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Engel words and the Restricted Burnside Problem

Pavel Shumyatsky · Jhone Caldeira Silva

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Abstract The following theorem is proved. For any positive integers n and k there exists a number $s = s(n, k)$ depending only on n and k such that the class of all groups G satisfying the identity $([x_1, {}_k y_1] \cdots [x_s, {}_k y_s])^n \equiv 1$ and having the verbal subgroup corresponding to the kth Engel word locally finite is a variety.

Keywords Groups · Varieties · Restricted Burnside Problem

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1 Introduction

Following Zelmanov's solution of the Restricted Burnside Problem [13,14] some new interesting varieties of groups have been descovered.

A variety is a class of groups defined by equations. More precisely, if W is a set of words in $x_1, x_2, \ldots,$ the class of all groups G such that $W(G) = 1$ is called the *variety* determined by W. By a well-known theorem of Birkhoff varieties are precisely classes of groups closed with respect to taking quotients, subgroups and cartesian products of

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P. Shumyatsky (✉) · J. C. Silva
Department of Mathematics, University of Brasilia, Brasilia-DF 70910-900, Brazil
e-mail: pavel@mat.unb.br

J. C. Silva
e-mail: jhone@mat.unb.br

their members. It is well-known that the solution of the Restricted Burnside Problem is equivalent to the following statement.

1.1 The class of locally finite groups of exponent n is a variety

If w is a word in variables $x_1, \dots, x_m$ we think of it primarily as a function of m variables defined on any given group G. We denote by $w(G)$ the verbal subgroup of G generated by the values of w. We are interested in the following question.

Problem 1.2 Let $n \geq 1$ and w a group-word. Consider the class of all groups G satisfying the identity $w^n \equiv 1$ and having $w(G)$ locally finite. Is that a variety?

According to the solution of the Restricted Burnside Problem the answer to the above question is positive if $w(x) = x$. In fact it is easy to see that the answer is positive whenever w is any non-commutator word (the word w is commutator if the sum of the exponents of any variable involved in w is zero).

Indeed, suppose $w(x_1, \dots, x_m)$ is such a word and that the sum of the exponents of x_i is $r \neq 0$. Now, given any group G satisfying the identity $w^n \equiv 1$, substitute the unit for all the variables except x_i and an arbitrary element $g \in G$ for x_i. We see that g^r is a w-value for all $g \in G$. Hence G satisfies the identity $x^{nr} \equiv 1$, that is, G has finite exponent dividing nr. Now positive answer to our problem follows easily from the positive solution of the Restricted Burnside Problem. Thus, the problem is essentially about commutator words.

A word w is called a *multilinear commutator of weight k* if it has form of a multilinear Lie monomial in precisely k independent variables. In other terminology these are called *outer commutator words*. Particular examples of multilinear commutators are the derived words, defined by the equations:

$$\delta_0(x) = x,$$
$$\delta_k(x_1, \dots, x_{2^k}) = [\delta_{k-1}(x_1, \dots, x_{2^{k-1}}), \delta_{k-1}(x_{2^{k-1}+1} \dots, x_{2^k})],$$

and the lower central words:

$$\gamma_1(x) = x,$$
$$\gamma_{k+1}(x_1, \dots, x_{k+1}) = [\gamma_k(x_1, \dots, x_k), x_{k+1}].$$

In [12] we proved the following theorem.

Theorem 1.3 *Let w be a multilinear commutator. For any positive integer n there exists t depending only on n such that the class of all groups G having $w(G)$ locally finite and satisfying the condition that the product of any t w-values is of order dividing n is a variety.*

The reader can consult the references list in [12] for other results on the case of multilinear commutators. The most relevant among commutator words that are not multilinear commutators are certainly the Engel words. These are defined inductively by

$$[x,{}_0 y] = x;\ \ [x,{}_k y] = [[x,{}_{k-1} y], y].$$

The following result was obtained in [11].

Theorem 1.4 *Let n be a prime-power and k a positive integer. There exists a number* $t = t(n, k)$ *depending only on k and n such that the class of groups G satisfying the identity* $([x_1,{}_k y_1]\dots[x_t,{}_k y_t])^n \equiv 1$ *and having the verbal subgroup corresponding to the kth Engel word locally finite is a variety.*

The goal of the present paper is to extend the above theorem to the case where n is not assumed to be a prime-power. It will be shown that the theorem holds for arbitrary n.

