

Foliations invariant under Lie group transverse actions

By

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Abstract. In this paper we study (smooth and holomorphic) foliations which are invariant under transverse actions of Lie groups.

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1. Introduction and main results

In the study of foliations it is very useful to consider the transverse structure. Among the simplest transverse structures are Lie group transverse structures, homogeneous transverse structures and Riemannian transverse structures. In the present work we consider a slightly different situation: foliations which are invariant under Lie group transverse actions. Another motivation for this work is the well-known result of Tischler [13] asserting that if a closed oriented manifold admits a (codimension one) foliation which is invariant under a transverse flow then the manifold is a fiber bundle over the circle. In this work we look for generalizations of this result for higher codimension foliations. All manifolds are assumed to be connected and oriented. All foliations are assumed to be smooth, oriented and transversely oriented.

Let M be a manifold, \mathcal{F} a codimension q foliation on M and G a Lie group of dimension $\dim G = \text{codim } \mathcal{F} = q$. We shall also say that \mathcal{F} is *invariant under a transverse action* of the group G , \mathcal{F} is G -i.u.t.a. for short, if there is an action $\Phi : G \times M \rightarrow M$ of G on M such that: (i) the action is *transverse to* \mathcal{F} , i.e., the orbits of this action have dimension q and intersect transversely the leaves of \mathcal{F} and (ii) Φ *leaves* \mathcal{F} *invariant*, i.e., the maps $\Phi_g : x \mapsto \Phi(g, x)$ take leaves of \mathcal{F} onto leaves of \mathcal{F} .

Let \mathcal{F} be a foliation on M such that \mathcal{F} is G -i.u.t.a. It is not difficult to prove the existence of a *Lie foliation structure for* \mathcal{F} *on* M *of model* G in the sense of Chap. III, Sect. 2 of [2]. We shall then say that \mathcal{F} *has* G -*transversal structure* and

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prove (with a self-contained proof) the existence of a *development* for \mathcal{F} as in Proposition 2.3, page 153 of [2] (Chap. III, Sect. 2). Indeed, we have a sort of strong form of this procedure in Sect. 4 with a self-contained proof (Proposition 3)

Indeed, from the proof of Proposition 3 we obtain an *algebraic model* for any foliated manifold (M, \mathcal{F}) assuming that \mathcal{F} is G -i.u.t.a. Given a leaf L of \mathcal{F} we define $H(L)$ as the set of $g \in G$ such that $\Phi(g, l) \in L$ for every (or equivalently for some) $l \in L$. Then $H(L)$ is a (not necessarily closed) subgroup of G which we provide with the discrete topology. We have the following algebraic model for the general foliation invariant under a transverse Lie group action.

Theorem 1 (Algebraic model). *Let \mathcal{F} be a foliation and G -i.u.t.a. Given a leaf L of \mathcal{F} there is a natural proper free action of $H(L)$ on $G \times L$ with a smooth quotient manifold $(G \times L)/H(L)$, which is G -equivariantly diffeomorphic to M . The leaves of \mathcal{F} are the sets $\Phi(p(\{g\} \times L))$, $g \in G$ where $p : G \times L \rightarrow (G \times L)/H(L)$ denotes the canonical projection.*

As a consequence of the above construction we have:

Theorem 2 (Fibration theorem). *Let M be a connected manifold and \mathcal{F} a foliation in M which is invariant under a transverse action of a Lie group G . Then the following statements are equivalent:*

- (a) \mathcal{F} has a leaf L which is closed in M .
- (b) $H(L)$ is a discrete (i.e., closed and zero-dimensional) subgroup of G .
- (c) The projection $\pi : G \times L \rightarrow G$ onto the first factor induces a smooth fibration $M \simeq G \times_H(L)L \rightarrow G/H(L)$, of which the fibers are the leaves of \mathcal{F} .

From Theorem 2 we immediately obtain:

Corollary 1. *If L is compact, then $H(L)$ is finite, and we have a fiber bundle over $G/H(L)$.*

Additional consequences of Theorem 2 are:

Corollary 2. *If $\pi_1(M, x)$ is finite, then $H(L)$ is closed and \mathcal{F} is a fibration over the base space $G/H(L)$. If moreover M is compact, then L and G are compact. In case G is simply connected the latter also implies that G is semi-simple.*

Corollary 3. *Let M be a compact manifold supporting a codimension two foliation \mathcal{F} invariant under a Lie group transverse action. If \mathcal{F} has a compact leaf then M is a fibre bundle over the two torus.*

A well-known consequence of (the proof of) Tischler’s theorem is that on a compact, connected, oriented manifold M with $\dim H_1(M, \mathbb{R}) \leq 1$ or, equivalently, with $\dim H_{\text{deRham}}^1(M) \leq 1$, any foliation \mathcal{F} of a codimension one foliation which admits an \mathbb{R} -transversal structure is defined by a fibration over a circle and in particular $\dim H^1(M) = 1$. This can be generalized as follows:

Corollary 4. *Let M be a compact smooth manifold and \mathcal{F} a foliation in M which admits a G -transversal structure. If $d := \dim H_{\text{deRham}}^1(M) \leq \dim \mathcal{L}G - \dim D\mathcal{L}G$, then we have equality and M fibers over a torus of dimension d , in*

such a way that the leaves of \mathcal{F} are contained in the fibers of this fibration. In particular, if $\mathcal{L}G$ is abelian and $\dim H^1_{\text{deRham}}(M) \leq \dim \mathcal{L}G$, then \mathcal{F} is a fibration over a torus.

Regarding (codimension two) foliations which are $\text{Aff}(\mathbb{R})$ -i.u.t.a. it is not difficult to prove that if $H^1(M, \mathbb{R}) = 0$ then \mathcal{F} is given by a submersion $F : M \rightarrow \text{Aff}(\mathbb{R})$. This fact admits the following strong form:

Theorem 3. *If M is connected, \mathcal{F} is a foliation which is invariant under the transverse action of a simply connected solvable Lie group G , and $H^1(M, \mathbb{R}) = 0$, then M is diffeomorphic to $G \times L$, where L is any leaf of \mathcal{F} , and the foliation is given by the projection to the first factor G , which is diffeomorphic to a vector space.*

1.1 Holomorphic foliations. In Sect. 8 we carry out the study of the holomorphic case and prove an analogue to Theorem 2 (Theorem 5). For the case M is a compact Kähler manifold we prove that if \mathcal{F} has a G -transverse structure then the universal covering of G is isomorphic to $(\mathbb{C}^q, +)$ (Proposition 7), if moreover \mathcal{F} has some compact leaf then $G = \mathbb{C}^q/H$ for some closed subgroup $H < \mathbb{C}^q$ (Proposition 8).

