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Complete hypersurfaces with constant scalar curvature in spheres

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Abstract To a given immersion $i : M^n \to \mathbb{S}^{n+1}$ with constant scalar curvature R, we associate the supremum of the squared norm of the second fundamental form $\sup|A|^2$. We prove the existence of a constant $C_n(R)$ depending on R and n so that $R \geq 1$ and $\sup|A|^2 = C_n(R)$ imply that the hypersurface is a $H(r)$-torus $\mathbb{S}^1(\sqrt{1-r^2}) \times \mathbb{S}^{n-1}(r)$. For $R > (n-2)/n$ we use rotation hypersurfaces to show that for each value $C > C_n(R)$ there is a complete hypersurface in $\mathbb{S}^{n+1}$ with constant scalar curvature R and $\sup|A|^2 = C$, answering questions raised by Q. M. Cheng.

Keywords Scalar curvature · Rotation hypersurfaces · Product of spheres

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1 Introduction

Let M^n be a complete hypersurface immersed by $i : M^n \to \mathbb{S}^{n+1}$ into the unit sphere $\mathbb{S}^{n+1}$ and R be the normalized scalar curvature of M (we recall the definitions in a moment). Many rigidity theorems have been obtained by imposing natural conditions to R. One of the first results in this respect was obtained by Cheng and Yau [4]. By using an adequate operator denoted here by L_1 they proved that a complete hypersurface in $\mathbb{S}^{n+1}$ with constant R and non-negative sectional curvature must be umbilical or isometric to a Riemannian product $\mathbb{S}^k(\sqrt{1-r^2}) \times \mathbb{S}^{n-k}(r)$, $1 \le k \le n-1$, where for example $\mathbb{S}^{n-k}(r)$ denotes a sphere of dimension $n-k$ and radius r.

Some years later, Li used L_1 in [8] to analyze a compact hypersurface of $\mathbb{S}^{n+1}$ with R constant, $R \ge (n-2)/n$ and second fundamental form A satisfying $|A|^2 \le C_n(R)$, where

$$C_n(R) = (n-1)\frac{n\bar{R}+2}{n-2} + \frac{n-2}{n\bar{R}+2}, \quad \bar{R} = R-1. \tag{1}$$

Under these conditions, he showed that M either satisfies $|A|^2 \equiv n\bar{R}$ and M is totally umbilical or $|A|^2 \equiv C_n(R)$ and M is isometric to a $H(r)$-torus given as $\mathbb{S}^1(\sqrt{1-r^2}) \times \mathbb{S}^{n-1}(r)$, where

$$r^2 = \frac{n-2}{nR} \le \frac{n-2}{n}.$$

In Li [9], he also remarked that for $R \ge 1$ a similar rigidity theorem may be obtained replacing "compact and $|A|^2 \le C_n(R)$" by "complete and $|A|^2 \le C_n(R) - \varepsilon$". In this paper we improve this result (see Theorem 1) by droping the number ε.

On the other hand, Cheng analyzed in [2] the case where $|A|^2 \ge C_n(R)$ and proved that a complete locally conformally flat hypersurface with such a condition must satisfy $R > (n-2)/n$.

When R is constant, $R \ne (n-2)/(n-1)$ and $|A|^2 \ge C_n(R)$, Cheng also proved that a complete hypersurface M with such restrictions must be again a $H(r)$-torus. In the same case $R \ne (n-2)/(n-1)$ he showed that there are no complete hypersurfaces in $\mathbb{S}^{n+1}$ with two principal curvatures of multiplicities $(n-1, 1)$ and $|A|^2 \ge C_n((n-2)/(n-1)) = n$.

In the same paper, Cheng [2] posed two problems:

Problem 1 Let M^n be an n-dimensional complete hypersurface with constant scalar curvature R in $\mathbb{S}^{n+1}$. If $R > (n-2)/n$ and $|A|^2 \le C_n(R)$, where $C_n(R)$ is given by (1). Is M isometric to either a totally umbilical hypersurface or to the Riemannian product of the form $\mathbb{S}^1(\sqrt{1-r^2}) \times \mathbb{S}^{n-1}(r)$?