Theorem 1.5 *There exists a number* $s = s(n, k)$ *depending only on k and n such that the class of all groups G satisfying the identity* $([x_1,{}_k y_1]\dots[x_s,{}_k y_s])^n \equiv 1$ *and having the verbal subgroup corresponding to the kth Engel word locally finite is a variety.*

The proof of the above theorem uses all the usual tools: the classification of finite simple groups, the Hall–Higman theory [3], Lie theory due to Zelmanov, etc. Segal's theorem [9] that in a finitely generated prosoluble group the derived group is closed is another important ingredient of the proof.

2 Bounding the Fitting height of a soluble group

We use the expression "$\{a, b, c, \dots\}$-bounded" to mean "bounded from above by some function depending only on $a, b, c, \dots$". Some arguments used in this paper are similar to those from [12]. The key idea is that Segal's theorem, combined with the other tools, can be used to bound the Fitting height of certain finite soluble groups. This enables us to reduce the problem to nilpotent groups at which point Theorem 1.4 becomes applicable.

Given a group G, an element $g \in G$ will be called a *k-Engel value* if there exist $x, y \in G$ such that $g = [x,{}_k y]$. Recall that the *Fitting subgroup* $F(G)$ of G is the product of all normal nilpotent subgroups of G. The *Fitting series* of G can be defined by the rules: $F_0(G) = 1$, $F_1(G) = F(G)$, $F_{i+1}(G)/F_i(G) = F(G/F_i(G))$ for $i = 1, 2, \dots$. If G is a finite soluble group, then the minimal number $h = h(G)$ such that $F_h(G) = G$ is called the *Fitting height* of G. The following important result is due to Segal [9].

Theorem 2.1 *If G is a finite soluble group generated by m elements* $a_1, a_2, \dots, a_m$, *then every element of the derived group of G is a product of an m-bounded number of commutators of the form* $[b, a]$, *where* $b \in G$ *and* $a \in \{a_1, a_2, \dots, a_m\}$.

In [6] Nikolov and Segal generalized the above theorem to the case where G is not required to be soluble but this will not be used in the present paper. From Segal's result we deduce the following lemma.

Lemma 2.2 *Let G be a finite soluble group generated by m elements $g_1, g_2, \ldots, g_m$ such that the order of each g_i divides a positive integer l. Then every element of G is a product of an $\{m, l\}$-bounded number $f = f(m, l)$ of conjugates of g_i.*

Proof By Theorem 2.1, any element of G' can be written as a product of r commutators $[b_i, a_i]$, where r is an m-bounded number, $b_i \in G$ and $a_i \in \{g_1, g_2, \ldots, g_m\}$. We now put $f = rl + m^l$. If $x \in G'$, we write

$$x = [b_1, a_1][b_2, a_2] \cdots [b_r, a_r] = (a_1^{-1})^{b_1} a_1 (a_2^{-1})^{b_2} a_2 \cdots (a_r^{-1})^{b_r} a_r$$
$$= a_1^{(l-1)b_1} a_1 a_2^{(l-1)b_2} a_2 \cdots a_r^{(l-1)b_r} a_r.$$

So x is a product of at most rl conjugates of g_i. Since G/G' is generated by elements of order dividing l and the order of G/G' is at most m^l, it follows that any element of G is a product of at most f conjugates of g_i. □

Throughout the rest of the paper $f(m, l)$ will stand for the $\{m, l\}$-bounded number as in the previous lemma.