2. Examples

In Sect. 5 we construct the algebraic model of the general foliation invariant under a Lie group transverse action. This provides a number of examples of foliated spaces with invariant foliations. Below we give some more concrete examples:

Example 1. The most trivial example of a foliation invariant under a Lie group transverse action is given by the product foliation on a manifold $M = G \times N$, product of a Lie group G by a manifold N . The leaves of the foliation are of the form $\{g\} \times N$ where $g \in G$.

Example 2. Let H be a closed (normal) subgroup of a Lie group G . We consider the action $\Phi : H \times G \rightarrow G$ given by $\Phi(h, g) = h.g$ and the quotient map $\pi : G \rightarrow G/H$ (a fibration) which defines a foliation \mathcal{F} on G . Given $x \in \mathcal{F}_g = \pi^{-1}(Hg)$ we have $\pi(x) = Hg$ and $\Phi_h(x) = h.x$. But $\pi(\Phi_h(x)) = \pi(h.x) = H.hx = Hx$ implies that $\Phi_h(x) \in \pi^{-1}(Hx) = \mathcal{F}_x$ and the orbit $\mathcal{O}(g) = Hg$ is transverse to the fiber $\pi^{-1}(Hg)$. Hence, \mathcal{F} is a foliation invariant under the transverse action Φ . Now let G be a simply-connected group, H a discrete subgroup of G and $\phi : H \rightarrow \text{Diff}(G)$ the natural representation given by $\phi(h) = L_h$. The universal covering of G/H is G with projection $\pi : G \rightarrow G/H$ and we have $\pi_1(G/H) \simeq H$ because $\pi \circ f(g) = Hf(g) = Hg$ for $f \in \text{Aut}(G)$, so $f(g) \simeq g$ implies that $f(g).g^{-1} \in H$ and $f(g) = h.g$, for some unique $h \in H$. Therefore $f = L_h$ and then we define $\text{Aut}(G) \rightarrow H; f \mapsto h$, which is an isomorphism. So, we may write $\phi : \pi_1(G/H) \rightarrow \text{Diff}(G)$ and $\Phi : \pi_1(G/H) \times G \rightarrow G$. The map $\Psi : H \times G \times G \rightarrow G \times G$ given by $\Psi(h, g_1, g_2) = (L_h(g_1), L_h(g_2))$ is a properly discontinuous action and defines a quotient manifold $M = \frac{G \times G}{\Psi}$, which equivalence classes are the orbits of Ψ . We have the following facts: (1) There exists a fibration $\sigma : M \rightarrow G/H$ with fiber G , induced by $\pi : G \rightarrow G/H$, and structural group isomorphic to $\phi(H) < \text{Diff}(G)$. (2)

The natural foliation $\overline{\mathcal{F}}$ on G given by classes Hg ; $g \in G$, is Φ -invariant, such that the product foliation $G \times \overline{\mathcal{F}}$ on $G \times G$ is Ψ -invariant and induces a foliation \mathcal{F}_0 on M , called suspension of $\overline{\mathcal{F}}$ for Φ , transverse to $\sigma : M \rightarrow G/H$.

Example 3. Let $G = \mathbb{P}SL(2, \mathbb{R})$ and $H = \text{Aff}(\mathbb{R}) \triangleleft G$. An element of G has the expression $x \mapsto \frac{ax+b}{cx+d} = \frac{a+\frac{b}{x}}{c+\frac{d}{x}}$. The group H is the isotropy group of ∞ , so $\frac{a}{c} = \infty \Leftrightarrow c = 0$ and an element of H is given by $x \mapsto \frac{ax+b}{d} \simeq \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}$. Since $H \triangleleft G$, G has dimension 3 and H has dimension 2, we conclude that G/H has dimension one. Thus we have a fibration $\mathbb{P}SL(2, \mathbb{R}) \rightarrow \mathbb{R}P(1) \simeq S^1$ which is invariant under an action of $\text{Aff}^+(\mathbb{R})$ on $\mathbb{P}SL(2, \mathbb{R})$ having leaves diffeomorphic to $\mathbb{R}^+ \times \mathbb{R} \simeq \mathbb{R}^2$.

3. Foliations with Lie transverse structure

Throughout this paper, except if explicitly mentioned otherwise, \mathbb{F} will denote the tangent bundle of the foliation \mathcal{F} . It is therefore an integrable subbundle of the tangent bundle TM of M and its connection form is flat, because of the integrability.

Definition 1 ([2], Chap. III, p. 152). Let G be a dimension q Lie group and $\overline{\mathcal{F}}$ a codimension q foliation on a differentiable manifold M . A *Lie transverse structure of model G for $\overline{\mathcal{F}}$* is given by: (1) An open cover $\{U_i\}_{i \in I}$ of M and a family of submersions $f_i : U_i \rightarrow G$ such that $\overline{\mathcal{F}}|_{U_i}$ is given by $f_i = \text{constant}$ and, (2) a family of locally constant maps $\gamma_{ij} : U_i \cap U_j \rightarrow \{\text{left translations on } G\}$ such that $f_i(x) = \gamma_{ij}(x).f_j(x)$, $\forall x \in U_i \cap U_j$.

According to Chap. III, Corollary 2.4 and Proposition 2.7 in [2], the existence of a G -transversal structure for $\overline{\mathcal{F}}$ is equivalent to the existence of a $\mathcal{L}G$ -valued smooth differential one-form ω on M , such that the tangent bundle \mathbb{F} of $\overline{\mathcal{F}}$ is equal to the kernel of ω , and $(d\omega)(u, v) = -[\omega(u), \omega(v)]$ for every pair of vector fields u, v . Here $\mathcal{L}G$ denotes the Lie algebra of the Lie group G , and, for every $x \in M$, $\mathbb{F}_x = T_x \overline{\mathcal{F}}_x$, the tangent space at x of the leaf passing through the point x . After the choice of a basis in $\mathcal{L}G$, this amounts to having the suitable systems of differential one-forms as follows:

Proposition 1. *Let M be a manifold equipped with a codimension q having transverse structure of model G . Then there exists an integrable system $\{\Omega_1, \dots, \Omega_q\}$ of one-forms defining $\overline{\mathcal{F}}$ on M with $d\Omega_k = \sum_{i < j} c_{ij}^k \Omega_i \wedge \Omega_j$, where $\{c_{ij}^k\}$ are the structure constants of the Lie algebra of G for a certain basis.*

Lie foliations exhibit the following structure (cf. [2] Proposition 2.9, p. 155):

Proposition 2 ([2]). *For a foliation $\overline{\mathcal{F}}$ on M the existing Lie transverse structures of model G are classified by: (i) A Galoisian covering map $\pi : P \rightarrow M$, (ii) a homomorphism $h : \pi_1(M) \rightarrow G$ such that $\pi^{-1}(e_G) \simeq \text{Aut}(\pi)$ and (iii) a submersion $\Theta : P \rightarrow G$ which is a first integral for the pull-back foliation $\pi^* \overline{\mathcal{F}}$ and is equivariant by h , that is, $\Theta(\alpha.x) = h(\alpha)\Theta(x)$, $\forall x \in M$, $\alpha \in \pi_1(M)$. We call (P, h, Θ) a development of $\overline{\mathcal{F}}$. Two developments (P_1, h_1, Θ_1) and (P_2, h_2, Θ_2) define the same Lie group transverse structure if and only if there is a diffeomorphism $\psi : P_1 \rightarrow P_2$ and an element $g \in G$ such that $h_2 = gh_1g^{-1}$ and $\Theta_2 \circ \psi = g \cdot \Theta_1$. The leaves of $\pi^* \overline{\mathcal{F}}$ are the connected components of $\Theta^{-1}(g)$, $g \in G$ and we have a*

submersion $\tilde{\Theta} : M \rightarrow G/H$ such that $\sigma \circ \Theta = \tilde{\Theta} \circ \pi$ where $\sigma : G \rightarrow G/H$ is the quotient map.

A proof of this proposition can be done (as in the classical way) by constructing a suitable system of differential forms on M which satisfy the same relations as the forms in a basis of the Lie algebra $\mathcal{L}(G)$ of G and then applying the classical Darboux-Lie theorem and Chap. III, Corollary 2.4 and Proposition 2.7 in [2].

4. Construction of a development

In this section \mathcal{F} is a foliation which is G -i.u.t.a. on a connected manifold M . Our aim is to give a self-contained proof of a version of Proposition 2 which is more adequate to our approach and purposes. Indeed, we prove:

Proposition 3. *Assume that \mathcal{F} is G -i.u.t.a. Then \mathcal{F} has a G -transversal structure and a complementary foliation. Moreover, given any leaf L of \mathcal{F} we have a development of \mathcal{F} as follows:*

- (i) A Galoisian covering map $\pi : P = G \times L \rightarrow M$,
- (ii) A homomorphism $h : \pi_1(M) \rightarrow G$ such that $\pi^{-1}(e_G) \simeq \text{Aut}(\pi)$,
- (iii) A submersion $\Theta : P \rightarrow G$ which is a first integral for the pull-back foliation $\pi^*\mathcal{F}$ and is equivariant by h , that is, $\Theta(\alpha.x) = h(\alpha)\Theta(x)$, $\forall x \in M, \alpha \in \pi_1(M)$.

Proof of Proposition 3. Choose a leaf L of \mathcal{F} . The restriction Φ_L to $G \times L$ of the action $\Phi : G \times M \rightarrow M$ has a bijective tangent mapping at every point, which implies that it is a local diffeomorphism, and that the image $\Phi(G \times L)$ is an open subset of M . In M we have the equivalence relation $x \sim y$ if and only if there exists a $g \in G$ such that $\Phi(x, g) \in L_y$, the leaf of \mathcal{F} through y , and the equivalence classes are the open sets $\Phi(G \times L)$ where $L \in \mathcal{F}$ is a leaf. Since M is connected there is only one equivalence class, that is $\Phi(G \times L) = M$. This shows that the local diffeomorphism $\Phi_L : G \times L \rightarrow M$ is surjective.

The assumption that the action of G maps leaves of \mathcal{F} to leaves of \mathcal{F} implies that for any $g \in G$ the following conditions are equivalent: (a) There exists an $x \in L$ such $\Phi(g, x) \in L$. (b) $g_M(L) = L$, if g_M denotes the mapping $x \mapsto \Phi(g, x)$. (The mapping $g \mapsto g_M$ is a homomorphism from G to the group of all diffeomorphisms of M). Let $H = H_L$ denote the set of $g \in G$ which satisfy (a) or (b). Then H is a subgroup of G , and we have that $\Phi_L(g, x) = \Phi_L(g', x')$ if and only if there exists an $h \in H$ such that $g' = gh^{-1}$ and $x' = \Phi(h, x)$. The mapping $h \mapsto ((g, x) \mapsto (gh^{-1}, \Phi(h, x)))$ defines an action of H on $G \times L$, which is free because the action on the first component is free. The mapping Φ_L induces a bijective mapping $\Psi_L : G \times_H L \rightarrow M$, which is the uniquely determined by the condition that $\Phi_L = \Psi_L \circ p$ where p denotes the canonical projection from $G \times L$ onto the space $G \times_H L \rightarrow M$ of H -orbits in $G \times L$.

The definition of $G \times_H L$ and Ψ_L in the previous paragraph was purely set-theoretic, let us now discuss the topological and smoothness aspects.

Claim 1. *If we provide H with the discrete topology, then the action of H on $G \times L$ is a proper mapping, i.e., the mapping $(h, (g, x)) \mapsto ((gh^{-1}, \Phi(h, x)), (g, x))$ is a proper mapping from $H \times (G \times L)$ to $(G \times L) \times (G \times L)$.*

