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# Rational forms of nilpotent Lie algebras and Anosov diffeomorphisms

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**Abstract.** We compute the set of all rational forms up to isomorphism for some real nilpotent Lie algebras of dimension 8. This is part of the classification of nilmanifolds admitting an Anosov diffeomorphism in dimension at most 8.

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## 1. Introduction

Let $N$ be a simply connected nilpotent Lie group and assume that $N$ admits a lattice $\Gamma$ (i.e. a cocompact discrete subgroup). It is then natural to study in the compact quotient $N/\Gamma$ (*nilmanifold*), dynamics of automorphisms of $N$ stabilizing $\Gamma$, or geometry if one equips $N$ with a left invariant Riemannian metric, complex structure, symplectic structure, etc. Dynamical and geometric properties of the nilmanifold often depend only on the commensurability class of the lattice $\Gamma$. It is then when one runs into the following problem:

$(*)$ To find all rational forms up to isomorphism of a given real nilpotent Lie algebra $\mathfrak{n}$.

There is not much on this question in the literature, and a complete answer seems quite difficult to obtain in explicit examples, even in low dimensional or 2-step nilpotent cases. In [5, Theorem 1.3], the set of isomorphism classes of rational forms of $\mathfrak{n}$ is described by using Galois cohomology of the group $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ with values in $\mathrm{Aut}(\mathfrak{n})$. The problem can also be described in terms of rational points in the orbit space of an algebraic variety (see [2, Section 5] and (4)).

A diffeomorphism $f$ of a compact differentiable manifold $M$ is called *Anosov* if the most perfect kind of global hyperbolic behavior for a dynamical system holds:

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the tangent bundle $\mathrm{T}M$ admits a continuous invariant splitting $\mathrm{T}M = E^+ \oplus E^-$ such that $\mathrm{d}f$ expands $E^+$ and contracts $E^-$ exponentially. Simple examples are obtained from the following algebraic construction. Let $\varphi$ be a *hyperbolic* automorphism of $N$ (i.e. all the eigenvalues of its derivative have absolute value different from 1) such that $\varphi(\Gamma) = \Gamma$ for some lattice $\Gamma$ of $N$. Then $\varphi$ defines an Anosov diffeomorphism on the nilmanifold $M = N/\Gamma$, which is called an *Anosov automorphism*.

Forty years have already passed since in the seminal paper [13], Smale raised the problem of classifying all compact manifolds (up to homeomorphism) which admit an Anosov diffeomorphism, and curiously enough, the only known examples so far up to finite quotient or topological conjugation are the Anosov automorphisms described above. It is conjectured that these exhaust the class of Anosov diffeomorphisms (see [12]). All this certainly highlights the problem of classifying nilmanifolds admitting Anosov automorphisms, which are easily seen to be in correspondence with the so called *Anosov Lie algebras* (see [6]), that is, rational nilpotent Lie algebras admitting a hyperbolic automorphism $A$ which is also *unimodular* (i.e. its matrix with respect to some basis belongs to $\mathrm{GL}_n(\mathbb{Z})$).

Unimodularity and hyperbolicity are, together, a quite strong condition to be satisfied by an automorphism of a Lie algebra, and therefore the existence of an Anosov automorphism only holds for very distinguished nilmanifolds. There is a construction in [6] of an Anosov rational form for any Lie algebra of the form $\mathfrak{n} \oplus \mathfrak{n}$, where $\mathfrak{n}$ is a real graded nilpotent Lie algebra admitting at least one rational form. Such an abundance turns the classification into a wild problem for large dimensions, but this is due to the great deal of nilpotent Lie algebras rather than to the Anosov condition.

An explicit classification of Anosov Lie algebras of dimension at most 8 is given in [8]. After a quite involved work carried out in [7, Section 4] and [8, Section 3] on the real level, it follows that the only real nilpotent Lie algebras of dimension at most 8 (without abelian factors) which admit at least one Anosov rational form are $\mathfrak{h}_3 \oplus \mathfrak{h}_3$, $\mathfrak{f}_3$, $\mathfrak{g}$, $\mathfrak{h}_3 \oplus \mathfrak{h}_5$, $\mathfrak{h}$ and $\mathfrak{l}_4 \oplus \mathfrak{l}_4$ (see Table 1). This is a really short list if we bear in mind that there exist several continuous families and hundreds of isolated examples of 7 and 8-dimensional nilpotent Lie algebras. After this, it was crucial to know for each of these Lie algebras, a complete list of all their rational forms up to isomorphism. This information has been obtained in the present paper (see Table 2) and is part of the classification of nilmanifolds admitting an Anosov diffeomorphism in dimension at most 8.

We recall that a *rational form* of $\mathfrak{n}$ is a rational subspace $\mathfrak{n}^{\mathbb{Q}}$ of $\mathfrak{n}$ such that $\mathfrak{n}^{\mathbb{Q}} \otimes \mathbb{R} = \mathfrak{n}$ and $[X,Y] \in \mathfrak{n}^{\mathbb{Q}}$ for all $X, Y \in \mathfrak{n}^{\mathbb{Q}}$. Two rational forms $\mathfrak{n}_1^{\mathbb{Q}}$, $\mathfrak{n}_2^{\mathbb{Q}}$ of $\mathfrak{n}$ are said to be *isomorphic* if there exists $A \in \mathrm{Aut}(\mathfrak{n})$ such that $An_1^{\mathbb{Q}} = \mathfrak{n}_2^{\mathbb{Q}}$, or equivalently, if they are isomorphic as Lie algebras over $\mathbb{Q}$. Not every real nilpotent Lie algebra admits a rational form. By a result due to Malcev, the existence of a rational form of $\mathfrak{n}$ is equivalent to the corresponding Lie group $N$ admits a lattice (see [9]). Another difference with the semisimple case is that sometimes $\mathfrak{n}$ has only one rational form up to isomorphism.

For a 2-step nilpotent Lie algebra $\mathfrak{n}$ with 2-dimensional center, Grunewald, Segal and Sterling [4], [5] gave an answer to $(*)$ in terms of isomorphism classes of binary forms. Such a binary form is the *Pfaffian form* of $\mathfrak{n}$, which is a homogeneous polynomial of degree $m$ in $k$ variables attached to any 2-step nilpotent Lie algebra $\mathfrak{n}$ of dimension $2m+k$ and $\dim[\mathfrak{n},\mathfrak{n}]=k$ (see Definition 2.2). The projective equivalence class of this form is an isomorphism invariant of $\mathfrak{n}$ (see also [10]).

In Section 4, we show how one can apply Pfaffian forms (Section 2), the results from [4], [5] and Scheuneman duality (Section 3), to solve problem $(*)$ in some cases. We compute explicitly the set of isomorphism classes of rational forms for the 2-step nilpotent Lie algebras $\mathfrak{h}_3\oplus\mathfrak{h}_3$, $\mathfrak{g}$, $\mathfrak{h}_3\oplus\mathfrak{h}_5$ and $\mathfrak{h}$. We also consider in Section 5 the 3-step nilpotent algebra $\mathfrak{l}_4\oplus\mathfrak{l}_4$, for which the above techniques do not apply.

## 2. Pfaffian form

Let $\mathfrak{n}$ be a Lie algebra over the field $K$, which is assumed from now on to be of characteristic zero. We are mainly interested in the cases $K=\mathbb{C},\mathbb{R},\mathbb{Q}$. Fix a non-degenerate symmetric $K$-bilinear form $\langle\cdot,\cdot\rangle$ on $\mathfrak{n}$ (i.e. an inner product). For each $Z\in\mathfrak{n}$ consider the $K$-linear transformation $J_Z:\mathfrak{n}\longrightarrow\mathfrak{n}$ defined by

$$\langle J_ZX,Y\rangle=\langle[X,Y],Z\rangle,\qquad\forall\,X,Y\in\mathfrak{n}.\tag{1}$$

Recall that $J_Z$ is skew symmetric with respect to $\langle\cdot,\cdot\rangle$ and the map $J:\mathfrak{n}\longrightarrow\mathfrak{so}(n,K)$ is $K$-linear, where $n$ is the dimension of $\mathfrak{n}$. Equivalently, we may define these maps by fixing a basis $\beta=\{X_1,\dots,X_n\}$ of $\mathfrak{n}$ rather than an inner product in the following way: $J_Z$ is the $K$-linear transformation whose matrix in terms of $\beta$ has entry $ij$ given by

$$\sum_{k=1}^{n}c_{ij}^{k}x_k,\qquad\text{where}\quad[X_i,X_j]=\sum_{k=1}^{n}c_{ij}^{k}X_k,\quad Z=\sum_{k=1}^{n}x_kX_k.$$

It is easy to see that this definition coincides with the first one if we let $\langle X_i,X_j\rangle=\delta_{ij}$.