Lemma 2.3 *Let G be a finite soluble group. Assume that the Fitting height of the verbal subgroup of G corresponding to the kth Engel word is h. Then one can choose h k-Engel values in G, $v_1, v_2, \ldots, v_h$, such that the subgroup $\langle v_1, v_2, \ldots, v_h \rangle$ has Fitting height precisely h.*

Proof Let R be the verbal subgroup of G corresponding to the kth Engel word. We denote $F_i(R)$ by F_i, $1 \leq i \leq h$. If $h = 1$, the lemma is obvious so we assume that $h \geq 2$. Since $F_h = R$, there exists a k-Engel value v_1 such that $v_1 \notin F_{h-1}$. Since $\langle v_1, F_{h-1} \rangle / F_{h-2}$ is not nilpotent, there exists $x_1 \in F_{h-1}$ such that $[x_1, {}_i v_1] \notin F_{h-2}$ for any positive integer i. Put $v_2 = [x_1, {}_k v_1]$. Note that v_2 is a k-Engel value. Moreover, $v_2 \in F_{h-1} \backslash F_{h-2}$.

Since $\langle v_2, F_{h-2} \rangle / F_{h-3}$ is not nilpotent, there exists $x_2 \in F_{h-2}$ such that $[x_2, {}_i v_2] \notin F_{h-3}$ for any positive integer i. Put $v_3 = [x_2, {}_k v_2]$. Note that v_3 is a k-Engel value and that $v_3 \in F_{h-2} \backslash F_{h-3}$.

Continuing this process, there exists $x_{r-1} \in F_{h-(r-1)}$ such that we can choose a k-Engel value v_r defined by $v_r = [x_{r-1}, {}_k v_{r-1}]$, where $1 \leq r \leq h$ and $v_r \in F_{h-(r-1)} \backslash F_{h-r}$.

Now let $H = \langle v_1, v_2, \ldots, v_h \rangle$. It remains only to show that the subgroup H has Fitting height precisely h. We have $v_h \in F(H)$. Note that $v_{h-1} \in F_2(H)$ but $v_{h-1} \notin F(H)$ (otherwise, there would exist j such that $[v_h, {}_j v_{h-1}] = 1$). More generally, $v_{h-r} \in F_{r+1}(H)$ but $v_{h-r} \notin F_r(H)$, where $0 \leq r \leq h - 1$. We conclude that H has Fitting height h, as required. □

Having fixed a positive integer k, the symbol w_j will denote the word that is the product of j k-Engel values. Thus, for example, the symbol w_2 will denote the word $[x_1, {}_k y_1][x_2, {}_k y_2]$. It is clear that if an element of a group G is a w_i-value, then it is also a w_j-value for any $i \leq j$. Therefore, for any $i \leq j$, the identity $w_j^n \equiv 1$ implies the identity $w_i^n \equiv 1$ in G. Obviously, the identity $w_j^n \equiv 1$ implies that all k-Engel values have order dividing n and this fact will be used freely without being explicitly mentioned.

A well-known corollary of the Hall–Higman theory says that the Fitting height of any finite soluble group of exponent n is bounded by the number of prime divisors of n, counting multiplicities. We will denote the number by $h(n)$.

Lemma 2.4 *Let j and n be positive integers with the property that $j \geq f(h(n)+1, n)$. Let G be a finite soluble group satisfying the identity $w_j^n \equiv 1$. Then the Fitting height of the verbal subgroup of G corresponding to the kth Engel word is at most $h(n)$.*

Proof Assume the lemma is false. Set $h = h(n)$. By Lemma 2.3 we can choose $h+1$ k-Engel values $v_1, v_2, \ldots, v_{h+1}$ in G such that the subgroup $H = \langle v_1, v_2, \ldots, v_{h+1}\rangle$ has Fitting height at least $h+1$. By Lemma 2.2 any element of H can be written as a product of at most $f(h+1, n)$ conjugates of v_i. Since conjugates of k-Engel values are again k-Engel values, by the hypothesis any such product has order dividing n. So H is of exponent n, whence $h(H) \leq h$, a contradiction. □

Corollary 2.5 *Under the hypothesis of Lemma 2.4, we have $h(G) \leq h(n) + 1$.*

Proof If R is the verbal subgroup of G corresponding to the kth Engel word then, by the previous lemma, $h(R) \leq h(n)$. The quotient G/R is a finite Engel group. By Zorn's Theorem [7, XII.3.4], G/R is nilpotent. Hence, $h(G) \leq h(n) + 1$. □

3 Main results

Following [3] we call a group G *monolithic* if it has a unique minimal normal subgroup which is non-abelian simple.