Proof of Claim 1. Let us show that given an infinite sequence $h_j \in H$, $g_j \in G$, $x_j \in L$, such that $(g_j h_j^{-1}, \Phi(h_j, x_j))$ converges in $G \times L$ to (g', x') and (g_j, x_j) converges in $G \times L$ to (g, x) , then a subsequence of the (h_j, g_j, x_j) converges in $H \times G \times L$ to some element of $H \times G \times L$. Here we use the leaf topology in L (notice that this is different from the M -topology in L if L is not a closed subset of M). Because we use the discrete topology in L , this amounts to the statement that $g_j \rightarrow g$ in G , $x_j \rightarrow x \in L$, $g_j h_j^{-1} \rightarrow g'$ in G , $\Phi(h_j, x_j) \rightarrow x'$ in L implies that the h_j have a constant subsequence. From the fact that the g_j and the $g_j h_j^{-1}$ converge in G , it follows that the $h_j = (g_j h_j^{-1})^{-1} g_j$ converge in G . There are open neighborhoods U and V of e_G and x in G and L , respectively, such that V is connected, and the restriction ψ of the G -action Φ to $U \times V$ is a diffeomorphism from $U \times V$ onto an open neighborhood W of x in M , in such a way that the integral manifolds of the restriction to W of the vector subbundle \mathbb{F} of TM , i.e., the “local leaves”, are equal to the sets of the form $\psi(\{u\} \times V)$, where u runs over U . That we have a diffeomorphism ψ follows from the fact that the tangent mapping of ψ at (e_G, x) is a bijective linear mapping from $\mathcal{L}G \times T_x L$ onto $T_x M$, and the statement about the local leaves follows from the fact that the action maps leaves to leaves and therefore the local action maps local leaves to local leaves. Let W_0 be a closed neighborhood of x in M such that $W_0 \subset W$. Because the h_j converge in G , and $x_j \rightarrow x$ in M , there exists an integer k such that $h_k^{-1} h_j \in U$, $x_j \in V$ and $y_j := \Phi(h_k^{-1} h_j, x_j) \in W_0$ whenever $j \geq k$. Because for $j \rightarrow \infty$ the y_j converge in the leaf topology to an element of $W_0 \subset W$, we conclude that there exists a k' such that all y_j for $j \geq k'$, belong to the same local leaf, which means that all $h_k^{-1} h_j$ are the same for all $j \geq k'$, which in turn implies that the h_j are the same for all $j \geq k'$. This completes the proof of the claim. \square

Because the action of H is proper and free, there is a unique structure of smooth manifold on the orbit space $G \times_H L$ for which the canonical projection $p : G \times L \rightarrow G \times_H L$ is a principal H -bundle, in which H is provided with the discrete topology. In other words, the canonical projection $G \times L \rightarrow G \times_H L$ is a Galois covering with group H . From the local triviality we obtain that the mapping $\Psi_L : G \times_H L \rightarrow M$ is smooth, because $\Phi_L = \Psi_L \circ p$, and Φ_L and p were local diffeomorphisms, we obtain that Ψ_L is a local diffeomorphism, and because Ψ_L is bijective, it follows that Ψ_L is a diffeomorphism from $G \times_H L$ onto M .

This completes the proof that (P, h, Θ) , in which $P := G \times L$, $\pi := \Psi_L \circ p$ with covering group H , $h :=$ the holonomy homomorphism from $\pi_1(M)$ onto the subgroup H of G and $\Theta : G \rightarrow G$ defined by $\Theta(g, x) = g$, is a development of \mathcal{F} . The uniqueness of developments is straightforward. \square

5. Algebraic model and proof of Theorem 2

Summarizing the discussion in the proof of Proposition 3 we have: Let \mathcal{F} be a foliation G -i.u.t.a. on M . Let L be a leaf of the foliation and let H be the set of $g \in G$ such that $\Phi(g, l) \in L$ for every (or equivalently for some) $l \in L$. Then H is a subgroup of G , not necessarily closed, which we endow with the discrete topology. Assuming that M is connected, the restriction of Φ to $G \times L$ is a covering map

$\Phi : G \times L \rightarrow M$. We have $\Phi(g, l) = \Phi(g', l')$ if and only if there is a uniquely determined $h \in H$ such that $g' = gh^{-1}$ and $l' = \Phi(h, l)$. On $G \times L$ we have the action of H in which $h \in H$ sends (g, l) to $(gh^{-1}, \Phi(h, l))$. This action is proper because the action of H on L is proper (and discrete), and the action is free because the right action of H on G is free. As a consequence the action is proper and free, we have a smooth quotient manifold $(G \times L)/H$, which is G -equivariantly diffeomorphic to M , where the diffeomorphism from $(G \times L)/H$ onto M is induced by Φ , and the G -equivariance is with respect to the action of G on $(G \times L)/H$ in which $g' \in G$ sends the H -orbit of (g, l) to the H -orbit of $(g'g, l)$. If we write $p : G \times L \rightarrow (G \times L)/H$ for the natural projection from $G \times L$ onto $(G \times L)/H$, then the leaves of the foliation in M are the sets $\Phi(p(\{g\} \times L))$, $g \in G$. That is:

The foliation \mathcal{F} in M corresponds to the foliation of the $p(\{g\} \times L)$, $g \in G$, in $(G \times L)/H$.

This proves Theorem 1, i.e., the *algebraic model* $(G \times L)/H$ of the general foliation invariant under a transverse Lie group action. In particular, all further analysis can be done in $(G \times L)/H$, in which H is a subgroup of G acting from the right on G and acting properly and discretely on the manifold L .

Now we can prove several results.

Proposition 4. *The following statements are equivalent.*

- (a) *The foliation has a closed leaf L .*
- (b) *H is a closed discrete subgroup of G .*
- (c) *H is a closed discrete subgroup of G and the projection $G \times L \rightarrow G$ onto the first factor exhibits $(G \times L)/H$ as a fiber bundle over G/H with fiber diffeomorphic to L .*

Proof. Suppose (a) holds and let h_j be a sequence in H which converges in G to some $g \in G$. Let $x \in L$. Then $\Phi(h_j, x) \rightarrow \Phi(g, x)$ in M and therefore $\Phi(g, x) \in L$ because L is closed in M . It follows that $g \in H$ proving that (a) implies (b). That (b) implies (c) is a general fact about closed subgroups H of a Lie group G , where H acts smoothly on a manifold L . Finally, that (c) implies (a) is obvious. □

Condition (c) means that the foliation in M is a G -invariant fibration. Thus we have:

Proof of Theorem 2. The theorem follows from Proposition 4 and the above construction of the algebraic model. □

Proof of Corollary 2. If $\pi_1(M, x)$ is finite, then $H = h(\pi_1(M))$ is finite, hence a closed subgroup of G , and the foliation is a fibration over the base space G/H . If moreover M is compact, then the fiber L as well as the group G is compact. If in addition G is simply-connected then G is semi-simple. □