If $\mathfrak{n}$ and $\mathfrak{n}'$ are two Lie algebras over $K$ and $J$, $J'$ are the corresponding maps, relative to the inner products $\langle\cdot,\cdot\rangle$ and $\langle\cdot,\cdot\rangle'$ respectively, then it is easy to see that a linear map $A:\mathfrak{n}\longrightarrow\mathfrak{n}'$ is a Lie algebra isomorphism if and only if

$$A^tJ'_ZA=J_{A^tZ},\qquad\forall\,Z\in\mathfrak{n}',\tag{2}$$

where $A^t:\mathfrak{n}'\longrightarrow\mathfrak{n}$ is given by $\langle A^tX,Y\rangle=\langle X,AY\rangle'$ for all $X\in\mathfrak{n}'$, $Y\in\mathfrak{n}$.

*Definition 2.1.* Consider the central descendent series of $\mathfrak{n}$ defined by $C^0(\mathfrak{n})=\mathfrak{n}$, $C^i(\mathfrak{n})=[\mathfrak{n},C^{i-1}(\mathfrak{n})]$. When $C^r(\mathfrak{n})=0$ and $C^{r-1}(\mathfrak{n})\neq0$, $\mathfrak{n}$ is said to be $r$-step nilpotent, and we denote by $(n_1,\dots,n_r)$ the *type* of $\mathfrak{n}$, where

$$n_i=\dim C^{i-1}(\mathfrak{n})/C^i(\mathfrak{n}).$$

We also take a decomposition $\mathfrak{n}=\mathfrak{n}_1\oplus\cdots\oplus\mathfrak{n}_r$, a direct sum of vector spaces, such that $C^i(\mathfrak{n})=\mathfrak{n}_{i+1}\oplus\cdots\oplus\mathfrak{n}_r$ for all $i$.

Assume now that $\mathfrak{n}$ is 2-step nilpotent, or equivalently of type $(n_1,n_2)$. Consider any direct sum decomposition of the form $\mathfrak{n}=V\oplus[\mathfrak{n},\mathfrak{n}]$, that is,

$\mathfrak{n}_1 = V$. If the inner product satisfies $\langle V, [\mathfrak{n},\mathfrak{n}]\rangle = 0$ then $V$ is $J_Z$-invariant for any $Z$ and $J_Z = 0$ if and only if $Z \in V$. We define $f : [\mathfrak{n},\mathfrak{n}] \longrightarrow K$ by

$$f(Z) = \mathrm{Pf}(J_Z|_V), \quad Z \in [\mathfrak{n},\mathfrak{n}],$$

where $\mathrm{Pf} : \mathfrak{so}(V,K) \longrightarrow K$ is the *Pfaffian*, that is, the only polynomial function satisfying $\mathrm{Pf}(B)^2 = \det B$ for all $B \in \mathfrak{so}(V,K)$ and $\mathrm{Pf}(J) = 1$ for

$$J = \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix}.$$

Roughly speaking, $f(Z) = (\det J_Z|_V)^{\frac{1}{2}}$, and so we need $\dim V$ to be even in order to get $f \neq 0$. For any $A \in \mathfrak{gl}(V,K)$, $B \in \mathfrak{so}(V,K)$ we have that $\mathrm{Pf}(ABA^t) = (\det A)\mathrm{Pf}(B)$.

*Definition 2.2.* We call $f$ the *Pfaffian form* of the 2-step nilpotent Lie algebra $\mathfrak{n}$.

If $\dim V = 2m$ and $\dim[\mathfrak{n},\mathfrak{n}] = k$ then $f = f(x_1,\dots,x_k)$ is a homogeneous polynomial of degree $m$ in $k$ variables with coefficients in $K$, where $Z = \sum_{i=1}^{k} x_i Z_i$ and $\{Z_1,\dots,Z_k\}$ is a fixed basis of $[\mathfrak{n},\mathfrak{n}]$. $f$ is also called a form of degree $m$, when $k = 2$ or 3 one uses the words binary or ternary and for $m = 2$ and 3, quadratic and cubic, respectively.

Let $P_{k,m}(K)$ denote the set of all homogeneous polynomials of degree $m$ in $k$ variables with coefficients in $K$. The group $\mathrm{GL}_k(K)$ acts naturally on $P_{k,m}(K)$ by

$$(A \cdot f)(x_1,\dots,x_k) = f(A^{-1}(x_1,\dots,x_k)),$$

that is, by linear substitution of variables, and thus the action determines the usual equivalence relation between forms, denoted by $f \simeq g$. In the present paper, we need to consider the following wider equivalence relation.

*Definition 2.3.* For $f, g \in P_{k,m}(K)$, we say that $f$ is *projectively equivalent* to $g$, and denote it by $f \simeq_K g$, if there exists $A \in \mathrm{GL}_k(K)$ and $c \in K^*$ such that

$$f(x_1,\dots,x_k) = cg(A(x_1,\dots,x_k)).$$

In other words, we are interested in projective equivalence classes of forms.

**Proposition 2.4.** *Let* $\mathfrak{n}, \mathfrak{n}'$ *be two-step nilpotent Lie algebras over the field* $K$*. If* $\mathfrak{n}$ *and* $\mathfrak{n}'$ *are isomorphic then* $f \simeq_K f'$*, where* $f$ *and* $f'$ *are the Pfaffian forms of* $\mathfrak{n}$ *and* $\mathfrak{n}'$*, respectively.*

*Proof.* Since $\mathfrak{n}$ and $\mathfrak{n}'$ are isomorphic we can assume that $\mathfrak{n} = \mathfrak{n}'$ and $[\mathfrak{n},\mathfrak{n}] = [\mathfrak{n}',\mathfrak{n}']$ as vector spaces, and then the decomposition $\mathfrak{n} = V \oplus [\mathfrak{n},\mathfrak{n}]$ is valid for both Lie brackets $[\,,\,]$ and $[\,,\,]'$. Any isomorphism satisfies $A[\mathfrak{n},\mathfrak{n}] = [\mathfrak{n}',\mathfrak{n}']'$, and it is easy to see that there is always an isomorphism $A$ between them satisfying $AV = V$. It follows from (2) that

$$A^t J'_Z A = J_{A^t Z}, \quad \forall\, Z \in [\mathfrak{n},\mathfrak{n}],$$

and since the subspaces $V$ and $[\mathfrak{n},\mathfrak{n}]$ are preserved by $A$ and $A^t$ we have that

$$f'(Z) = cf(A_2^t Z),$$

where $A_2 = A|_{[\mathfrak{n},\mathfrak{n}]}$ and $c^{-1} = \det A|_V$. This shows that $f \simeq_K f'$. □

The above proposition says that the projective equivalence class of the form $f(x_1,\dots,x_k)$ is an isomorphism invariant of the Lie algebra $\mathfrak{n}$. We note that this invariant was actually introduced by J. Scheuneman in [10], from a different point of view.

What is known about the classification of forms? Unfortunately, much less than one could naively expect. The case $K=\mathbb{C}$ is as usual the most developed one, and in such a case the understanding of the ring of invariant polynomials $\mathbb{C}[P_{k,m}]^{\mathrm{SL}_k(\mathbb{C})}$ is crucial. A set of generators and their relations for such a ring is known only for small values of $k$ and $m$, for instance for $k=2$ and $m\leqslant 8$, or $k=3$ and $m\leqslant 3$. We refer to [1] and the references therein for several explicit classification results.

The following result, which is easy to prove, will help us to distinguish between projective equivalence classes of forms, and in view of Proposition 2.4, to recognize non-isomorphic two-step nilpotent Lie algebras.