In Cheng [2], solved problem 1 affirmatively for $R = (n-2)/(n-1)$. Also, in Cheng et al. [3], answered it affirmatively for compact hypersurfaces with $R > (n-2)/n$, $R \ne (n-2)/(n-1)$ and two principal curvatures. Here we further analyze this problem, proving that the answer to problem 1 is affirmative also in the complete non-compact case with $R \ge 1$.

Theorem 1 *Let M^n be a n-dimensional complete hypersurface of $\mathbb{S}^{n+1}$ with constant scalar curvature $R \geq 1$. If $|A|^2 \leq C_n(R)$ everywhere, then either*

1. $|A|^2 \equiv n(R-1)$ *and M is totally umbilical, or*
2. $\sup |A|^2 = C_n(R)$. *If $\sup |A|^2$ is attained at some point in M, then M is the $H(r)$–torus $\mathbb{S}^1(\sqrt{1-r^2}) \times \mathbb{S}^{n-1}(r)$.*

Hence, problem 1 remains open for $(n-2)/n < R < 1$.

Problem 2 Let M^n be an n-dimensional complete hypersurface with constant scalar curvature $R = (n-2)/(n-1)$ in $\mathbb{S}^{n+1}$. If M has only two distinct principal curvatures, one of which is simple, is M isometric to the Clifford torus $\mathbb{S}^1(\sqrt{1/n}) \times \mathbb{S}^{n-1}(\sqrt{(n-1)/n})$?

In order to make some educated guesses to answer this question, we have at hand some examples where geometric quantities as R and $|A|^2$ can be calculated or easily estimated.

These examples are the rotation hypersurfaces defined and studied in space forms in [5]. See also [7], where Leite made a detailed analysis of rotation hypersurfaces with constant scalar curvature in space forms.

In this paper, we will analyze these hypersurfaces and describe the variation of $|A|^2$ in terms of R, to obtain the following result.

Theorem 2 *Let R, C be constants such that $R > (n-2)/n$ and $C \geq C_n(R)$. Then there exists a complete n-dimensional hypersurface of $\mathbb{S}^{n+1}$ with constant scalar curvature R such that $\sup |A|^2 = C$.*

Taking $R = (n-2)/(n-1)$, this result shows that the answer to problem 2 is negative.

We may represent graphically our results by using a plane $(R, \sup |A|^2)$ as in Fig. 1.

Our paper is organized as follows. Section 2 contains all prerequisite material. In Sect. 3 we will prove Theorem 1. In Sect. 4 and for completeness, we will describe the rotation hypersurfaces with constant scalar curvature (with a detailed study in [7]), analyzing the corresponding values of $|A|^2$, thus proving Theorem 2.

A final comment is in order. Our theorems show that for $R \geq 1$ fixed, $\sup |A|^2$ varies in the set $\{n\bar{R}\} \cup [C_n(R), \infty)$. The analysis of the behavior of $|A|^2$ for the case of rotation hypersurfaces show that for $(n-2)/n < R < 1$, $\sup |A|^2$ varies at least in $[C_n(R), \infty)$. Thus our examples suggest an affirmative answer to Cheng's problem 1 also when $(n-2)/n < R < 1$.

2 Preliminaries

Let M^n be a n-dimensional complete orientable manifold. Denote by $f : M^n \to \mathbb{S}^{n+1}$ a immersion of M^n into the $(n+1)$-dimensional unit sphere $\mathbb{S}^{n+1}$. Choose a local orthonormal frame field $E_1, \ldots, E_{n+1}$ such that at each point $p \in M$, $E_1(p), \ldots, E_n(p)$ is an orthonormal basis of T_pM.