Proof of Corollary 3. In this case G is a two-dimensional connected Lie group thus G must be isomorphic to \mathbb{R}^2 , $S^1 \times \mathbb{R}$, $\text{Aff}^+(\mathbb{R}) \simeq \mathbb{R}^+ \times \mathbb{R}$ or $S^1 \times S^1$. The subgroup $H = h(\pi_1(M))$ must be discrete. If $H < \mathbb{R}^2$ then H is isomorphic to the trivial group, $0 \times \mathbb{Z}$ or \mathbb{Z}^2 , so $G/H \simeq \mathbb{R}^2$, $\mathbb{R}^* \times \mathbb{R}$ or T^2 . If $H < S^1 \times \mathbb{R}$ then H is

isomorphic to the trivial group, $\mathbb{Z} \times 0$, $0 \times \mathbb{Z}$ or \mathbb{Z}^2 so $G/H \simeq S^1 \times \mathbb{R}$, $\frac{S^1 \times \mathbb{R}}{\mathbb{Z} \times 0}$, $S^1 \times S^1$, $\frac{S^1 \times \mathbb{R}}{\mathbb{Z} \times \mathbb{Z}}$. And if $H < \text{Aff}^+(\mathbb{R})$ then H is isomorphic to the trivial group, $\mathbb{N} \times 0$, $0 \times \mathbb{Z}$ or $\mathbb{N} \times \mathbb{Z}$. Since M is compact, G/H is compact therefore we have $G/H \simeq T^2$. □

6. Foliations invariant under a transverse local action

Proposition 1 states that the existence of a G -transversal structure for a foliation \mathcal{F} is equivalent to the existence of a suitable system of differential forms $\{\Omega_j\}_{j=1}^q$ satisfying the same structure equations of a given basis of the Lie algebra $\mathcal{L}(G)$. Let us now prove this and give an interpretation of the invariance of \mathcal{F} under a G -transversal action in a way that motivates a generalization. Thus, in what follows we assume that \mathcal{F} is a foliation on M which is G -i.u.t.a. For any $X \in \mathcal{L}G$, let X_M denote vector field on M which defines the infinitesimal action of X on M . The assumption that the G -orbits have the same dimension as G is equivalent to the condition that the action is locally free, which in turn is equivalent to the condition that for each $x \in M$ the mapping $X \mapsto X_{M,x}$ is injective from $\mathcal{L}G$ to T_xM . Denote the image space by $\mathcal{L}G_{M,x}$, this can be viewed as the tangent space at x to the orbit through x . The transversality condition means that, for any $x \in M$, T_xM is equal to the direct sum of \mathbb{F}_x and $\mathcal{L}G_{M,x}$. Therefore, there is a unique $\mathcal{L}G$ -valued one-form Θ on M , such that $\Theta = 0$ on \mathbb{F} and $\Theta_x(X_{M,x}) = X$ for every $X \in \mathcal{L}G$. This $\mathcal{L}G$ -valued one-form Θ on M is called the *connection form* of the infinitesimal connection \mathbb{F} for the infinitesimal action of $\mathcal{L}G$ on M . The form Θ is automatically smooth. The fact that the infinitesimal connection is flat, meaning that \mathbb{F} is integrable, is equivalent to the condition that $(d\Theta)(u, v) = -[\Theta(u), \Theta(v)]$ for every pair of vector fields u, v on M . The one-forms Ω_j appearing in Proposition 2 are exactly the components of the connection form. Thus we can prove Proposition 1 by a repetition of the proof that integrability of \mathbb{F} implies that $(d\Theta)(u, v) = -[\Theta(u), \Theta(v)]$, or equivalently $d\Omega_k = \sum_{i < j} c_{ij}^k \Omega_i \wedge \Omega_j$. Now, a construction of the mappings f_i and γ_{ij} in the definition of Lie transverse structure (Definition 1) is not clear at a first moment. This is quite obvious once we observe that this is equivalent to a construction of a local action of G on the leaf space and that the local action of G maps local leaves to local leaves.

Motivated by this we observe that a weaker assumption than the invariance under a Lie group transverse action is the following:

Definition 2. We say that a foliation \mathcal{F} in M is *invariant under a transverse local action* of a Lie group G (G -i.u.t.l.a.) if there is a locally free local action of G on M , the tangent mappings of which leave \mathbb{F} invariant, and such that the $\mathcal{L}G_{M,x}$ are complementary subspaces to the \mathbb{F}_x in T_xM .

The relation between the two notions in Definitions 1 and 2 is given below:

Proposition 5. *Given \mathcal{F} on M the following conditions are equivalent:*

- (i) \mathcal{F} is G -i.u.t.l.a.
- (ii) \mathcal{F} has a transversal G -structure and a complementary foliation.

Proof. Assume that \mathcal{F} is G -i.u.t.l.a. Because the $\mathcal{L}G_{M,x}$ are the tangent spaces to the local G -orbits, they define an integrable vector subbundle $\mathcal{L}G_M$ of TM , which is complementary to \mathbb{F} . As we have already observed above this weaker condition already implies that there is a G -transverse structure. Conversely, if K is an integrable vector subbundle of TM (i.e., defining a foliation in M) which is complementary to \mathbb{F} , and we have a transversal G -structure to \mathbb{F} , defined by a $\mathcal{L}G$ -valued one-form ω as in 1), then for each $X \in \mathcal{L}G$ and $x \in M$ there is a unique $X_{M,x} \in K_x$ such that $\omega_x(X_{M,x}) = X$. This defines a smooth vector field X_M on M and the equation $(d\omega)(u, v) = -[\omega(u), \omega(v)]$ in combination with the integrability of K implies that $x \mapsto X_M$ is a homomorphism of Lie algebras from $\mathcal{L}G$ to the Lie algebra of smooth vector fields in M . In other words, in this way we obtain an infinitesimal, and hence a local action of G , which also maps local leaves to local leaves. It is the unique infinitesimal action of $\mathcal{L}G$ for which ω is equal to the connection form and K is tangent to the orbits. \square

Remark 1. The assumption of having a transverse G -action which maps leaves of \mathcal{F} to leaves of \mathcal{F} is equivalent to the weaker assumption that \mathcal{F} is G -i.u.t.l.a., but with the additional assumption that the local action of G on M can be extended to a global one on M . (If such an extension exists, it is unique.) If M is compact, an extension to a global action always exists. Therefore, if M is compact, then \mathcal{F} is G -i.u.t.a. iff \mathcal{F} is G -i.u.t.l.a., i.e., the weaker assumption in Definition 2 is equivalent to the fact that \mathcal{F} is G -i.u.t.a. Clarifying how much the assumption “ \mathcal{F} is G -i.u.t.a.” is stronger than “ \mathcal{F} is G -i.u.t.l.a.” might help in understanding which consequences are typical consequences of the first and not only of the existence of a G -transverse structure. In a rough manner, a G -transversal structure for the foliation \mathcal{F} is something like a locally free local action of G on the leaf space M/\mathcal{F} , in which the latter has to be treated as a sort of non-Hausdorff manifold. Finally, as it was pointed-out by the referee, any Lie algebra action can be extended to a Lie group action on a larger manifold (which might be non-Hausdorff) where also the foliation extends to the larger manifold ([6]).