**Proposition 2.5.** *If $f,g\in P_{k,m}(K)$ satisfy*

$$f(x_1,\dots,x_k)=cg(A(x_1,\dots,x_k))$$

*for some $A\in \mathrm{GL}_k(K)$ and $c\in K^*$, then*

$$Hf(x_1,\dots,x_k)=c^k(\det A)^2Hg(A(x_1,\dots,x_k)),$$

*where the Hessian $Hf$ of the form $f$ is defined by*

$$Hf(x_1,\dots,x_k)=\det\left[\frac{\partial^2 f}{\partial x_i\partial x_j}\right].$$

Pfaffian forms are also a very useful tool to study Anosov Lie algebras.

**Proposition 2.6** [8]. *Let $\mathfrak{n}$ be a real 2-step nilpotent Lie algebra with* $\dim[\mathfrak{n},\mathfrak{n}]=k$ *and admitting a rational form $\mathfrak{n}^{\mathbb{Q}}$ which is Anosov. Then, if $f$ is the Pfaffian form of $\mathfrak{n}^{\mathbb{Q}}$, for any $p\in\mathbb{Z}$ the set*

$$S_p=\{(x_1,\dots,x_k)\in\mathbb{Z}^k : f(x_1,\dots,x_k)=p\}$$

*is either empty or infinite.*

The first non-abelian example of an Anosov Lie algebra, due to Borel (see [13]), is a rational form of $\mathfrak{h}_3\oplus\mathfrak{h}_3$. Rational Lie algebras of type (4,2) are parametrized by the set of square free numbers $k\in\mathbb{Z}$ and their Pfaffian forms are $f_k(x,y)=x^2-ky^2$ (see paragraph before Proposition 3.2). Thus the set of solutions

$$\{(x,y)\in\mathbb{Z}^2 : f_k(x,y)=1\}$$

is infinite if and only if $k>1$ or $k=0$ (Pell equation). By Proposition 2.6, the Lie algebra $\mathfrak{n}_k^{\mathbb{Q}}$ can never be Anosov for $k<0$ or $k=1$. This implies that the rational form $\mathfrak{n}_1^{\mathbb{Q}}$ of $\mathfrak{h}_3\oplus\mathfrak{h}_3$ is not Anosov and also that nilmanifolds covered by the nilpotent Lie group with Lie algebra $\mathfrak{h}_3^{\mathbb{C}}$ can never admit Anosov automorphisms.

## 3. Rational forms

Let $\mathfrak{n}$ be a nilpotent Lie algebra over $\mathbb{R}$ of dimension $n$.

*Definition 3.1.* A *rational form* of $\mathfrak{n}$ is an $n$-dimensional rational subspace $\mathfrak{n}^{\mathbb{Q}}$ of $\mathfrak{n}$ such that

$$[X,Y]\in\mathfrak{n}^{\mathbb{Q}},\quad \forall\, X,Y\in\mathfrak{n}^{\mathbb{Q}}.$$

Two rational forms $\mathfrak{n}_1^{\mathbb{Q}}$, $\mathfrak{n}_2^{\mathbb{Q}}$ of $\mathfrak{n}$ are said to be *isomorphic* if there exists $A\in\mathrm{Aut}(\mathfrak{n})$ such that $A\mathfrak{n}_1^{\mathbb{Q}}=\mathfrak{n}_2^{\mathbb{Q}}$, or equivalently, if they are isomorphic as Lie algebras over $\mathbb{Q}$ (recall that $\mathfrak{n}^{\mathbb{Q}}\otimes\mathbb{R}=\mathfrak{n}$). In an analogous way, by considering $\mathbb{R}$ and $\mathbb{C}$ (resp. $\mathbb{Q}$ and $\mathbb{C}$) instead of $\mathbb{Q}$ and $\mathbb{R}$, one defines a *real form* (resp. a *rational form*) of a complex Lie algebra.

The problem of finding all isomorphism classes of rational forms for a given real nilpotent Lie algebra is a very difficult one, even in the low dimensional or two-step cases. Very little is known about this problem in the literature (see [2, Section 5] and [11]).

We now give a first example on how to use Pfaffian forms to study rational forms of 2-step nilpotent Lie algebras. Let $\mathfrak{n}^{\mathbb{Q}}$ be a rational nilpotent Lie algebra of type (4, 2). If $\mathfrak{n}^{\mathbb{Q}}=\mathfrak{n}_1\oplus\mathfrak{n}_2$ is the decomposition such that $\dim\mathfrak{n}_1=4$, $\dim\mathfrak{n}_2=2$ and $[\mathfrak{n}^{\mathbb{Q}},\mathfrak{n}^{\mathbb{Q}}]=\mathfrak{n}_2$, then we consider the Pfaffian form $f$ of $\mathfrak{n}^{\mathbb{Q}}$. Thus $f$ is a binary quadratic form, say $f(x,y)=ax^2+bxy+cy^2$, with $a,b,c\in\mathbb{Q}$. It is proved in [4] that the converse of Proposition 2.4 is valid in this case, that is, there is a one-to-one correspondence between isomorphism classes of non-degenerate (i.e. with center equal to $\mathfrak{n}_2$) rational Lie algebras of type (4,2) and projective equivalence classes of binary quadratic forms with coefficients in $\mathbb{Q}$. It is well known that these last classes can be parametrized by

$$\{f_k(x,y)=x^2-ky^2 : k \text{ is a square free integer number}\}.$$

Recall that an integer number is called *square free* if $p^2\nmid k$ for any prime $p$. The set of all square free numbers parametrizes the equivalence classes of the relation in $\mathbb{Q}$ defined by $r\equiv s$ if and only if $r=q^2s$ for some $q\in\mathbb{Q}^*$. We are considering $k=0$ a square free number too. If $f_k\simeq_K f_{k'}$ then it follows from Proposition 2.5 that $-4k=-4q^2k'$ for some $q\in\mathbb{Q}^*$, which implies that $k=k'$ if $k$ and $k'$ are square free.

It is easy to check that the Pfaffian form of the Lie algebra $\mathfrak{n}_k^{\mathbb{Q}}=\mathfrak{n}_1\oplus\mathfrak{n}_2$ defined by

$$[X_1,X_3]=Z_1,\quad [X_1,X_4]=Z_2,\quad [X_2,X_3]=kZ_2,\quad [X_2,X_4]=Z_1 \tag{3}$$

is $f_k$. For $K=\mathbb{R}$, these Lie algebras can be distinguished only by the sign of the discriminant of $f_k$, which says that there are only three real Lie algebras of type (4, 2), namely, those of the form $\mathfrak{n}_k^{\mathbb{Q}}\otimes\mathbb{R}$ with $k>0$, $k=0$ and $k<0$, respectively. It is easy to check that $\mathfrak{n}_1^{\mathbb{Q}}\otimes\mathbb{R}\simeq\mathfrak{h}_3\oplus\mathfrak{h}_3$, where $\mathfrak{h}_3$ denotes the real 3-dimensional Heisenberg algebra and $\mathfrak{n}_{-1}^{\mathbb{Q}}\otimes\mathbb{R}\simeq\mathfrak{h}_3^{\mathbb{C}}$, the complex 3-dimensional Heisenberg algebra viewed as real. On the other hand, we obtain that there are only two complexifications $\mathfrak{n}_k^{\mathbb{Q}}\otimes\mathbb{C}$, those with $k\neq 0$ and the one with $k=0$.

