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Generalized skew derivations with nilpotent values on Lie ideals

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Abstract Let R be a prime ring and L a noncommutative Lie ideal of R. Suppose that f is a right generalized β-derivation of R associated with a β-derivation δ such that $f(x)^n = 0$ for all $x \in L$, where n is a fixed positive integer. Then $f = 0$.

Keywords Skew derivation · Generalized skew derivation · Automorphism · Prime ring · Generalized polynomial identity (GPI) · Lie ideal

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In [12] Herstein proved that if D is an inner derivation of a prime ring R such that $D(x)^n = 0$ (resp. $D(x)^n$ is central) for all $x \in R$, then $D = 0$ (resp. $\dim_C RC \leq 4$ where the field C is defined below). Later Giambruno and Herstein [10] extended this result to an arbitrary derivation D for the nilpotent case and then Herstein [13] took care of the case where powers are central. The author extended this result to the case of an (α, β)-derivation [2,3] and to the case of a right generalized (α, β)-derivation case [5] (see also [18] for the nilpotent case). On the other hand, Carini and Giambruno [1] proved that if d is a derivation of a ring R such that $d(x)^{n(x)} = 0$ for all $x \in L$, where L is a Lie ideal of R, then $d(L) = 0$ provided that R has no nonzero nil right ideals and char $R \neq 2$. They also obtained the same conclusion when $n(x) = n$ is fixed and R is a semiprime ring having no 2-torsion. In [15] Lanski then removed both the bound on the indices of nilpotence and the characteristic assumption on R. In [17] Lee extended this result further to the case of generalized derivations when R is a prime

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ring. In this paper we extend [17, Theorem 7] further to the so-called right generalized skew derivations. Of course, our result also covers the case of a skew derivation.

Throughout this paper, R is always a prime ring with center Z. For $x, y \in R$, set $[x, y] = xy - yx$. For two subsets A and B of R, $[A, B]$ is defined to be the additive subgroup of R generated by all elements $[a, b]$ with $a \in A$ and $b \in B$. An additive subgroup L of R is said to be a Lie ideal if $[l, r] \in L$ for all $l \in L$ and $r \in R$. A Lie ideal L is said to be noncommutative if $[L, L] \neq 0$. Let L be a noncommutative Lie ideal of R. It is well known that $[R[L, L]R, R] \subseteq L$ (see the proof of [11, Lemma 1.3]). Since $[L, L] \neq 0$, we have $0 \neq [I, R] \subseteq L$ for $I = R[L, L]R$ a nonzero ideal of R.

Let β be an automorphism of R. A β-derivation of R is an additive mapping $\delta : R \to R$ satisfying $\delta(xy) = \delta(x)y + \beta(x)\delta(y)$ for all $x, y \in R$; sometimes β-derivations are called skew derivations. When $\beta = 1$, the identity map of R, β-derivations are merely ordinary derivations. If $\beta \neq 1$, then $1 - \beta$ is a β-derivation. An additive mapping $f : R \to R$ is a right generalized β-derivation if there exists a β-derivation $\delta : R \to R$ such that $f(xy) = f(x)y + \beta(x)\delta(y)$ for all $x, y \in R$. The right generalized β-derivations generalize both β-derivations and generalized derivations. If $a, b \in R$ and $\beta \neq 1$, then $f(x) = ax - \beta(x)b$ is a right generalized β-derivation. Moreover, if δ is a β-derivation of R, then $f(x) = ax + \delta(x)$ is a right generalized β-derivation.

We let ${}_F R$ denote the right Martindale quotient ring of R and Q the two-sided Martindale quotient ring of R. Let C be the center of Q and ${}_F R$, which is called the extended centroid of R. Note that Q and ${}_F R$ are also prime rings and C is a field. It is known that automorphisms, derivations and β-derivations of R can be uniquely extended to Q and ${}_F R$. From [4], we know that right generalized β-derivations of R can also be uniquely extended to ${}_F R$. Indeed, if f is a right generalized β-derivation of R, then $f(x) = f(1)x + \delta(x)$ for all $x \in R$, where δ is a β-derivation of R (see [4, Lemma 2]).

A β-derivation δ of R is called X-inner if $\delta(x) = bx - \beta(x)b$ for some $b \in Q$, and δ is called X-outer if it is not X-inner. An automorphism β is called X-inner if $\beta(x) = uxu^{-1}$ for some invertible element $u \in Q$, and β is called X-outer if it is not X-inner.