7. Solvable groups

Let us consider the Lie group of affine maps of the real line $\text{Aff}(\mathbb{R}) = \{x \mapsto a.x + b; a \in \mathbb{R}^* \text{ and } b \in \mathbb{R}\} \simeq \mathbb{R}^* \times \mathbb{R}$. Let \mathcal{F} be a foliation of codimension two on M invariant under a transverse action of $\text{Aff}(\mathbb{R})$. We assume that \mathcal{F} is transversely oriented so that indeed \mathcal{F} is $\text{Aff}^+(\mathbb{R})$ -i.u.t.a., where $\text{Aff}^+(\mathbb{R})$ is the subgroup of orientation preserving affine maps of the real line. Notice that as a manifold we have $\text{Aff}^+(\mathbb{R}) = (0, +\infty) \times \mathbb{R}$ so that it is simply-connected, also it is solvable as a group. According to Proposition 1 there is an integrable system of two one-forms ω, η defining \mathcal{F} on M such that $d\omega = \omega \wedge \eta, d\eta = 0$. If $H_1(M, \mathbb{R}) = 0$ then $\eta = dh$, for some differentiable function $h : M \rightarrow \mathbb{R}$, and we define $f = e^h : M \rightarrow \mathbb{R}^*$, thus $\eta = \frac{1}{f}df$. A straightforward computation then shows that $d(f\omega) = 0$ and therefore $\omega = f^{-1}dg$ for some differentiable function $g : M \rightarrow \mathbb{R}$. So there exists a fibration $F = (f, g) : M \rightarrow \text{Aff}(\mathbb{R})$ whose fibers are the leaves of \mathcal{F} . Assume now that M is compact. In this case according to Tischler’s Theorem

[13], since η is a nonsingular closed one-form in M , there exists a fibration $f : M \rightarrow S^1$. We have proved:

Proposition 6. *Let M be a connected manifold with a foliation \mathcal{F} which is $\text{Aff}(\mathbb{R})$ -i.u.t.a. Then \mathcal{F} has an $\text{Aff}(\mathbb{R})$ -transverse structure and we have:*

- (i) *If $H^1(M, \mathbb{R}) = 0$ then \mathcal{F} is given by a submersion $F : M \rightarrow \text{Aff}(\mathbb{R})$. In particular, in this case M is not compact.*
- (ii) *If M is compact then it admits a fibration $f : M \rightarrow S^1$.*

This proposition is a particular case of Theorem 3 of which proof is given below:

Proof of Theorem 3. Let $D\mathcal{L}G$ denote the *derived Lie algebra* of $\mathcal{L}(G)$, i.e., the linear subspace of $\mathcal{L}G$ which is generated by all $[X, Y]$ such that $X, Y \in \mathcal{L}G$. It is known that $D\mathcal{L}(G)$ is a normal Lie subalgebra of $\mathcal{L}G$ ([10]). Let $\xi \in (D\mathcal{L}G)^0 \simeq (\mathcal{L}G/D\mathcal{L}G)^*$ be a linear form on $\mathcal{L}G$ which is equal to zero on $D\mathcal{L}G$. If Θ denotes the connection form introduced in Sect. 6 then $\omega := \omega_\xi := \xi \circ \Theta$ is a closed one-form on M , hence $\omega = df$ for a smooth real-valued function $f = f_\xi$ on M , which we can let depend linearly on ξ . It follows that $x \mapsto (\xi \mapsto f_\xi(x))$ defines a smooth mapping f from M to $((\mathcal{L}G/D\mathcal{L}G)^*)^* = \mathcal{L}G/D\mathcal{L}G$. It is a submersion, the leaves of \mathcal{F} are contained in the fibers of f , and also the fibers of f are invariant under the action of the group DG generated by $D\mathcal{L}G$. According to Theorem 3.18.1 in [14] if G is simply connected, the analytic subgroup DG of G defined by $D\mathcal{L}G$ is a closed normal subgroup of G , therefore G/DG is an abelian and simply connected Lie group.

The abelian group G/DG is isomorphic with $\mathcal{L}G/D\mathcal{L}G$ and acts on it by translations. Using lifts by elements of G acting on M , we obtain that f is surjective and defines a topologically trivial fibration. In particular the first cohomology group H^1 of each fiber of f is equal to zero as well. It is also clear that an element g of G leaves a fiber of f fixed if and only if $g \in DG$. Because any $h \in H$ leaves L fixed, and therefore also leaves the f -fiber containing L fixed, we conclude that $h \in DG$, i.e., we obtain that $H \subset DG$.

This means that in the fibers of f , we have the same situation again, with $H \subset DG$, connected f -fibers and the H^1 of the f -fibers equal to zero. Since the Lie algebra $\mathcal{L}G$ is *solvable* (because G is solvable), then the repeated derived Lie algebras $D^i \mathcal{L}G$ terminate at zero (cf. p. 201 in [14]), and we arrive at the conclusion that the group H is trivial. In view of Theorem 2 this means that M is isomorphic to $G \times L$ and the foliation is defined by the projection onto the first factor. Using Theorem 3.18.11 in [14] we conclude that (since G is simply connected and solvable) G is diffeomorphic to an Euclidean space. In the above we have used the fact that any fiber bundle over a contractible space is trivial. This can be found in Corollary 11.6 in [12]. □

We have the following generalization of Theorem 3:

Theorem 4. *Let \mathcal{F} be a smooth foliation in a connected smooth manifold M . Assume that \mathcal{F} is invariant under a transverse action of a Lie group G and assume that $H^1(M, \mathbb{R}) = 0$. Let p be the smallest nonnegative integer such that $D^{p+1} \mathcal{L}G = D^p \mathcal{L}G$ and let $D^p G$ denote the analytic Lie subgroup of G with Lie algebra equal to $D^p \mathcal{L}G$. Then $H \subset D^p G$.*