**Proposition 3.2.** *The set of isomorphism classes of rational forms of the Lie algebras* $\mathfrak{h}_3 \oplus \mathfrak{h}_3$, $\mathfrak{h}_3^{\mathbb{C}}$ *and* $\mathfrak{n}_0^{\mathbb{Q}} \otimes \mathbb{R}$ *is respectively parametrized by*

$$\{\mathfrak{n}_k^{\mathbb{Q}} : k > 0 \, \textit{is square free}\}, \qquad \{\mathfrak{n}_{-k}^{\mathbb{Q}} : k > 0 \, \textit{is square free}\}, \qquad \{\mathfrak{n}_0^{\mathbb{Q}}\}.$$

*Proof.* The Lie bracket of $\mathfrak{h}_3 \oplus \mathfrak{h}_3$ is

$$[X_1, X_2] = Z_1, \qquad [X_3, X_4] = Z_2,$$

and one can easily check that the rational subspace generated by the set

$$\{X_1 + X_3,\ \sqrt{k}(X_1 - X_3),\ \sqrt{k}(X_2 + X_4),\ X_2 - X_4,\ \sqrt{k}(Z_1 + Z_2),\ Z_1 - Z_2\},$$

is a rational subalgebra of $\mathfrak{h}_3 \oplus \mathfrak{h}_3$ isomorphic to $\mathfrak{n}_k^{\mathbb{Q}}$. For $\mathfrak{h}_3^{\mathbb{C}}$, we argue in an analogous way by using $\{\sqrt{-k}X_1, X_2, X_3, \sqrt{-k}X_4, \sqrt{-k}Z_1, -kZ_2\}$. □

We now describe the results in [5] for the general case (see also [3]). Consider $\mathfrak{n} = \mathfrak{n}_1 \oplus \mathfrak{n}_2$ a vector space over $K$ such that $\mathfrak{n}_1$ and $\mathfrak{n}_2$ are subspaces of dimension $n$ and 2 respectively. Every 2-step nilpotent Lie algebra of dimension $n + 2$ with a 2-dimensional center can be represented by a bilinear form $\mu : \mathfrak{n}_1 \times \mathfrak{n}_1 \longrightarrow \mathfrak{n}_2$ which is non-degenerate in the following way: for any nonzero $X \in \mathfrak{n}_1$ there exists $Y \in \mathfrak{n}_1$ such that $\mu(X, Y) \neq 0$. If we fix bases $\{X_1, \dots, X_n\}$ and $\{Z_1, Z_2\}$ of $\mathfrak{n}_1$ and $\mathfrak{n}_2$ respectively, then each $\mu$ has an associated Pfaffian binary form $f_\mu$ defined by

$$f_\mu(x, y) = \operatorname{Pf}(J^\mu_{xZ_1 + yZ_2})$$

(see Definition 2.2). A *central decomposition* of $\mu$ is given by a decomposition of $\mathfrak{n}_1$ in a direct sum of subspaces $\mathfrak{n}_1 = V_1 \oplus \cdots \oplus V_r$ such that $\mu(V_i, V_j) = 0$ for all $i \neq j$. We say that $\mu$ is *indecomposable* when the only possible central decomposition has $r = 1$. Every $\mu$ has a central decomposition into indecomposables constituents and such a decomposition is unique up to an automorphism of $\mu$; in particular, the constituents $V_i \oplus \mathfrak{n}_2$ are unique up to isomorphism.

There is only one indecomposable $\mu$ for $n$ odd and it can be defined by

$$J^\mu_{xZ_1 + yZ_2} = \left[\begin{array}{c|c} 0 & \begin{matrix} -x & -y & & 0 \\ 0 & -x & -y & \\ & & \ddots & \ddots \\ 0 & & -x & -y \end{matrix} \\ \hline \begin{matrix} x & 0 & & 0 \\ y & x & & \\ 0 & y & \ddots & \\ & & \ddots & x \\ 0 & & & y \end{matrix} & 0 \end{array}\right].$$

Recall that $f_\mu = 0$ in this case. When $n$ is even the situation is much more abundant: two indecomposables $\mu$ and $\lambda$ are isomorphic if and only if $f_\mu \simeq_K f_\lambda$. If $n = 2m$ and $f_\mu(x, y) = x^m - a_1 x^{m-1} y - \cdots - a_m y^m$, then

$$J^\mu_{xZ_1 + yZ_2} = \begin{bmatrix} 0 & -B^t \\ B & 0 \end{bmatrix},$$

where

$$B = \begin{bmatrix} x & y & 0 & \cdots & 0 \\ 0 & x & y & & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & x & y \\ a_m y & a_{m-1} y & \cdots & a_2 y & a_1 y + x \end{bmatrix}.$$

We note that here $f_\mu$ is always nonzero, and in order to get $\mu$ indecomposable one needs the form $f_\mu$ to be primitive (i.e. a power of an irreducible one). For decomposable $\mu$ and $\lambda$ with respective central decompositions $\mathfrak{n}_1 = V_1 \oplus \cdots \oplus V_r$ and $\mathfrak{n}_1 = W_1 \oplus \cdots \oplus W_s$ into indecomposables constituents, we have that $\mu$ is isomorphic to $\lambda$ if and only if $r = s$ and after a suitable reordering one has that

(i) for some $t \leqslant r$, $\dim V_i = \dim W_i$ for all $i = 1, \dots, t$ and they are all even numbers;

(ii) if $\mu_i = \mu|_{V_i \times V_i}$, $\lambda_i = \lambda|_{W_i \times W_i}$ then there exist $A \in \mathrm{GL}_2(K)$ and $c_1, \dots, c_t \in K^*$ such that

$$f_{\mu_i}(x,y) = c_i f_{\lambda_i}(A(x,y)) \quad \forall\ i = 1, \dots, t;$$

(iii) $\dim V_i = \dim W_i$ is odd for all $i = t+1, \dots, r$.

Concerning our search for all rational forms up to isomorphism of a given real nilpotent Lie algebra, these results say that the picture in the 2-step nilpotent with 2-dimensional center case is as follows. Let $(\mathfrak{n}^{\mathbb{Q}} = \mathfrak{n}_1 \oplus \mathfrak{n}_2, \mu)$ be one of such Lie algebras over $\mathbb{Q}$, and consider the corresponding Pfaffian form $f_\mu \in P_{2,m}(\mathbb{Q})$. The isomorphism classes of rational forms of $\mathfrak{n}^{\mathbb{Q}} \otimes \mathbb{R}$ are then parametrized by

$$((\mathbb{R}^* \times \mathrm{GL}_2(\mathbb{R})) \cdot f_\mu \cap P_{2,m}(\mathbb{Q}))/(\mathbb{Q}^* \times \mathrm{GL}_2(\mathbb{Q})). \tag{4}$$

In other words, the rational points of the orbit $(\mathbb{R}^* \times \mathrm{GL}_2(\mathbb{R})) \cdot f_\mu$ ($f_\mu$ viewed as an element of $P_{2,m}(\mathbb{R})$) is a $(\mathbb{Q}^* \times \mathrm{GL}_2(\mathbb{Q}))$-invariant set and we have to consider the orbit space for this action. Such a description shows the high difficulty of the problem. Recall that we have to consider the action of $\mathbb{R}^* \times \mathrm{GL}_2(\mathbb{R})$ instead of just that of $\mathrm{GL}_2(\mathbb{R})$ only when $m$ is even.

We now describe a duality for 2-step nilpotent Lie algebras over any field of characteristic zero introduced by J. Scheuneman [10] (see also [3] and [4, Section 8]), which assigns to each Lie algebra of type $(n,k)$ another one of type $\left(n, \frac{n(n-1)}{2} - k\right)$. The *dual* of a Lie algebra $\mathfrak{n} = \mathfrak{n}_1 \oplus \mathfrak{n}_2$ of type $(n,k)$ can be defined as follows: consider the maps $\{J_Z : Z \in \mathfrak{n}_2\} \subset \mathfrak{so}(n)$ corresponding to a fixed inner product $\langle \cdot, \cdot \rangle$ on $\mathfrak{n}$ (see (1)). Let $\tilde{\mathfrak{n}}_2 \subset \mathfrak{so}(n)$ be the orthogonal complement of the $k$-dimensional subspace $\{J_Z : Z \in \mathfrak{n}_2\}$ in $\mathfrak{so}(n)$ relative to the inner product $(A,B) = -\mathrm{tr}\, AB$. Now, we define the 2-step nilpotent Lie algebra $\tilde{\mathfrak{n}} = \mathfrak{n}_1 \oplus \tilde{\mathfrak{n}}_2$ whose Lie bracket is determined by

$$([X,Y],Z) = \langle Z(X), Y \rangle, \quad Z \in \tilde{\mathfrak{n}}_2.$$

In other words, the maps $\widetilde{J}_Z$'s for this Lie algebra are the $Z$'s themselves. Recall that $\dim \tilde{\mathfrak{n}}_2 = \frac{n(n-1)}{2} - k$, and so the dual $\tilde{\mathfrak{n}}$ of $\mathfrak{n}$ is of type $\left(n, \frac{n(n-1)}{2} - k\right)$. It is proved in [10] that $\mathfrak{n}_1$ is isomorphic to $\mathfrak{n}_2$ if and only if $\tilde{\mathfrak{n}}_1$ is isomorphic to $\tilde{\mathfrak{n}}_2$, so that any classification for algebras of type $(n,k)$ simultaneously determines the algebras of type $\left(n,\frac{n(n-1)}{2}-k\right)$.