In the sequel, the following conventions on indices are used:

$$A, B, C, \ldots = 1, \ldots, n+1; \quad i, j, k, \ldots = 1, \ldots, n.$$

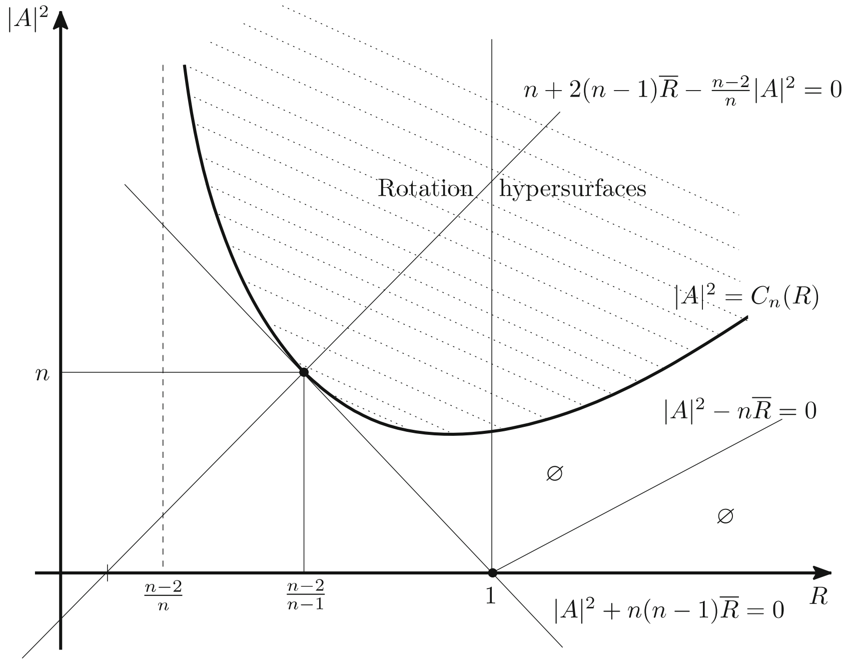

Fig. 1 The coordinate plane $(R, \sup|A|^2)$, where the line $|A|^2 - n(R-1) = 0$, crossing the R-axis at $R = 1$, represents the totally umbilical hypersurfaces in $\mathbb{S}^{n+1}$. The curve $|A|^2 = C_n(R)$ represents the $H(r)$-torus $\mathbb{S}^{n-1}(\sqrt{1-r^2}) \times \mathbb{S}^1(r)$ with constant scalar curvature $R > (n-2)/n$. Our Theorem 1 shows that there are no complete hypersurfaces in $\mathbb{S}^{n+1}$ with scalar curvature R and $\sup|A|^2$ at the regions marked with the empty set symbol. On the other hand, Theorem 2 shows that for each point (R, C) over the curve $\sup|A|^2 = C_n(R)$ there is a rotation hypersurface with such scalar curvature R and $\sup|A|^2 = C$

Let $\omega_1, \ldots, \omega_{n+1}$ be the dual forms associated to $E_1, \ldots, E_{n+1}$ and ω_{AB} the corresponding connection forms, so that the following structure equations for $\mathbb{S}^{n+1}$ hold:

$$d\omega_A = \sum_B \omega_{AB} \wedge \omega_B, \quad \omega_{AB} + \omega_{BA} = 0,$$

$$d\omega_{AB} = \sum_C \omega_{AC} \wedge \omega_{CB} - \frac{1}{2} \sum_{C,D} \bar{R}_{ABCD}\, \omega_C \wedge \omega_D.$$

where as usual $\bar{R}_{ABCD} = \bar{R}_{ABDC}$. The coefficients

$$\bar{R}_{ABCD} = \delta_{AC}\delta_{BD} - \delta_{AD}\delta_{BC}$$

are the components of the curvature tensor of $\mathbb{S}^{n+1}$. Similarly, the structure equations for M may be written as

$$d\omega_i = \sum_j \omega_{ij} \wedge \omega_j, \quad \omega_{ij} + \omega_{ji} = 0,$$

$$d\omega_{ij} = \sum_k \omega_{ik} \wedge \omega_{kj} - \frac{1}{2} \sum_{k,l} R_{ijkl}\, \omega_k \wedge \omega_l,$$

where R_{ijkl} are the components of the curvature tensor of M with respect to the induced metric. As $\omega_{n+1} = 0$ restricted to M, we have

$$\omega_{i,n+1} = \sum_j h_{ij}\omega_j, \quad h_{ij} = h_{ji}.$$

Here h_{ij} are the coefficients of the second fundamental form of M,

$$A = \sum h_{ij}\omega_i \wedge \omega_j.$$

The squared norm of the second fundamental form $|A|^2$ and the mean curvature H are defined respectively by

$$|A|^2 = \sum_{i,j} h_{ij}^2, \quad H = \frac{1}{n}\sum_i h_{ii},$$

while the Ricci curvature is

$$(n-1)\mathrm{Ric}(v) = \sum_{i<n} R_{inin} = \sum_{i<n}\left(1 + h_{ii}h_{nn} - h_{in}^2\right), \quad v = e_n. \tag{2}$$