Proof. By passing to the universal covering of G , we may assume that G is simply connected. Theorem 3.18.12 in [14] states that in this case every analytic subgroup of G is closed and simply connected. Then the result follows from the proof of Theorem 3. □

Proof of Corollary 4. We shall use the same notation of the proof of Theorem 3. Let M be a compact smooth manifold and \mathcal{F} a foliation in M which admits a G -transversal structure. Then the mapping $\xi \mapsto [\xi \circ \Theta]$ from $(\mathcal{L}G/D\mathcal{L}G)^*$ to $H_{\text{de Rham}}^1(M)$ is injective. Moreover, if $d := \dim H_{\text{de Rham}}^1(M) \leq \dim \mathcal{L}G - \dim D\mathcal{L}G$, then we have equality here and M fibers over a d -dimensional torus in such a way that the leaves of \mathcal{F} are contained in the fibers of this fibration. If $\mathcal{L}G$ is abelian and $\dim H_{\text{de Rham}}^1(M) \leq \dim \mathcal{L}G$, then \mathcal{F} is a fibration over a torus. □

8. Holomorphic foliations

In this section we study holomorphic foliations which are invariant under transverse actions of complex Lie groups.

Theorem 5. *Let M be a connected complex manifold and \mathcal{F} a holomorphic foliation invariant under a holomorphic transverse action of a complex Lie group G of dimension $\dim G = \text{codim } \mathcal{F}$. The following conditions are equivalent:*

- (a) \mathcal{F} has a leaf L which is closed in M .
- (b) $H(L)$ is a discrete subgroup of G .
- (c) The projection $\pi : G \times L \rightarrow G$ onto the first factor induces a holomorphic fibration $M \simeq G \times_H(L)L \rightarrow G/H(L)$, of which the fibers are the leaves of \mathcal{F} .

Remark 2. In the above statement the fibration $M \simeq G \times_H(L)L \rightarrow G/H(L)$ is a holomorphic fibration, in the sense that its local trivializations are biholomorphic maps. According to Ehresmann’s Theorem [2], any proper C^r , $r \geq 2$ submersion defines a C^r -locally trivial fiber bundle. This is not true for proper holomorphic submersions. Indeed, the analytic type of the fiber may vary. On the other hand, Grauert-Fischer’s theorem [1] asserts that this is the only obstruction: a proper holomorphic submersion is a holomorphic fibration (i.e., a locally trivial holomorphic fiber bundle) if and only if the fibers are holomorphically equivalent. Thus Theorem 5 follows from Theorem 2 and Grauert-Fischer’s theorem. Nevertheless, we give a “simpler” self-contained proof in Sect. 8.

Proof of Theorem 5. We already know (Theorem 2) that M is a C^∞ fibre bundle over the homogenous space G/H . Since the fibers are holomorphically equivalent (by biholomorphisms $\Phi_g : M \rightarrow M$), Grauert-Fischer’s Theorem [1] states that the submersion $M \rightarrow G/H$ is a locally holomorphically trivial fibration. Nonetheless, we can give a more self-contained proof as follows: we know that H is a closed and zero-dimensional subgroup of G and that $\pi : M \rightarrow G/H$ is a G -equivariant holomorphic mapping. Let $g \in G$ and write $L = \pi^{-1}(\{gH\})$ for the fiber over the point gH in G/H . Then there exists an open neighborhood U of the origin in $\mathcal{L}G$ such that $X \mapsto \exp(X)gH$ is a holomorphic diffeomorphism from U onto an open neighborhood V of gH in G/H . The mapping $(X, x) \mapsto \Phi(\exp(X), x)$ is a holomorphic diffeomorphism from $U \times L$ onto $\pi^{-1}(V)$, and it yields the desired holomorphic trivialization. □

Proposition 7. *Let M be a compact Kähler manifold, G a complex simply connected Lie group, and \mathcal{F} a holomorphic foliation of codimension q with a (holomorphic) G -transverse structure. Then $G \simeq \mathbb{C}^q$. If moreover $\dim H^1(M, \mathbb{R}) \leq 2q$, then $\dim H^1(M, \mathbb{R}) = 2q$ and the foliation \mathcal{F} is a fibration over a real $2q$ -dimensional torus.*

Proof. According to [3] p. 110 any holomorphic q -form on a compact Kähler manifold is closed. Applying this to the connection form (which is holomorphic), we conclude that this is closed, which in turn implies that $\mathcal{L}G$ and therefore G is abelian. □

A natural holomorphic version of Proposition 3 implies the following:

Proposition 8. *Let M be a compact Kähler manifold with a holomorphic codimension q foliation \mathcal{F} invariant under a Lie group transverse action of G . Then the universal covering of G is isomorphic to $(\mathbb{C}^q, +)$. If moreover \mathcal{F} has a compact leaf then we have $G = \mathbb{C}^q/H$ for some closed subgroup $H < \mathbb{C}^q$.*

Remark 3. Since an algebraic manifold is always Kähler, Proposition 8 is valid for any projective manifold.

8.1. Codimension one algebraic foliations. By definition such an *algebraic foliation* \mathcal{F}_0 on \mathbb{C}^n is given by a polynomial one-form $\Omega = \sum_{j=1}^n P_j dz_j$, where the P_j are polynomials in the affine variables $(z_1, \dots, z_n) \in \mathbb{C}^n$, satisfying the integrability condition $\Omega \wedge d\Omega = 0$. Such a foliation admits a unique extension to a holomorphic foliation with singularities \mathcal{F} on $\mathbb{C}\mathbb{P}^n$. Conversely, any foliation of $\mathbb{C}\mathbb{P}^n$ is obtained this way. Assume now that \mathcal{F}_0 is \mathbb{C} -i.u.t.a., i.e., invariant by a holomorphic flow in \mathbb{C}^n . The foliation is then given by a closed holomorphic one-form ω on \mathbb{C}^n . Thus we have $\omega = dF$ for an entire function F on \mathbb{C}^n .