## 4. Applications

In this section, we determine the set of all rational forms up to isomorphism for some 2-step 8-dimensional nilpotent Lie algebras, as an application of the results described in Sections 2 and 3. We refer to Tables 1 and 2 for a summary of the results obtained.

Let $\mathfrak{g}$ be the 8-dimensional 2-step nilpotent Lie algebra of type (6, 2) defined by

$$[X_1,X_2]=Z_1,\quad [X_1,X_3]=Z_2,\quad [X_4,X_5]=Z_1,\quad [X_4,X_6]=Z_2. \tag{5}$$

It is easy to see that its Pfaffian form $f$ is zero. Let $\mathfrak{g}^{\mathbb{Q}}$ be a rational form of $\mathfrak{g}$, for which we can assume that $\mathfrak{g}^{\mathbb{Q}}=\langle X_1,\dots,X_6\rangle_{\mathbb{Q}}\oplus\langle Z_1,Z_2\rangle_{\mathbb{Q}}$. Since the Pfaffian form $g$ of $\mathfrak{g}^{\mathbb{Q}}$ satisfies $g\simeq_{\mathbb{R}}f=0$ we obtain that $g=0$. It follows that $\mathfrak{g}^{\mathbb{Q}}$ can not be indecomposable, and so $\langle X_1,\dots,X_6\rangle_{\mathbb{Q}}=V_1\oplus\cdots\oplus V_r$ with $[V_i,V_j]=0$ for all $i\neq j$. Now, $\langle X_1,\dots,X_6\rangle_{\mathbb{R}}=V_1\otimes\mathbb{R}\oplus\cdots\oplus V_r\otimes\mathbb{R}$ is also a central decomposition for $\mathfrak{g}$, proving that $r=2$ and $\dim V_1=\dim V_2=3$ by the uniqueness of such a decomposition. But 3 is odd, and hence we obtain the following result.

**Proposition 4.1.** *The Lie algebra* $\mathfrak{g}$ *of type* (6, 2) *given in* (5) *has only one rational form up to isomorphism, denoted by* $\mathfrak{g}^{\mathbb{Q}}$.

*Remark 4.2.* Clearly, the same proof is valid if one need to find all real forms of the complex Lie algebra $\mathfrak{g}_{\mathbb{C}}=\mathfrak{g}\otimes\mathbb{C}$. Thus $\mathfrak{g}$ is the only real form of $\mathfrak{g}_{\mathbb{C}}$ up to isomorphism.

As another application of the correspondence with binary forms given above, we now study rational forms of the real Lie algebra $\mathfrak{h}_3\oplus\mathfrak{h}_5$ of type (6, 2). It has central decomposition $\mathfrak{n}_1=V_1\oplus V_2\oplus V_3$ with $\dim V_i=2$ for all $i$ as a real Lie algebra and its Pfaffian form is $f(x,y)=xy^2$. Let $\mu:\mathfrak{n}_1\times\mathfrak{n}_1\longrightarrow\mathfrak{n}_2$ be a rational form of $\mathfrak{h}_3\oplus\mathfrak{h}_5$ with Pfaffian form $f_\mu$. If $\mu$ is decomposable then $\mathfrak{n}_1=W_1\oplus W_2$, $\dim W_1=2$, $\dim W_2=4$; or $\mathfrak{n}_1=W_1\oplus W_2\oplus W_3$, $\dim W_i=2$ for all $i$. In any case, $f_{\mu_i}\simeq_{\mathbb{Q}}x,y$ or $y^2$ proving that $\mu$ must be isomorphic to the canonical rational form

$$\mu_0(X_1,X_2)=Z_1,\quad \mu_0(X_3,X_4)=Z_2,\quad \mu_0(X_5,X_6)=Z_2,$$

for which $f_{\mu_0}=f$. We then assume that $\mu$ is indecomposable. We shall prove that there is only one $\mathrm{GL}_2(\mathbb{Q})$-orbit of rational points in $\mathrm{GL}_2(\mathbb{R})\cdot f$, and so $\mu$ will have

**Table 1.** Notation for some real nilpotent Lie algebras

| Notation | Type | Lie brackets |
|---|---|---|
| $\mathfrak{h}_{2k+1}$ | $(2k,1)$ | $[x_1,x_2]=z_1,\dots,[x_{2k-1},x_{2k}]=z_1$ |
| $\mathfrak{f}_3$ | $(3,3)$ | $[x_1,x_2]=z_1,\ [x_1,x_3]=z_2,\ [x_2,x_3]=z_3$ |
| $\mathfrak{g}$ | $(6,2)$ | $[x_1,x_2]=z_1,\ [x_1,x_3]=z_2,\ [x_4,x_5]=z_1,\ [x_4,x_6]=z_2$ |
| $\mathfrak{h}$ | $(4,4)$ | $[x_1,x_3]=z_1,\ [x_1,x_4]=z_2,\ [x_2,x_3]=z_3,\ [x_2,x_4]=z_4$ |
| $\mathfrak{l}_4$ | $(2,1,1)$ | $[x_1,x_2]=x_3,\ [x_1,x_3]=x_4$ |

to be isomorphic to $\mu_0$. There exists $A \in \mathrm{GL}_2(\mathbb{R})$ such that $f_\mu = A^{-1} \cdot f$, that is,

$$f_\mu(x,y) = ac^2x^3 + c(2ad+bc)x^2y + d(ad+2bc)xy^2 + bd^2y^3, \qquad A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

Since $\mu$ is rational we have that

$$q := ac^2, \qquad r := c(2ad+bc), \qquad s := d(ad+2bc), \qquad t := bd^2$$

are all in $\mathbb{Q}$. If $c=0$ then $q=r=0$ and $s=ad^2$, $t=bd^2$, which implies that $s\neq 0$ and hence

$$f_\mu = B^{-1}\cdot f, \quad \text{for} \quad B = \begin{bmatrix} s & t \\ 0 & 1 \end{bmatrix} \in \mathrm{GL}_2(\mathbb{Q}).$$

If $c \neq 0$ then one can check by a straightforward computation that

$$\frac{d}{c} = \frac{9qst + rs^2 - 6r^2t}{6qs^2 - r^2s - 9qrt} \in \mathbb{Q}.$$

There must be a simpler formula for $\frac{d}{c}$ in terms of $q,r,s,t$, but unfortunately we were not able to find it. By putting $u := \frac{d}{c}$ we have that

$$f_\mu = B^{-1}\cdot f, \quad \text{for} \quad B = \begin{bmatrix} q & t/u^2 \\ 1 & u \end{bmatrix} \in \mathrm{GL}_2(\mathbb{Q}).$$

Recall that $\det B = qu - \frac{t}{u^2} = c(ad-bc) = c\det A \neq 0$. We then obtain that in any case $f_\mu \simeq_{\mathbb{Q}} f$ and so $\mu$ is isomorphic to $\mu_0$.

**Proposition 4.3.** *Up to isomorphism, the real Lie algebra* $\mathfrak{h}_3 \oplus \mathfrak{h}_5$ *of type* (6, 2) *has only one rational form, which will be denoted by* $(\mathfrak{h}_3 \oplus \mathfrak{h}_5)^{\mathbb{Q}}$.

*Remark 4.4.* It is easy to check that the above proof is also valid if we replace $\mathbb{Q}$ and $\mathbb{R}$ by $\mathbb{R}$ and $\mathbb{C}$, obtaining in this way that the only real form of $(\mathfrak{h}_3 \oplus \mathfrak{h}_5)_{\mathbb{C}}$ is $\mathfrak{h}_3 \oplus \mathfrak{h}_5$.