Proposition 3.1 *Let j, j_1 and n be positive integers with the property that $j \geq j_1 n+1$ and let G be a finite group satisfying the identity $w_j^n \equiv 1$. Assume that G has no non-trivial normal soluble subgroups. Then G possesses a normal subgroup L such that L is residually monolithic and G/L residually belongs to the class of finite groups satisfying an identity $w_{j_1}^{n/p} \equiv 1$ for some prime divisor p of n.*

Proof Let M be a minimal normal subgroup of G. We know that $M \cong S_1 \times S_2 \times \cdots \times S_r$, where $S_1, S_2, \ldots, S_r$ are isomorphic simple groups. The group G acts on M by permuting the simple factors so we obtain a representation of G by permutations of the set $\{S_1, S_2, \ldots, S_r\}$. Let L_M be the kernel of the representation. Choose in S_1 a non-trivial element b of the form w_1. This is possible because S_1 is not soluble. Let p be a prime divisor of the order of b. We want to show that G/L_M satisfies the identity $w_{j_1}^{n/p} \equiv 1$. Suppose this is not true. Let $q = p^\alpha$ be the largest power of p dividing n. Since the identity $w_{j_1}^{n/p} \equiv 1$ does not hold in G/L_M, there exists an element $a \in G$ of the form w_{j_1} such that q divides the order of a modulo L_M. Write $n = n_1 q$ and $a_1 = a^{n_1}$. Then a_1 is an element of the form $w_{j_1 n_1}$ that has order q and permute regularly some q factors in $\{S_1, S_2, \ldots, S_r\}$. Without any loss of generality we will assume that S_1 is one of those factors. Write

$$(ba_1)^q = bb^{a_1^{-1}} b^{a_1^{-2}} \cdots b^{a_1}.$$

Since each of the elements $b^{a_1^{-i}}$ belongs to a different subgroup S_i, the product $bb^{a_1^{-1}}b^{a_1^{-2}}\cdots b^{a_1}$ has the same order as b. Thus, $(ba_1)^q$ has order divisible by p so the order of ba_1 is divisible by $p^{\alpha+1}$. However ba_1 is of the form w_j so its order must divide n. This contradiction shows that indeed G/L_M satisfies the identity $w_{j_1}^{n/p} \equiv 1$.

Let now L be the intersection of all the subgroups L_M, where M ranges through the minimal normal subgroups of G. The previous paragraph implies that the proof of the proposition will be completed once it is shown that L is residually monolithic. If T is the product of the minimal normal subgroups of G, it is clear that T is the product of pairwise commuting simple groups $S_1, S_2, \ldots, S_t$ and that L is the intersection of the normalizers of S_i. Since G has no non-trivial normal soluble subgroups, it follows that $C_G(T) = 1$ and therefore any element of L induces a non-trivial automorphism of some the S_i. Let ρ_i be the natural homomorphism of L into the group of automorphisms of S_i. It is easy to see that the image of ρ_i is monolithic and that the intersection of the kernels of all ρ_i is trivial. Hence L is residually monolithic. □

According to the solution of the Restricted Burnside Problem the order of any finite m-generated group of exponent n is $\{m, n\}$-bounded. In this section we present some generalizations of this result.

The next proposition was proved in [11] using Lie-theoretical techniques that Zelmanov created in his solution of the Restricted Burnside Problem.

Proposition 3.2 *There exists a function $t = t(n, k)$ with the following property. Let G be a finite group generated by m elements $a_1, \ldots, a_m$, each of order dividing l. Suppose that G satisfies the identity $w_t^n \equiv 1$, where n is a prime-power. Then the order of G is $\{k, m, n, l\}$-bounded.*

Let $t = t(n, k)$ have the same meaning as in Proposition 3.2. Let us now choose a function $s(x, y)$ defined for any positive integers x, y with the following properties.

1. $s(x, y) \geq f(h(x) + 1, x)$ for all x, y;
2. $s(x, y) \geq t(x, y)$ whenever x is a prime-power;
3. $s(x, y) \geq x \cdot s(z, y) + 1$ for all x, y, z such that z is a proper divisor of x.