Also, the normalized scalar curvature R is given by

$$n(n-1)R = \sum_{i,j} R_{ijij}.$$

With these notations, Gauss equation takes the form

$$R_{ijkl} = \left(\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}\right) + \left(h_{ik}h_{jl} - h_{il}h_{jk}\right).$$

We will denote $\bar{R} = R - 1$, so that Gauss equation may be written as

$$n(n-1)\bar{R} = (nH)^2 - |A|^2. \tag{3}$$

If f is a C^2-function on M, we define the gradient df, the hessian (f_{ij}) and the Laplacian Δf of f as

$$df = \sum_i f_i\omega_i, \quad \sum_j f_{ij}\omega_j = df_i + \sum_j f_j\omega_{ji}, \quad \Delta f = \sum_i f_{ii}.$$

We introduce the operator L_1 acting on differentiable functions f defined on M by

$$L_1(f) = \sum_{ij}(nH\delta_{ij} - h_{ij})f_{ij}$$

Locally, we may choose $E_1, \ldots, E_n$ so that $h_{ij} = \kappa_i \delta_{ij}$. By a standard calculation,

$$\begin{aligned} L_1(nH) &= \sum_{ij} (nH - \kappa_i)\delta_{ij}(nH)_{ij} \\ &= nH\Delta(nH) - \sum_i \kappa_i (nH)_{ii} \\ &= \frac{1}{2}\Delta(nH)^2 - |\nabla(nH)|^2 - \sum_i \kappa_i (nH)_{ii}. \end{aligned}$$

Using Gauss equation (3) and the well-known Simons formula (see, for example, [10])

$$\frac{1}{2}\Delta(nH)^2 = \frac{1}{2}\Delta|A|^2 = |\nabla A|^2 + \sum_i \kappa_i (nH)_{ii} + \frac{1}{2}\sum_{i,j} R_{ijij}(\kappa_i - \kappa_j)^2, \tag{4}$$

we have

$$L_1(nH) = |\nabla A|^2 - |\nabla(nH)|^2 + \frac{1}{2}\sum_{i,j} R_{ijij}\left(\kappa_i - \kappa_j\right)^2. \tag{5}$$

The following inequality was proved by Alencar and do Carmo (see also Li [8]), but we include it for completeness.

Lemma 1 [1, p. 1226] *Let M^n be a immersed hypersurface in $\mathbb{S}^{n+1}$. Then*

$$\frac{1}{2}\sum_{i,j} R_{ijij}\left(\kappa_i - \kappa_j\right)^2 \geq |\phi|^2 \left(-|\phi|^2 - \frac{n(n-2)}{\sqrt{n(n-1)}}|H|\,|\phi| + n(H^2+1)\right) \tag{6}$$

where

$$|\phi|^2 = \frac{n-1}{n}\left(|A|^2 - n\bar{R}\right).$$

Equality holds whenever $(n-1)$ of the principal curvatures are equal to $\pm\sqrt{(n-1)/n}|\phi|$.