Let $\mathfrak{h}$ be the Lie algebra of type (4, 4) which is dual to $\mathfrak{h}_3 \oplus \mathfrak{h}_3$ (of type (4, 2)). The Lie bracket of $\mathfrak{h}_3 \oplus \mathfrak{h}_3$ is

$$[X_1, X_2] = Z_1, \qquad [X_3, X_4] = Z_2,$$

and hence

$$J_{Z_1} = \begin{bmatrix} 0 & -1 & & \\ 1 & 0 & & \\ & & 0 & 0 \\ & & 0 & 0 \end{bmatrix}, \qquad J_{Z_2} = \begin{bmatrix} 0 & 0 & & \\ 0 & 0 & & \\ & & 0 & -1 \\ & & 1 & 0 \end{bmatrix}.$$

The orthogonal complement $\tilde{\mathfrak{n}}_2$ of $\{J_Z : Z \in \mathfrak{n}_2\}$ is then linearly generated by

$$\begin{bmatrix} & & -1 & 0 \\ & & 0 & 0 \\ 1 & 0 & & \\ 0 & 0 & & \end{bmatrix}, \quad \begin{bmatrix} & & 0 & -1 \\ & & 0 & 0 \\ 0 & 0 & & \\ 1 & 0 & & \end{bmatrix}, \quad \begin{bmatrix} & & 0 & 0 \\ & & -1 & 0 \\ 0 & 1 & & \\ 0 & 0 & & \end{bmatrix}, \quad \begin{bmatrix} & & 0 & 0 \\ & & 0 & -1 \\ 0 & 0 & & \\ 0 & 1 & & \end{bmatrix},$$

which determines the Lie bracket for $\mathfrak{h}$ given by

$$[X_1,X_3]=Z_1,\quad [X_1,X_4]=Z_2,\quad [X_2,X_3]=Z_3,\quad [X_2,X_4]=Z_4. \tag{6}$$

Scheuneman duality allows us to find all the rational forms of $\mathfrak{h}$; namely, the dual of the rational form of $\mathfrak{h}_3\oplus\mathfrak{h}_3$, already computed in Proposition 3.2.

**Proposition 4.5.** *For any $k\in\mathbb{Z}$ let $\mathfrak{h}_k^{\mathbb{Q}}$ be the rational Lie algebra of type* (4, 4) *defined by*

$$\begin{aligned} [X_1,X_2]&=Z_1, & [X_2,X_3]&=-Z_3,\\ [X_1,X_3]&=Z_2, & [X_2,X_4]&=-Z_2,\\ [X_1,X_4]&=kZ_3, & [X_3,X_4]&=Z_4. \end{aligned}$$

*Then the set of isomorphism classes of rational forms of the Lie algebra $\mathfrak{h}$ defined in* (6) *is parametrized by*

$$\{\mathfrak{h}_k^{\mathbb{Q}} : k \textit{ is a square free natural number}\}.$$

*Proof.* For the rational form $\mathfrak{n}_k^{\mathbb{Q}}$ of $\mathfrak{h}_3\oplus\mathfrak{h}_3$ (see (3)) we have that

$$J_{Z_1}=\begin{bmatrix} & & -1 & 0\\ & & 0 & -1\\ 1 & 0 & & \\ 0 & 1 & & \end{bmatrix},\qquad J_{Z_2}=\begin{bmatrix} & & 0 & -1\\ & & -k & 0\\ 0 & k & & \\ 1 & 0 & & \end{bmatrix}.$$

A basis of the orthogonal complement of $\langle J_{Z_1},J_{Z_2}\rangle_{\mathbb{Q}}$ is then given by

$$\begin{bmatrix} 0 & -1 & & \\ 1 & 0 & & \\ & & 0 & 0\\ & & 0 & 0 \end{bmatrix},\quad \begin{bmatrix} & & -1 & 0\\ & & 0 & 1\\ 1 & 0 & & \\ 0 & -1 & & \end{bmatrix},\quad \begin{bmatrix} & & 0 & -k\\ & & 1 & 0\\ 0 & -1 & & \\ k & 0 & & \end{bmatrix},\quad \begin{bmatrix} 0 & 0 & & \\ 0 & 0 & & \\ & & 0 & -1\\ & & 1 & 0 \end{bmatrix},$$

which determines the Lie bracket for $\mathfrak{h}_k^{\mathbb{Q}}$. To conclude the proof, one can easily check that the rational subspace generated by

$$\begin{gathered}\{\sqrt{k}(X_1-X_3),X_1+X_3,X_2+X_4,\sqrt{k}(X_2-X_4),\\ 2\sqrt{k}Z_1,\sqrt{k}(Z_2+Z_3),Z_3-Z_2,-2\sqrt{k}Z_4\},\end{gathered}$$

is closed under the Lie bracket of $\mathfrak{h}$ and isomorphic to $\mathfrak{h}_k^{\mathbb{Q}}$. □

An alternative proof of the non-isomorphism between the $\mathfrak{h}_k^{\mathbb{Q}}$'s without using Scheuneman duality may be given as follows: from the form of $J_{Z_1},\dots,J_{Z_4}$ for $\mathfrak{h}_k^{\mathbb{Q}}$ in the above proof it follows that

$$J_{xZ_1+yZ_2+zZ_3+wZ_4}=\begin{bmatrix} 0 & -x & -y & -kz\\ x & 0 & z & y\\ y & -z & 0 & -w\\ kz & -y & w & 0 \end{bmatrix},$$

and so the Pfaffian form of $\mathfrak{h}_k^{\mathbb{Q}}$ is given by $f_k(x,y,z,w)=xw+y^2-kz^2$. Now, if $\mathfrak{h}_k^{\mathbb{Q}}$ is isomorphic to $\mathfrak{h}_{k'}^{\mathbb{Q}}$ then $f_k\simeq_{\mathbb{Q}} f_{k'}$ (see Proposition 2.4), which implies that

**Table 2.** Set of rational forms up to isomorphism for some real nilpotent Lie algebras. In all cases $k$ runs over all square-free natural numbers

| Real Lie algebra | Type | Rational forms | Reference |
|---|---|---|---|
| $\mathfrak{h}_3 \oplus \mathfrak{h}_3$ | $(4,2)$ | $\mathfrak{n}_k^{\mathbb{Q}},\ k \geqslant 1$ | Prop. 3.2 |
| $\mathfrak{f}_3$ | $(3,3)$ | $\mathfrak{f}_3^{\mathbb{Q}}$ | – |
| $\mathfrak{g}$ | $(6,2)$ | $\mathfrak{g}^{\mathbb{Q}}$ | Prop. 4.1 |
| $\mathfrak{h}_3 \oplus \mathfrak{h}_5$ | $(6,2)$ | $(\mathfrak{h}_3 \oplus \mathfrak{h}_5)^{\mathbb{Q}}$ | Prop. 4.3 |
| $\mathfrak{h}$ | $(4,4)$ | $\mathfrak{h}_k^{\mathbb{Q}},\ k \geqslant 1$ | Prop. 4.5 |
| $\mathfrak{l}_4 \oplus \mathfrak{l}_4$ | $(4,2,2)$ | $\mathfrak{l}_k^{\mathbb{Q}},\ k \geqslant 1$ | Prop. 5.1 |

$k = q^2k'$ for some $q \in \mathbb{Q}^*$ by applying Proposition 2.5 (recall that $Hf_k = 4k$). Thus $k = k'$ since they are square free.

