1, p_2 \rangle$. Since $\varphi(\infty) = 0$, the inverse image H' of H is a circle through 0. On the other hand, $\varphi(2x) = 2x$. Therefore the circle H' passes through $2x$. Further on, the plane $(\mathbb{M}, \| \cdot \|)$ is strictly convex, and this means that $2x$ is the unique common point of $\mathcal{C}$ and H. Hence there do not exist common points of H and H' except for $2x$, and the line H appears to be supporting H' at $2x$. Thus the strict convexity of $(\mathbb{M}, \| \cdot \|)$ implies that the center of H' lies on $\langle 0, 2x \rangle$. But $0, 2x \in H'$, and therefore $H' = x + \frac{1}{2}\mathcal{C}$. Let $\langle p_1, p_2 \rangle \cap (x + \frac{1}{2}\mathcal{C}) = \{q_1, q_2\}$, $[0, q_1\rangle \cap H = \{q_1'\}$, and $[0, q_2\rangle \cap H = \{q_2'\}$; see again Fig. 5. For the points q_1' and q_2' the equations

$$\|2x - q_1'\| = 1, \quad \|2x - q_2'\| = 1 \tag{7}$$

hold. Since the inversive image of any line through 0 is the same line, we have $\varphi(q_1, q_2) = \{q_1', q_2'\}$. Consider $G' = \varphi(G)$, where $G = \langle p_1, p_2 \rangle$. Clearly, G' is a circle through $0, p_1, p_2, q_1', q_2'$. Thus we obtain that $G' = 2x + \mathcal{C}$; see (7). Therefore

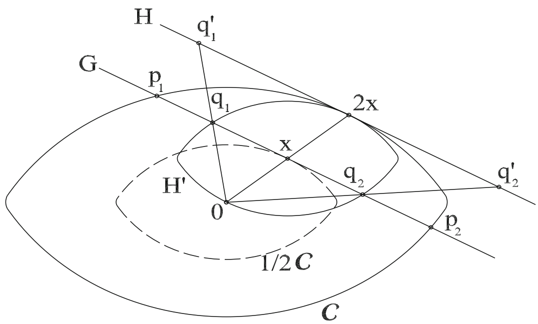

Fig. 5 Proof of Theorem 4.2

the quadrangle with vertices 0, p_1, $2x$, and p_2 is a metric parallelogram (i.e., a quadrangle with opposite sides of equal lengths). But in a strictly convex normed plane every metric parallelogram is a parallelogram (see [11, Proposition 12]), i.e., we get that x is the midpoint of $[p_1, p_2]$. Thus, in view of Theorem 4.1 the proof is complete. □

Theorem 4.3 *If Miquel's theorem holds in a strictly convex, smooth normed plane* $(\mathbb{M}, \|\cdot\|)$*, then this plane is Euclidean.*

Proof Consider $\Sigma = (\mathbb{M}, \|\cdot\|)$ as a Möbius plane. According to the Theorem of Smid and van der Waerden (see [6, § 5] and [14]), this plane is isomorphic to a Möbius plane $\Sigma' = \mathrm{Mo}(\mathbb{F}, \mathbb{E})$, where $\mathbb{F}$ is a commutative field and $\mathbb{E}$ is a quadratic extension of $\mathbb{F}$. Denote by θ the corresponding homography from Σ to Σ'. The plane $(\mathbb{M}, \|\cdot\|)$ is an (F)-plane (see Theorem 2.2), therefore Σ' is also an (F)-plane. But Theorem 5 in [5] states that if Σ' is an (F)-plane, then $\mathbb{E}$ is a separable extension of $\mathbb{F}$. If $\mathcal{C}$ is the unit circle of $(\mathbb{M}, \|\cdot\|)$, then let $\theta(\mathcal{C}) = \mathcal{C}'$. Since $\mathbb{E}$ is a separable extension of $\mathbb{F}$, there exists exactly one homography ψ of Σ' being an involution that fixes only the points of $\mathcal{C}'$; see [8], but also [6, §4.7]. Therefore $\theta^{-1}\psi\theta$ is the inversion with respect to $\mathcal{C}$. Thus Theorem 4.2 implies that $(\mathbb{M}, \|\cdot\|)$ is Euclidean. □

Remark 4.2 Consider four pairs from the set of points $p_1, \ldots, p_8$ such that the points in every such pair are different and every point belongs to exactly one pair, e.g.,

$$(p_1, p_2), (p_3, p_4), (p_5, p_6), (p_7, p_8).$$

These four pairs can be combined as pairs in six ways. Thus we obtain the quadruples

$$\begin{aligned}(p_1, p_2, p_3, p_4), (p_1, p_2, p_5, p_6), (p_1, p_2, p_7, p_8),\\ (p_3, p_4, p_5, p_6), (p_3, p_4, p_7, p_8), (p_5, p_6, p_7, p_8).\end{aligned} \tag{8}$$

If five quadruples in (8) are concyclic, then such a configuration is called a *Bundle configuration*. The statement that *all six* quadruples in a Bundle configuration are concyclic is known as the *Bundle theorem*. If Miquel's theorem holds in a Möbius plane, then also the Bundle theorem holds there; see [6, § 5] and [14]. But the converse implication is not always true; see again [6, § 5]. In [13] the following theorem is given: if in a strictly convex, smooth normed plane $(\mathbb{M}, \|\cdot\|)$ the Bundle theorem holds, then $(\mathbb{M}, \|\cdot\|)$ is Euclidean. Thus we can derive another proof of our Theorem 4.3.

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