Substituting (6) in (5), using Gauss equation (3) and the above expression for $|\phi|^2$, we obtain

$$L_1(nH) + |\nabla(nH)|^2 \geq \frac{n-1}{n}\left(|A|^2 - n\bar{R}\right)P_R\left(|A|^2\right), \tag{7}$$

where

$$P_R(x) = n + 2(n-1)\bar{R} - \frac{n-2}{n}\left(x + \sqrt{(x + n(n-1)\bar{R})(x - n\bar{R})}\right). \tag{8}$$

Lemma 2 *Let R, x be real numbers, $R \geq (n-2)/(n-1)$ and x such that $P_R(x)$ is defined. Then $P_R(x)$ is a decreasing function of x for R fixed. Moreover, $P_R(x) \geq 0$ if and only if $x \leq C_n(R)$, where $C_n(R)$ is the (positive) constant given explicitly by (1). Also, $P_R(x) = 0$ if and only if $x = C_n(R)$.*

Proof The proof that $P_R(x)$ is a decreasing function of x uses standard techniques, so we omit it. $P_R(x) \geq 0$ if and only if

$$n + 2(n-1)\bar{R} - \frac{n-2}{n}x \geq \frac{n-2}{n}\sqrt{(x + n(n-1)\bar{R})(x - n\bar{R})}.$$

We will consider the region of the (R, x)-plane where the left hand side of this inequality is non-negative. (See Fig. 1, where we depicted the line $n + 2(n-1)\bar{R} - \frac{n-2}{n}x = 0$.) This region contains the set where $R \geq (n-2)/(n-1)$ and $x \leq C_n(R)$. Moreover, in this region the above inequality is equivalent to that between the squares of the corresponding terms, which in turn is equivalent to $x \leq C_n(R)$. □

We will also use Omori's classical version of the maximum principle at infinity for complete manifolds.

Theorem 3 [11] *Let M^n be an n-dimensional complete Riemannian manifold whose sectional curvatures are bounded from below. Let f be a C^2-function bounded from above on M^n. Then there exists a sequence of points $p_k \in M$ such that*

$$\lim_{k\to\infty} f(p_k) = \sup f, \quad \lim_{k\to\infty} |\nabla f(p_k)| = 0 \quad \text{and} \quad \limsup_{k\to\infty} \max_{|X|=1} \Delta f(p_k)(X, X) \leq 0.$$

3 The gap in the case $R \geq 1$

We will need the following result to assure that Omori's principle may be applied.

Lemma 3 *Let M be an n-dimensional complete hypersurface in $\mathbb{S}^{n+1}$. If $|A|^2$ is bounded from above, then the sectional curvatures of M are bounded.*

Proof By hypothesis, $|A|^2$ is bounded from above by a constant, say C. Following the notation in the Preliminaries,

$$\kappa_i^2 \leq |A|^2 \leq C,$$

so that $|\kappa_i| \leq \sqrt{C}$ for all i, j. From Gauss equation we have that $R_{ijij} = 1 + \kappa_i \kappa_j$, so

$$1 - C \leq R_{ijij} \leq 1 + C,$$

and the lemma follows. □

We are ready to prove Theorem 1.

Proof of Theorem 1 As $R \geq 1$, Gauss equation (3) implies that nH does not change sign on M, so we may suppose $H > 0$. Moreover, the same Eq. (3) and the condition $|A|^2 \leq C_n(R)$ imply that $(nH)^2$ is bounded, so nH is bounded from above. By Lemma 3, the sectional curvatures of M are bounded from below, so we may apply Omori's principle to the function $f = nH$, thus obtaining a sequence of points p_k in M such that

$$(nH)(p_k) \to \sup(nH), \quad |\nabla(nH)(p_k)| \to 0 \quad \text{and} \quad \limsup_{k\to\infty} \max_{|X|<1} \Delta(nH)(p_k)(X,X) \leq 0.$$

We have $(nH)^2(p_k) \to \sup(nH)^2$ so that Gauss equation implies

$$|A|^2(p_k) \to \sup|A|^2. \tag{9}$$

Evaluating $L_1(nH)$ at the points p_k, we have

$$L_1(nH)(p_k) \leq \limsup_{k\to\infty} \left(\sum_i (nH - \kappa_i)(nH)_{ii} \right)(p_k).$$

Note that $\kappa_i^2 \leq |A|^2 \leq (nH)^2$, so that $nH - \kappa_i \geq 0$. As noted before, $nH - \kappa_i$ is also bounded from above by, say C. Then

$$L_1(nH)(p_k) \leq \limsup_{k\to\infty} C\Delta(nH)(p_k).$$

Substituting in (7) and taking the lim sup we have

$$0 \leq \frac{n-1}{n}\left(\sup|A|^2 - n\bar{R}\right) P_R(\sup|A|^2) \leq \limsup_{k\to\infty} \left(L_1(nH) + |\nabla(nH)|^2\right)(p_k) \leq 0.$$

If $\sup|A|^2 - n\bar{R} = 0$, then $|A|^2 \equiv n\bar{R}$, so that M is totally umbilical. On the other hand, if $P_R(\sup|A|^2) = 0$ we have $\sup|A|^2 = C_n(R)$, as claimed.