## 5. A 3-step nilpotent case

We compute in this section the rational forms of $\mathfrak{l}_4 \oplus \mathfrak{l}_4$, where $\mathfrak{l}_4$ is the 4-dimensional real Lie algebra with Lie bracket

$$[Y_1, Y_2] = Y_3, \qquad [Y_1, Y_3] = Y_4.$$

Notice that $\mathfrak{l}_4 \oplus \mathfrak{l}_4$ is 3-step nilpotent, and therefore Pfaffian forms and duality can not be used as tools to distinguish or classify rational forms. For each $k \in \mathbb{Z}$, consider the 8-dimensional rational nilpotent Lie algebra $\mathfrak{l}_k^{\mathbb{Q}}$ with basis $\{X_1, X_2, X_3, X_4, Z_1, Z_2, Z_3, Z_4\}$ and Lie bracket defined by

$$\begin{aligned} &[X_1, X_3] = Z_1, \qquad &&[X_2, X_3] = Z_2,\\ &[X_1, X_4] = Z_2, &&[X_2, X_4] = kZ_1,\\ &[X_1, Z_1] = Z_3, &&[X_2, Z_2] = kZ_3,\\ &[X_1, Z_2] = Z_4, &&[X_2, Z_1] = Z_4. \end{aligned} \tag{7}$$

**Proposition 5.1.** *Let* $\{X_1, X_2, X_3, X_4, Z_1, Z_2, Z_3, Z_4\}$ *be a basis of the Lie algebra* $\mathfrak{l}_4 \oplus \mathfrak{l}_4$ *of type* (4, 2, 2) *with structure coefficients*

$$\begin{aligned} &[X_1, X_3] = Z_1, \quad &&[X_2, X_4] = Z_2,\\ &[X_1, Z_1] = Z_3, &&[X_2, Z_2] = Z_4. \end{aligned}$$

*For each* $k \in \mathbb{N}$ *the rational subspace generated by the set*

$$\{X_1 + X_2, \sqrt{k}(X_1 - X_2), X_3 + X_4, \sqrt{k}(X_3 - X_4),$$
$$Z_1 + Z_2, \sqrt{k}(Z_1 - Z_2), Z_3 + Z_4, \sqrt{k}(Z_3 - Z_4)\}$$

*is a rational form of* $\mathfrak{l}_4 \oplus \mathfrak{l}_4$ *isomorphic to the Lie algebra* $\mathfrak{l}_k^{\mathbb{Q}}$ *defined in* (7). *Moreover, the set*

$$\{\mathfrak{l}_k^{\mathbb{Q}} : k \text{ is a square-free natural number}\}$$

*parametrizes the set all the rational forms of* $\mathfrak{l}_4 \oplus \mathfrak{l}_4$ *up to isomorphism.*

*Proof.* It is easy to see that the Lie brackets of the basis of the rational subspace coincides with the one of $\mathfrak{l}_k^{\mathbb{Q}}$ by renaming the basis as $\{X_1,\dots,Z_4\}$ with the same order. In particular, such a subspace is a rational form of $\mathfrak{l}_4\oplus\mathfrak{l}_4$. If $k'=q^2k$ then one can easily check that $A:\mathfrak{l}_{k'}^{\mathbb{Q}}\longrightarrow\mathfrak{l}_k^{\mathbb{Q}}$ given by the diagonal matrix with entries $(1,q,1,q,1,q,1,q)$ is an isomorphism of Lie algebras.

Conversely, assume that $A:\mathfrak{l}_k^{\mathbb{Q}}\longrightarrow\mathfrak{l}_{k'}^{\mathbb{Q}}$ is an isomorphism. We will show that $k'=q^2k$ for some $q\in\mathbb{Q}^*$. Let $\{J'_Z\}$, $\{J_Z\}$ be the maps defined at the beginning of this section corresponding to $\mathfrak{l}_{k'}^{\mathbb{Q}}$ and $\mathfrak{l}_k^{\mathbb{Q}}$, respectively. If $Z=xZ_1+yZ_2+zZ_3+wZ_4$ we have that

$$J_Z=\left[\begin{array}{cccccccc}
0 & 0 & -x & -y & -z & -w & 0 & 0\\
0 & 0 & -y & -kx & -w & -kz & 0 & 0\\
x & y & 0 & & \cdots & & & 0\\
y & kx & & & & & & \\
z & w & \vdots & & & & \vdots & \vdots\\
w & kz & & & & & & \\
0 & 0 & & & & & & \\
0 & 0 & 0 & & \cdots & & & 0
\end{array}\right],$$

and $J'_Z$ is obtained just by replacing $k$ with $k'$. It follows from (2) that $A^tJ'_ZA=J_{A^tZ}$ for all $Z\in\langle Z_3,Z_4\rangle_{\mathbb{Q}}$, and since this subspace is $A$-invariant we get that the subspace

$$\bigcap_{Z\in\langle Z_3,Z_4\rangle_{\mathbb{Q}}}\operatorname{Ker}J_Z=\bigcap_{Z\in\langle Z_3,Z_4\rangle_{\mathbb{Q}}}\operatorname{Ker}J'_Z=\langle X_3,X_4,Z_3,Z_4\rangle_{\mathbb{Q}}$$

is also $A$-invariant. Thus $A$ has the form

$$A=\begin{bmatrix}
A_1 & 0 & 0 & 0\\
\star & A_2 & 0 & 0\\
\star & 0 & A_3 & 0\\
\star & \star & \star & A_4.
\end{bmatrix}\tag{8}$$

(recall that $C^1(\mathfrak{l}_k^{\mathbb{Q}})=C^1(\mathfrak{l}_{k'}^{\mathbb{Q}})=\langle Z_1,Z_2,Z_3,Z_4\rangle_{\mathbb{Q}}$ and $C^2(\mathfrak{l}_k^{\mathbb{Q}})=C^2(\mathfrak{l}_{k'}^{\mathbb{Q}})=\langle Z_3,Z_4\rangle_{\mathbb{Q}}$ are always $A$-invariant), and now it is easy to prove that

$$A_3^t\begin{bmatrix} z & w\\ w & k'z\end{bmatrix}A_1=\begin{bmatrix} az+bw & cz+dw\\ cz+dw & k'(az+bw)\end{bmatrix},\quad\text{where } A_4^t=\begin{bmatrix} a & bw\\ c & d\end{bmatrix}.$$

We compute the determinant of both sides getting

$$qf'(z,w)=f(A_4^t(z,w)),\quad\forall\ (z,w)\in\mathbb{Q}^2,$$

where $q=\det A_3A_1\in\mathbb{Q}^*$ and $f(z,w)=kz^2-w^2$, $f'(z,w)=k'z^2-w^2$. By Proposition 2.5 we have that

$$4k'=q^{-2}(\det A_4)^2 4k,$$

and so $k=k'$ as long as they are square free numbers, as we wanted to show.

To conclude the proof, it remains to show that these are all the rational forms up to isomorphism. Let $\mathfrak{n}^{\mathbb{Q}}$ be a rational form of $\mathfrak{l}_4 \oplus \mathfrak{l}_4$. Since $\mathfrak{n}^{\mathbb{Q}}/[\mathfrak{n}^{\mathbb{Q}}, [\mathfrak{n}^{\mathbb{Q}}, \mathfrak{n}^{\mathbb{Q}}]]$ is of type (4, 2), we can use the classification of rational Lie algebras of this type given in (3) to get linearly independent vectors $X_1, \dots, Z_2$ such that

$$[X_1, X_3] = Z_1, \quad [X_1, X_4] = Z_2, \quad [X_2, X_3] = Z_2, \quad [X_2, X_4] = kZ_1, \tag{9}$$

where $k$ is a square free integer number. Jacobi condition is equivalent to

$$\begin{aligned} [X_1, Z_2] &= [X_2, Z_1], & [X_3, Z_2] &= [X_4, Z_1], \\ k[X_1, Z_1] &= [X_2, Z_2], & k[X_3, Z_1] &= [X_4, Z_2]. \end{aligned} \tag{10}$$

We will consider the following two cases separately:

(I) $Z_3 := [X_1, Z_1]$ and $Z_4 := [X_1, Z_2]$ are linearly independent,
(II) $[X_1, Z_1], [X_1, Z_2] \in \mathbb{Q}Z_3$ for some nonzero $Z_3 \in \mathfrak{n}^{\mathbb{Q}}$.