Claim 2. *If the hyperplane $\mathbb{C}\mathbb{P}_\infty^{n-1} = \mathbb{C}\mathbb{P}^n \setminus \mathbb{C}^n$ is not \mathcal{F} -invariant then F is a polynomial first integral on $\mathbb{C}\mathbb{P}^n$.*

Proof. Fixed a generic point $q \in \mathbb{C}\mathbb{P}_\infty^{n-1}$ we may consider a “flow box” (i.e., a distinguished neighborhood for \mathcal{F}) V containing q with coordinates $(z_1, \dots, z_n) \in V$, such that $\mathbb{C}\mathbb{P}_\infty^{n-1} \cap V = \{z_1 = 0\}$ and $\mathcal{F}|_V$ is given by $dz_n = 0$. Let $V^* = V \setminus (V \cap \mathbb{C}\mathbb{P}(n-1)_\infty) = V \setminus \{z_1 = 0\}$. In V^* we have $\omega \wedge dz_n \equiv 0$, i.e., $dF \wedge dz_n \equiv 0$. Therefore $F|_{V^*} = F(z_n)$ is depends only on the variable z_n . On the other hand, $F|_{V^*}$ is holomorphic. Therefore F extends meromorphically to V . Then Hartogs’ theorem [4] implies that F is meromorphic on $\mathbb{C}\mathbb{P}^n$. Liouville’s theorem [5] then shows that F is a rational function and since it is holomorphic on \mathbb{C}^n we conclude that F is a polynomial on \mathbb{C}^n . □

Proposition 9. *Let \mathcal{F}_0 be an algebraic codimension one foliation on \mathbb{C}^n , $n \geq 2$. Suppose that \mathcal{F} is \mathbb{C} -i.u.t.a. Then the hyperplane at infinity $\mathbb{C}\mathbb{P}(n-1)_\infty$ is \mathcal{F} -invariant.*

Proof. If $\mathbb{C}\mathbb{P}_\infty^{n-1}$ is not invariant then by the above claim \mathcal{F} has a polynomial first integral F on \mathbb{C}^n . However, as a general fact for meromorphic first integrals, the polar set $\{F = \infty\}$ and the zero set $\{F = 0\}$ are invariant. Since the polar set is $\mathbb{C}\mathbb{P}_\infty^{n-1}$ the proposition follows. □

Using techniques introduced in [8] one may be able to go further in the classification of \mathcal{F} in this case.

9. Complements

9.1. The realization problem. In [7] the author discusses, mainly for solvable Lie groups, the “*Realization problem*”, which is the following question of Haefliger:

Question 1. *Which subgroups H of a given Lie group G can occur as $h(\pi_1(M))$ for a development (P, h, Θ) of a G -transverse structure on a compact manifold M ?*

Under the additional hypothesis that the “ G -transverse structure” is followed by the fact that the “the foliation is invariant under a transverse action of G on M ”, then the realization problem asks for the subgroups H of G for which there exists a smooth connected manifold L such that H acts smoothly on L , the right-left-action of H on $G \times L$ is proper (here H is provided with the discrete topology) and the quotient $G \times_H L$ is compact. This seems to imply that H is finitely generated. Also, for closed subgroups H of G the answer is that H is a discrete subgroup of G , G/H is compact, and one can take any smooth action of H (for instance, the trivial action) on any compact connected manifold L . Nevertheless, in the theory of foliations one is mostly interested in foliations which are not fibrations, which corresponds to the case that H is not closed in G . Such cases can occur, as shown by the example of orbit foliations of non-closed subgroups of tori. It may be helpful to restrict to solvable Lie groups G .

9.2. More on the algebraic model. Let us say a few more words about the algebraic model $G \times_H L$ for a foliated manifold with transverse Lie group action. The main motivation comes from the book [10], more precisely from its Sect. 2.4 where the authors introduce the associated fiber bundle $X \times_H Y$, in which X and Y are manifolds, H is a Lie group acting on X by $(h, x) \mapsto x \cdot h^{-1}$, on Y by $(h, y) \mapsto h \cdot y$, and it is assumed that the action of H on X is proper and free. This then implies that the action $(h, (x, y)) \mapsto (x \cdot h^{-1}, h \cdot y)$ of H on $X \times Y$ is proper and free, which then makes that the orbit space $X \times_H Y$ is a smooth manifold, and that the projection $X \times Y \rightarrow X \times_H Y$ is a principal H -bundle. Moreover, the projection onto the first factor induces a fibration $X \times_H Y \rightarrow X/H$, with fibers isomorphic to Y . In the statements that the orbit spaces = quotients are smooth manifolds, the properness assumption is quite essential.

If $X = G$ is equal to a Lie group and H is a Lie subgroup of G , then the right action of H is proper if and only if H is closed in G . In this case there is a unique action of G on $G \times_H Y$ such that the projection $G \times Y \rightarrow G \times_H Y$ intertwines the left action of G on $G \times Y$ (defined by means of the left action of G on the first factor) with the action of G on $G \times_H Y$. Furthermore the projection $G \times_H Y \rightarrow G/H$ intertwines the action of G on $G \times_H Y$ with the transitive left action of G on G/H . This leads to a well-understood model of a G -homogeneous bundle over a G -homogeneous base manifold.

The point of this construction in the aforementioned book was that for any proper Lie group action every orbit has an invariant open neighborhood which, as a manifold with Lie group action, is isomorphic to some $G \times_H Y$, where H is the stabilizer subgroup of a point in the given orbit and Y is a so-called slice for the G -action. This is the Tube theorem 2.4.1 in the book [10], originally found in [9].

It is then clear that set-theoretically the G -space M is of the form $G \times_H L$, in which L is a leaf of your foliation and H is the subgroup of G which maps L to itself. Moreover, in order that $G \times_H L$ inherits the structure of a smooth manifold from its construction as the space of H -orbits (which is equivalent to saying that the mapping $G \times L \rightarrow M$, defined by the restriction to $G \times L$ of the G -action in M , is a principal H -bundle), it is sufficient to assume that the H -action on $G \times L$ is proper. Finally, in our situation this properness follows from your assumptions (in the more general situation that the G -action maps leaves to leaves and is transversal, but X_M tangent to a leaf for nonzero elements X of the Lie algebra, then one would need quite technical additional assumptions on the action in order to obtain that the action of H on $G \times L$ is proper). Because H is discrete in our case, the principal H -fibration $G \times L \rightarrow M$ is a Galois covering (actually Galois coverings are nothing else than principal fiber bundles with discrete group actions).

From the background sketched above Theorem 2 is clear, i.e., L is closed in M if and only if H is closed in G if and only if M is a G -homogeneous fiber bundle over the homogeneous space G/H , where the fibers are equal to the leaves of the foliation.

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