Suppose that $\sup|A|^2$ is attained and let L be the operator acting on C^2-functions by

$$L(f) = L_1 f + \langle \nabla f, \nabla f \rangle.$$

Note that L satisfies a sufficient condition (eq. (10.36) of [6, p. 277]) to apply an extended version of the maximum principle for quasilinear operators. As $L(nH) \geq 0$ and $\sup(nH)$ is attained, we have that nH is constant. Hence, by Gauss equation, we have also that $|A|^2$ is constant. Thus equality holds in (6) and the corresponding hypersurface has $n-1$ equal constant principal curvatures. By a known result (in [12], for example), M is isometric to a $H(r)$-torus $\mathbb{S}^1(\sqrt{1-r^2}) \times \mathbb{S}^{n-1}(r)$. □

4 Hypersurfaces with $\sup |A|^2 > C_n(R)$

In this section, we prove Theorem 2 by analyzing the variation of $|A|^2$ for the class of rotation hypersurfaces with constant scalar curvature, introduced by Leite [7]. We recall that a rotation hypersurface $M^n \subset \mathbb{S}^{n+1}$ is an $O(n)$–invariant hypersurface, where $O(n)$ is considered as a subgroup of isometries of $\mathbb{S}^{n+1}$.

$O(n)$ fixes a given geodesic γ (the rotation axis) and rotates a profile curve α parameterized by arc length s. We denote by $d(s)$ the minimum distance from $\alpha(s)$ to γ, realized by a point $P(s)$ in γ and $h(s)$ the height of $P(s)$ measured from a fixed point in γ. With these notations, the principal curvatures of the rotation hypersurface M are given by

$$\lambda = \kappa_i = \frac{\sqrt{1 - r'^2 - r^2}}{r}, \quad i = 1, \ldots, n-1, \quad \text{and} \quad \mu = \kappa_n = -\frac{r'' + r}{\sqrt{1 - r'^2 - r^2}},$$

where $r(s) = \sin(d(s))$. Thus, the scalar curvature R of M is given by

$$(n-2)\lambda^2 + 2\lambda\mu = n(R-1), \tag{10}$$

or

$$R = -\frac{2r''}{nr} + \frac{(n-2)(1-r'^2)}{nr^2}.$$

Under the hypothesis of R being constant, this equation is equivalent to its first order integral

$$G_R(r, r') = r^{n-2}(1 - r'^2 - Rr^2) = K,$$

where K is a constant. As in [7], we will study the rotation hypersurfaces through this function G_R; for example, it is shown in [7] that every level curve of G_R contained in the region $r^2 + r'^2 \leq 1$ of the (r, r')-plane gives rise to a complete rotation hypersurface with constant scalar curvature R.

Let us also consider the null set $G_R(r, r') = 0$, which is given by the union of the r'-axis and the conic $1 - r'^2 - Rr^2 = 0$. As we will be interested in the case $R > (n-2)/n$, this conic is always an ellipse, which lies outside (coincides with, is entirely contained in) the unit circle whenever $(n-2)/n < R < 1$ ($R = 1$, $R > 1$ respectively). In cases $R \geq 1$, this curve is associated to a totally umbilical hypersurface, since in this case the principal curvatures satisfy

$$\lambda = \frac{\sqrt{1 - r'^2 - r^2}}{r} = \sqrt{R-1}.$$

Thus, from (10), we obtain that $\mu = \sqrt{R-1}$ and M is umbilical.