In both cases we will make use of the following isomorphism invariant for real 3-step nilpotent Lie algebras:

$$U(\mathfrak{n}) := \{X \in \mathfrak{n}/[\mathfrak{n}, [\mathfrak{n}, \mathfrak{n}]] : \dim \mathrm{Im}(\mathrm{ad}\ X) = 1\} \cup \{0\}.$$

Clearly, if $A : \mathfrak{n} \longrightarrow \mathfrak{n}'$ is an isomorphism then $AU(\mathfrak{n}) = U(\mathfrak{n}')$. Under the presentation of $\mathfrak{l}_4 \oplus \mathfrak{l}_4$ given in the statement of the theorem, it is easy to see that

$$U(\mathfrak{l}_4 \oplus \mathfrak{l}_4) = \langle X_3, Z_1 \rangle_{\mathbb{R}} \cup \langle X_4, Z_3 \rangle_{\mathbb{R}}. \tag{11}$$

In case (I), it follows from (10) that we also have

$$[X_2, Z_1] = Z_4, \qquad [X_2, Z_2] = kZ_3.$$

Therefore, in order to get that $\mathfrak{n}^{\mathbb{Q}}$ is isomorphic to $\mathfrak{l}_k^{\mathbb{Q}}$ (see (7)), it is enough to show that the vectors in $\langle Z_3, Z_4 \rangle_{\mathbb{R}}$ given by

$$Z := k[X_3, Z_1] = [X_4, Z_2], \qquad Z' := [X_3, Z_2] = [X_4, Z_1]$$

are both zero (see (10)). Let us compute the cone $U(\mathfrak{n})$ for $\mathfrak{n} = \mathfrak{n}^{\mathbb{Q}} \otimes \mathbb{R}$. Recall that $U(\mathfrak{n})$ has to be the union of two disjoint planes as $\mathfrak{n} \simeq \mathfrak{l}_4 \oplus \mathfrak{l}_4$ (see (11)). If $X = aX_1 + bX_2 + cX_3 + dX_4 + eZ_1 + fZ_2$ then

$$\begin{aligned} [X_1, X] &= cZ_1 + dZ_2 + eZ_3 + fZ_4, \\ [X_2, X] &= dkZ_1 + cZ_2 + fkZ_3 + eZ_4, \\ [X_3, X] &= -aZ_1 - bZ_2 + \frac{e}{k}Z + fZ', \\ [X_4, X] &= -bkZ_1 - aZ_2 + fZ + eZ', \\ [Z_1, X] &= -aZ_3 - bZ_4 - \frac{c}{k}Z - dZ', \\ [Z_2, X] &= -bkZ_3 - aZ_4 - dZ - cZ'. \end{aligned}$$

Assume that $\mathrm{Im}(\mathrm{ad}\ X) = \mathbb{R}X_0$, $X_0 \neq 0$. If $k \leqslant 0$ then it follows easily from $[X_1, X] = \lambda[X_2, X]$ and $[X_3, X] = \mu[X_4, X]$ for some $\lambda, \mu \in \mathbb{R}$ that $a = b = c = d = e = f = 0$, which implies that $U(\mathfrak{n}) = \{0\}$, a contradiction.

*Remark 5.2*. Since $k$ has to be positive one can also get by an easy adaptation of this proof that the only real form of $(\mathfrak{l}_4 \oplus \mathfrak{l}_4)_{\mathbb{C}}$ is $\mathfrak{l}_4 \oplus \mathfrak{l}_4$.

We then have that $k > 0$ and $a = \pm\sqrt{k}b$, $c = \pm\sqrt{k}d$, $e = \pm\sqrt{k}f$, where $c$ and $e$ have the same sign. This implies that

$$X = b(\pm\sqrt{k}X_1 + X_2) + d(\pm\sqrt{k}X_3 + X_4) + f(\pm\sqrt{k}Z_1 + Z_2)$$

and

$$\begin{aligned}
[X_1, X] &= d(\pm\sqrt{k}Z_1 + Z_2) + f(\pm\sqrt{k}Z_3 + Z_4),\\
[X_2, X] &= \sqrt{k}[X_1, X],\\
[X_3, X] &= -b(\pm\sqrt{k}Z_1 + Z_2) + f\left(\pm\frac{1}{\sqrt{k}}Z + Z'\right),\\
[X_4, X] &= \sqrt{k}[X_3, X],\\
[Z_1, X] &= -b(\pm\sqrt{k}Z_3 + Z_4) - d\left(\pm\frac{1}{\sqrt{k}}Z + Z'\right),\\
[Z_2, X] &= \sqrt{k}[Z_1, X].
\end{aligned}$$

If $b \neq 0$ then $d \neq 0$ and $a$ has the same sign as $c$ and $e$, and since $X_0$ has a nonzero component in $\langle Z_1, Z_2\rangle_{\mathbb{R}}$ we get $[Z_1, X] = 0$, that is, $-\frac{b}{d}(\pm\sqrt{k}Z_3 + Z_4) = \pm\frac{1}{\sqrt{k}}Z + Z'$. In any case we obtain a subset of $U(\mathfrak{n})$ of the form

$$\{b(\pm\sqrt{k}X_1 + X_2) + d(\pm\sqrt{k}X_3 + X_4) + f(\pm\sqrt{k}Z_1 + Z_2) : b, d \neq 0\}$$

with the same sign in all the terms, which is a contradiction since $U(\mathfrak{n})$ is the union of two planes. Thus $b = 0$ and so

$$U(\mathfrak{n}) = \langle\sqrt{k}X_3 + X_4, \sqrt{k}Z_1 + Z_2\rangle_{\mathbb{R}} \cup \langle -\sqrt{k}X_3 + X_4, -\sqrt{k}Z_1 + Z_2\rangle_{\mathbb{R}}.$$

This clearly implies that $\frac{1}{\sqrt{k}}Z + Z' = -\frac{1}{\sqrt{k}}Z + Z' = 0$, that is $Z = Z' = 0$, as was to be shown.

Concerning case (II), we can assume that

$$[X_1, Z_2] = rZ_3, \quad k[X_1, Z_1] = sZ_3, \quad [X_3, Z_2] = tZ_4, \quad k[X_3, Z_1] = uZ_4,$$

where $Z_3, Z_4$ are linearly independent and $(s, r), (u, t) \neq (0, 0)$. By using (10), for $X = aX_1 + bX_2 + cX_3 + dX_4 + eZ_1 + fZ_2$ we have that

$$\begin{aligned}
[X_1, X] &= cZ_1 + dZ_2 + \left(\frac{e}{k}s + fr\right)Z_3,\\
[X_2, X] &= dkZ_1 + cZ_2 + (fs + er)Z_3,\\
[X_3, X] &= -aZ_1 - bZ_2 + \left(\frac{e}{k}u + ft\right)Z_4,\\
[X_4, X] &= -bkZ_1 - aZ_2 + (fu + et)Z_4,\\
[Z_1, X] &= -\left(\frac{a}{k}s + br\right)Z_3 - \left(\frac{c}{k}u + dt\right)Z_4,
\end{aligned}$$

$$[Z_2, X] = -\left(\frac{b}{k}s + ar\right)Z_3 - \left(\frac{d}{k}u + ct\right)Z_4.$$

If $a = 0$ then $b = c = d = 0$. We also obtain that $e^2 = kf^2$, since either

$$\begin{bmatrix} \frac{e}{k} & f \\ f & e \end{bmatrix}\begin{bmatrix} s \\ r \end{bmatrix} = 0 \qquad \text{or} \qquad \begin{bmatrix} \frac{e}{k} & f \\ f & e \end{bmatrix}\begin{bmatrix} u \\ t \end{bmatrix} = 0.$$

We do not get any plane in $U(\mathfrak{n})$ in this way and therefore there must be an $X \in U(\mathfrak{n})$ with $a \neq 0$, which implies that $b, c, d \neq 0$ and $a^2 = kb^2$, $c^2 = kd^2$. Thus $[Z_1, X] = [Z_2, X] = 0$ and so $\mathrm{Im}(\mathrm{ad}\, X) \subset \langle Z_1, Z_2 \rangle_{\mathbb{R}}$. This implies that $e^2 = kf^2$ and then the 3-dimensional subspace

$$\langle \sqrt{k}X_1 + X_2, \sqrt{k}X_3 + X_4, \sqrt{k}Z_1 + Z_2 \rangle_{\mathbb{R}} \subset U(\mathfrak{n}),$$

which is a contradiction, proving that case (II) is not possible. This concludes the proof of the proposition. □

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