We are now ready to state the main result:

Main Theorem. *Let R be a prime ring and L a noncommutative Lie ideal of R. Suppose that f is a right generalized β-derivation of R associated with a β-derivation δ such that $f(x)^n = 0$ for all $x \in L$, where n is a fixed positive integer. Then $f = 0$.*

We begin with

Lemma 1 *Let R be a noncommutative prime ring and $b, c \in R$.*

(i) *If $(b[x, y])^n = 0$ for all $x, y \in R$, where n is a fixed positive integer, then $b = 0$.*
(ii) *If $([x, y]c)^n = 0$ for all $x, y \in R$, where n is a fixed positive integer, then $c = 0$.*

Proof (i) Suppose that $(b[x, y])^n = 0$ for all $x, y \in R$. Assume on the contrary that $b \neq 0$. Then $(b[x, y])^n = 0$ is a nontrivial generalized polynomial identity (GPI) of R and hence is also a nontrivial GPI of Q by Chuang [6]. Therefore Q is isomorphic to a dense subring of $\mathrm{End}({}_D V)$ where D is a finite-dimensional division ring over C

and V is a left vector space over D. If $\dim_D V = 1$, then $Q \simeq D$ and $b[x, y] = 0$ for all $x, y \in Q$. Since R is not commutative, $b = 0$, a contradiction. So we may assume that $\dim_D V \geq 2$. If there exists $v \in V$ such that v and vb are D-independent, then by the density of Q, there exist $x, y \in Q$ such that

$$vx = -v, \quad vbx = v, \quad vy = 0, \quad vby = v.$$

Therefore, $vb[x, y] = v$ and hence $0 = v(b[x, y])^n = v$, a contradiction.

Now we assume that v and vb are D-dependent for all $v \in V$. It is well known that there exists $\lambda \in C$ such that $vb = \lambda v$ for all $v \in V$. This implies $b = \lambda \in C$ since Q acts faithfully on V. So $(b[x, y])^n = (\lambda[x, y])^n = \lambda^n[x, y]^n = 0$ for all $x, y \in R$, and hence $[x, y]^n = 0$ for all $x, y \in R$. By Giambruno and Herstein [10], R is commutative, which is a contradiction. This proves (i).

(ii) Suppose $([x, y]c)^n = 0$ for all $x, y \in R$. Then $0 = c([x, y]c)^n[x, y] = (c[x, y])^{n+1}$ for all $x, y \in R$. By (i), $c = 0$ and this proves (ii). □

The following result extends [2, Lemma 1] to the Lie ideal case.

Lemma 2 *Let R be a noncommutative prime ring and $a, b, c \in R$ with a invertible in R. Suppose that $(a(bx - xc))^n = 0$ for all $x \in [R, R]$, where n is a fixed positive integer. Then $b = c \in Z$, the center of R.*

Proof Since R is noncommutative, $[R, R]$ is a noncommutative Lie ideal of R. If R is not a PI ring, then $(a(bx - xc))^n = 0$ for all $x \in R$ by Lee [16, Theorem 2]. Applying [2, Lemma 1], we have $b = c \in Z$ and we are done. Hence we may assume that R is a PI-ring and so RC is a finite-dimensional central simple C-algebra. Clearly, we may assume that RC is not a division ring and hence RC contains nontrivial idempotents. By Chuang [6, Theorem 2], we have

$$(a(bx - xc))^n = 0 \tag{1}$$

for all $x \in [RC, RC]$. Let e be an idempotent in RC and $x, y \in RC$. Applying (1) we have

$$(a(b[a^{-1}(1-e)xe, a^{-1}(1-e)ye] - [a^{-1}(1-e)xe, a^{-1}(1-e)ye]c))^n = 0.$$

Right-multiplying the above equation by $1 - e$ yields that

$$(1-e)((yea^{-1}(1-e)x - xea^{-1}(1-e)y)ec(1-e))^n = 0.$$

By Xu et al. [20, Remark 2.1], it follows that one of the following holds:

(i) $ec(1-e) = 0$;
(ii) $ec(1-e)yea^{-1}(1-e) = 0$ and $ea^{-1}(1-e)yec(1-e) = 0$ for all $y \in RC$; or
(iii) $(1-e)yea^{-1}(1-e) = 0$ for all $y \in RC$.

By the primeness of RC, we can conclude that if e is any idempotent of RC then either $ea^{-1}(1-e) = 0$ or $ec(1-e) = 0$.