As another example, from corollary 2.3 in [7], we see that for every $R > \frac{n-2}{n}$, G_R has one critical point of the form $(r, 0)$, $r^2 = \frac{n-2}{nR}$, corresponding to the torus

$\mathbb{S}^1(\sqrt{1-r^2}) \times \mathbb{S}^{n-1}(r)$. The critical point is a maximum of G_R, which implies the existence of a whole family of closed level curves of G_R, obtained by taking into account *non-negative* values of K. These level curves surround the critical point, growing until they reach the null set $G_R(r, r') = 0$. Every level curve outside the null set escapes from the region $r^2 + r'^2 \leq 1$ and thus the corresponding hypersurface is not complete.

In the rest of this section, we analyze the behavior of $|A|^2$ for each level curve of G_R, first for $R \geq 1$ and then for $\frac{n-2}{n} < R < 1$.

Proof of Theorem 2 First case. Suppose $R \geq 1$. Since $|A|^2 = (n-1)\lambda^2 + \mu^2$ and $(n-2)\lambda^2 + 2\lambda\mu = n(R-1)$, we may write $|A|^2$ as a function of λ alone. Also, fixing a level curve of G_R, so that $r^{n-2}(1 - r'^2 - Rr^2) = K$, we write λ in terms of r, obtaining

$$|A|^2(r) = (n-1)\lambda^2 + \left(\frac{n(R-1) - (n-2)\lambda^2}{2\lambda}\right)^2, \tag{11}$$

where

$$\lambda = \frac{\sqrt{1 - r'^2 - r^2}}{r} = \frac{\sqrt{\frac{K}{r^{n-2}} + Rr^2 - r^2}}{r} = \sqrt{\frac{K}{r^n} + (R-1)}. \tag{12}$$

Observe that $K \geq 0$, so $\lambda \geq \sqrt{R-1}$. Differentiating $|A|^2$ with respect to r, we have

$$\frac{d|A|^2}{dr} = \frac{d|A|^2}{d\lambda}\frac{d\lambda}{dr} = \frac{n^2}{2}\left(\lambda - \frac{(R-1)^2}{\lambda^3}\right)\frac{-Kn}{2r^{n+1}\lambda}. \tag{13}$$

The derivative of $|A|^2$ is non-positive. In fact, as $\lambda \geq \sqrt{R-1}$, the derivative can vanish if and only if $\lambda = \sqrt{R-1}$, which is equivalent in this case to $K = 0$. Thus, for a fixed level curve of G_R, $|A|^2$ attains its extreme values at the intersections of the level curve with the r-axis. As the interior of the level curve contains the critical point of G_R, the interval of variation of $|A|^2$ contains the value of $|A|^2$ for the critical point; namely, the constant $C_n(R)$ defined in (1).

Consider the variation of $|A|^2(r)$, $0 < r < 1/\sqrt{R}$ (this value corresponding to $G_R(r, 0) = 0$). From (11) and (12) we obtain

$$\lim_{r\to 0} \lambda(r, 0) = +\infty, \quad \text{so that} \quad \lim_{r\to 0} |A|^2(r, 0) = +\infty,$$

and

$$\lim_{r\to 1/\sqrt{R}} \lambda(r, 0) = \sqrt{R-1}, \quad \text{so that} \quad \lim_{r\to 1/\sqrt{R}} |A|^2(r, 0) = n(R-1).$$

By continuity, $|A|^2(r, 0)$ assumes all values in $[n(R-1), +\infty)$. We may classify the associated hypersurfaces as follows:

1. Totally umbilical hypersurfaces (corresponding to the null set of G_R): $|A|^2$ is constant and equal to $n(R-1)$.
2. Product of spheres (corresponding to the critical points of G_R): $|A|^2$ is constant and equal to $C_n(R)$.
3. Rotation hypersurfaces associated with closed level curves near the critical point of G_R have $|A|^2$ varying in a closed interval containing $C_n(R)$ in its interior. When the closed level curves approach the null set of G_R, $\inf |A|^2 \to n(R-1)$, while $\sup |A|^2 \to \infty$.

Therefore, we have that for each $C \geq C_n(R)$ there is a rotation hypersurface satisfying $\sup |A|^2 = C$, which proves Theorem 2 in the case $R \geq 1$.

Second case. Suppose $(n-2)/n < R < 1$. The analysis in this case is quite similar, so we just point out the differences. In this case, the null set of G_R lies outside of the region $r^2 + r'^2 \leq 1$, so we don't have umbilical hypersurfaces for these values of R and G_R is everywhere positive.

To study the variation of $|A|^2$, we may use the expressions (11)–(13). Nevertheless, we must analyze carefully the condition $\lambda^4 = (R-1)^2$ for the derivative to vanish, since now $\lambda^2 = 1 - R$. Evaluating (12) at the points $(r, 0)$, we have

$$\lambda = \frac{\sqrt{1-r^2}}{r}, \quad \text{so that} \quad 1 - R = \lambda^2 = \frac{1-r^2}{r^2}, \quad \text{or} \quad r^2 = \frac{1}{2-R},$$

which means that the critical point of G_R lies inside the region $r^2 + r'^2 \leq 1$. Once again, it is easy to show that G_R attains a maximum at this point. By the way, this point lies to the left of (coincides with, lies to the right of) the critical point of G_R if R is less than (equal to, greater than, respectively) $(n-2)/(n-1)$. By making an analysis of the variation of $|A|^2$ similar to that in the first case $R \geq 1$, we may summarize our results as follows.

1. The minimum value of $|A|^2$ is $n(n-1)(1-R)$, which coincides with $C_n(R)$ only for $R = (n-2)/(n-1)$.
2. As every closed level curve of G_R contains the critical point of G_R in its interior, $|A|^2$ varies in a closed interval containing $C_n(R)$ in its interior if $R \neq (n-2)/(n-1)$. If $R = (n-2)/(n-1)$, $|A|^2$ varies in a closed interval with $C_n(R)$ as its left extreme value.
3. The level curve contains the critical point of $|A|^2$ (in the closure of its interior) if and only if $|A|^2$ varies in a closed interval with left extreme value equal to $n(n-1)(1-R)$.
4. When the level curves approach $(0, 0)$, $\sup |A|^2 \to \infty$.

In short, $\sup |A|^2$ varies from $C_n(R)$ to $+\infty$, which implies again the existence of a hypersurface with constant scalar curvature R and $\sup |A|^2 = C$ for each $C \geq C_n(R)$, which finishes the proof of Theorem 2 for this second and last case $(n-2)/n < R < 1$. $\square$

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References

1. Alencar, H., do Carmo, M.P.: Hypersurfaces with constant mean curvature in spheres. Proc. AMS **120**(4), 1223–1229 (1994)
2. Cheng, Q.M.: Hypersurfaces in a unit sphere $\mathbb{S}^{n+1}$ with constant scalar curvature. J. Lond. Math. Soc. **64**(2), 755–768 (2001)
3. Cheng, Q.M., Shu, S., Suh, Y.J.: Compact hypersurfaces in a unit sphere. Proc. Royal Soc. Edinburgh **135**, 1129–1137 (2005)
4. Cheng, S.Y., Yau, S.T.: Hypersurfaces with constant scalar curvature. Math. Ann. **225**, 195–204 (1977)
5. do Carmo, M.P., Dajczer, M.: Rotational hypersurfaces in spaces of constant curvature. Trans. Am. Math. Soc. **277**(2), 685–709 (1983)
6. Gilbarg, D., Trudinger, N.S.: Elliptic Partial Differential Equations of Second Order. Springer, New York (1977, 1983)
7. Leite, M.L.: Rotational hypersurfaces of space forms with constant scalar curvature. Manuscr. Math. **67**, 285–304 (1990)
8. Li, H.: Hypersurfaces with constant scalar curvature in space forms. Math. Ann. **305**, 665–672 (1996)
9. Li, H.: Global rigidity theorems of hypersurfaces. Ark. Mat. **35**, 327–351 (1997)
10. Okumura, M.: Hypersurfaces and a pinching problem on the second fundamental tensor. Am. J. Math. **96**, 207–213 (1974)
11. Omori, H.: Isometric Immersions of Riemannian manifolds. J. Math. Soc. Jpn. **19**, 205–214 (1967)
12. Otsuki, T.: Minimal hypersurfaces in a Riemannian manifold of constant curvature. Am. J. Math. **92**, 145–173 (1970)