Assume that $e \in RC$ is an idempotent such that $ea^{-1}(1-e) = 0$. Then for $x, y \in RC$, we get from (1) that

$$\begin{aligned}0 &= (a(b[a^{-1}(1-e)xe, ye] - [a^{-1}(1-e)xe, ye]c))^n(1-e)\\ &= (1-e)(xeyec(1-e))^n.\end{aligned}$$

This implies $(xeyec(1-e))^{n+1} = 0$ for all $x, y \in RC$. By Levitzki's lemma [11, Lemma 1.1] we have $eyec(1-e) = 0$ for all $y \in RC$ and hence $ec(1-e) = 0$ since RC is a prime ring. So $ec(1-e) = 0$ for all idempotents in RC. Now if $e \in RC$ is an idempotent, then both $ec(1-e) = 0$ and $(1-e)ce = 0$ hold, and hence $ec = ece = ce$. Denote by T the additive subgroup of RC generated by all the idempotents in RC. It follows that $[c, T] = 0$. In view of [11, Corollary p. 18], we have $[RC, RC] \subseteq T$ and hence $[c, [RC, RC]] = 0$. This implies $c \in Z$. Now we can reduce (1) to $(a(b-c)x)^n = 0$ for all $x \in [RC, RC]$. Since R is not commutative, by Lemma 1 we have $a(b-c) = 0$. Since a is invertible in R we have $b = c \in Z$, as asserted. □

The following result extends [5, Lemma 4] (or [18, Lemma 2.6]) to the Lie ideal case.

Lemma 3 *Let R be a prime ring and L a noncommutative Lie ideal of R. Let $b, c \in R$ and let β be an automorphism of R. Suppose that $(bx - \beta(x)c)^n = 0$ for all $x \in L$, where n is a fixed positive integer. Then $bx - \beta(x)c = 0$ for all $x \in R$.*

Proof Set $I = R[L, L]R$. Then $0 \neq [I, R] \subseteq L$. By the assumption, we have

$$(bx - \beta(x)c)^n = 0 \tag{2}$$

for all $x \in [I, R]$. Suppose that β is X-inner and write $\beta(x) = gxg^{-1}$ for all $x \in R$, where g is invertible in Q. Thus (2) becomes

$$(bx - \beta(x)c)^n = (bx - gxg^{-1}c)^n = (g(g^{-1}bx - xg^{-1}c))^n = 0 \tag{3}$$

for all $x \in [I, R]$. Since I, R and Q satisfy the same generalized polynomial identities over Q, (3) still holds for all $x \in [Q, Q]$. So we may assume that $R = I = Q$ and R is a centrally closed prime ring. Also,

$$(g(g^{-1}bx - xg^{-1}c))^n = 0 \tag{4}$$

for all $x \in [R, R]$. By Lemma 2 we have $g^{-1}b = g^{-1}c \in Z$ and hence $bx - \beta(x)c = 0$ for all $x \in R$, as asserted.

Suppose next that β is X-outer. By Chuang [8, Theorem 1], I, R and Q satisfy the same generalized polynomial identity with automorphisms and hence (2) still holds for all $x \in [Q, Q]$. So we may assume that $R = I = Q$ and R is a centrally closed prime ring. Also,

$$(bx - \beta(x)c)^n = 0 \tag{5}$$

for all $x \in [R, R]$. If one of b and c is zero, we are done by Lemma 1. So we may assume that both b and c are nonzero. By Chuang [7, Main Theorem], R is a GPI-ring.

Thus R is a primitive ring with nonzero socle by Martindale III [19, Theorem 3]. If R is a domain, then $bx-\beta(x)c=0$ for all $x\in[R,R]$. In particular, $b[x,y]-[\beta(x),\beta(y)]c=0$ for all $x,y\in R$. Since β is X-outer, by Kharchenko [14], we have $b[x,y]-[z,u]c=0$ for all $x,y,z,u\in R$. Since both b and c are nonzero, R is commutative, a contradiction. Now we assume that R is not a domain. Then R has nontrivial idempotents. Let e be an idempotent of R. Applying (5) we have

$$\begin{aligned}0&=(b[\beta^{-1}(1-e)xe,\beta^{-1}(1-e)ye]-[(1-e)\beta(x)\beta(e),\\&\quad(1-e)\beta(y)\beta(e)]c)^n(1-e)\\&=([(1-e)\beta(x)\beta(e),(1-e)\beta(y)\beta(e)]c)^n(1-e)\\&=(1-e)((\beta(x)\beta(e)(1-e)\beta(y)-\beta(y)\beta(e)(1-e)\beta(x))\beta(e)c(1-e))^n\end{aligned}$$

for all $x,y\in R$. This implies $(1-e)((y\beta(e)(1-e)x-x\beta(e)(1-e)y)\beta(e)c(1-e))^n=0$ for all $x,y\in R$. By Xu et al. [20, Remark 2.1(1)], it follows that one of the following holds:

(i) $\beta(e)c(1-e)=0$;
(ii) $\beta(e)c(1-e)y\beta(e)(1-e)=0$ and $\beta(e)(1-e)y\beta(e)c(1-e)=0$ for all $y\in R$; or
(iii) $(1-e)y\beta(e)(1-e)=0$ for all $y\in R$.

Since R is a prime ring, this implies that if e is an idempotent of R, then either $\beta(e)(1-e)=0$ or $\beta(e)c(1-e)=0$.

Assume that e is an idempotent of R with $\beta(e)(1-e)=0$. Then for $x,y\in R$, we have again from (5) that

$$\begin{aligned}0&=(b[\beta^{-1}(1-e)xe,ye]-[(1-e)\beta(x)\beta(e),\beta(y)\beta(e)]c)^n(1-e)\\&=(1-e)(\beta(x)\beta(e)\beta(y)\beta(e)c(1-e))^n.\end{aligned}$$

Thus $(\beta(x)\beta(e)\beta(y)\beta(e)c(1-e))^{n+1}=0$ for all $x,y\in R$. By Levitzki's lemma, we have $\beta(e)R\beta(e)c(1-e)=0$ and hence $\beta(e)c(1-e)=0$. That is, if e is an idempotent of R, then $\beta(e)c(1-e)=0$. Similarly, $\beta(1-e)ce=0$. Thus $ce=\beta(e)ce=\beta(e)c$ for all idempotents of R. Again, let T be the additive subgroup of R generated by all idempotents of R. Thus $ct=\beta(t)c$ for all $t\in T$. By Herstein [11, Corollary p. 18], we have $[R,R]\subseteq T$. Hence $cx=\beta(x)c$ for all $x\in[R,R]$, which leads a contradiction as before. This proves the lemma. □

Proof of the main theorem Set $I=R[L,L]R$. Then $0\neq[I,R]\subseteq L$. It is known that there exists $s=f(1)\in {}_FR$ such that $f(x)=sx+\delta(x)$ for all $x\in R$, where δ is a β-derivation of ${}_FR$. By the assumption, we have

$$(sx+\delta(x))^n=0 \tag{6}$$

for all $x\in L$ and hence for all $x\in[I,R]$. By Chuang and Lee [9, Theorem 2], I, R and ${}_FR$ satisfy the same GPI with single skew derivation over ${}_FR$ and hence (6) holds for all $x\in[{}_FR,{}_FR]$. So we may assume that $R=I={}_FR$. If δ is X-inner, then

$\delta(x) = cx - \beta(x)c$ for all $x \in R$, where $c \in Q$. We rewrite (6) as

$$((s+c)x - \beta(x)c)^n = 0$$

for all $x \in [R, R]$. So we are done by Lemma 3. Finally, we assume that δ is X-outer. By (6) we have

$$\begin{aligned} 0 &= (s[x, y] + \delta[x, y])^n = (s[x, y] + \delta(xy) - \delta(yx))^n \\ &= (s[x, y] + \delta(x)y + \beta(x)\delta(y) - \delta(y)x - \beta(y)\delta(x))^n \end{aligned}$$

for all $x, y \in R$. By Chuang and Lee [9, Theorem 1], we have $(s[x, y] + zy + \beta(x)u - ux - \beta(y)z)^n = 0$ for all $x, y, z, u \in R$. Setting $z = u = 0$, we have then $(s[x, y])^n = 0$ for $x, y \in R$ and hence $s = 0$ by Lemma 1. Therefore, we have $(zy + \beta(x)u - ux - \beta(y)z)^n = 0$ for all $x, y, z, u \in R$. Setting $u = 0$ and using [5, Lemma 4] (or [18, Lemma 2.6]), we have $zy - \beta(y)z = 0$ for all $y, z \in R$. In particular, $\beta(y) = y$ for all $y \in R$ and hence $zy = yz$ for $y, z \in R$. Therefore R is commutative, which is a contradiction. This proves the theorem. □

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