Such a function can be constructed using induction on x. Indeed, fix a positive integer y and define $s(1, y)$ to be the maximum of the numbers $f(1, 1)$ and $t(1, y)$. Now suppose that $s(x, y)$ is defined for all $x \leq n - 1$. If n is a prime-power, put $s(n, y) = t(n, y)$. Otherwise, let M be the maximal value among $f(h(n) + 1, n)$ and $n \cdot s(z, y) + 1$, where z ranges through the set of all proper divisors of n. Put $s(n, y) = M$. This can be performed for any y, thus establishing the existence of a function with the desired properties. Eventually, it will be shown that the chosen function satisfies the hypothesis of Theorem 1.5.

Proposition 3.3 *Let m, n, l be positive integers and $s = s(n, k)$. Let G be a finite group satisfying the identity $w_s^n \equiv 1$. Assume that G can be generated by m elements $g_1, g_2, \ldots, g_m$ such that each g_i and each commutator of the form $[g, x]$, where $g \in \{g_1, g_2, \ldots, g_m\}$, $x \in G$, have order dividing l. Then the order of G is $\{k, m, n, l\}$-bounded.*

Proof If $n = 1$, G is a k-Engel group. By a result of Burns and Medvedev [1], there exist numbers $c(k)$ and $e(k)$ depending only on k with the property that G has a normal subgroup N such that N is of exponent dividing $e(k)$ and G/N is a nilpotent group of class at most $c(k)$. It is easy to see that G/N has $\{k, m, n, l\}$-bounded order. The minimal number of generators of N is bounded in terms of m and $|G : N|$. The positive solution of the Restricted Burnside Problem allows us to conclude that $|N|$ and, therefore, $|G|$ is $\{k, m, n, l\}$-bounded.

We will now use induction on n, the case $n = 1$ being covered in the previous paragraph. Suppose that $n \geq 2$ and that the proposition is true for groups satisfying an identity $w_{s(n/p,k)}^{n/p} \equiv 1$ for a prime divisor p of n. In other words, the induction hypothesis is that there exists a $\{k, m, n, l\}$-bounded number N_0 such that if G is a finite group satisfying the identity $w_{s(n/p,k)}^{n/p} \equiv 1$ that can be generated by m elements $g_1, g_2, \ldots, g_m$ such that each g_i and each commutator of the form $[g, x]$ have order dividing l, then $|G| \leq N_0$.

Suppose for a moment that G has no non-trivial normal soluble subgroups. Since $s(n, k) \geq n \cdot s(n/p, k) + 1$, Proposition 3.1 tells us that G possesses a normal subgroup L such that L is residually monolithic and G/L residually belongs to the class of finite groups satisfying an identity $w_{s(n/p,k)}^{n/p} \equiv 1$ for some prime divisor p of n. It follows that G/L is residually of order at most N_0. Since G/L is m-generated, by Theorem 7.2.9 of [2], the number of normal subgroups of index at most N_0 in G/L is $\{m, N_0\}$-bounded. Therefore $|G/L|$ is $\{k, m, n, l\}$-bounded. In particular, it follows that L can be generated by r elements for some $\{k, m, n, l\}$-bounded number r.

A result of Jones [5] says that any infinite family of finite simple groups generates the variety of all groups. It follows that up to isomorphism there exist only finitely many monolithic groups satisfying the identity $w_s^n \equiv 1$. Let $N_1 = N_1(n, s)$ be the maximum of their orders. Then L is residually of order at most N_1. Since L is r-generated, the number of distinct normal subgroups of index at most N_1 in L is $\{r, N_1\}$-bounded. Therefore L has $\{k, m, n, l\}$-bounded order. We conclude that $|G|$ is $\{k, m, n, l\}$-bounded.

Now let us drop the assumption that G has no non-trivial normal soluble subgroups. Let S be the product of all normal soluble subgroups of G. The above paragraph shows that G/S has $\{k, m, n, l\}$-bounded order. Since $s(n, k) \geq f(h(n) + 1, n)$, by Corollary 2.5 the Fitting height of S is $\{k, m, n, l\}$-bounded. Let $F = F(G)$ be the Fitting subgroup of G. Using induction on the Fitting height of S, we assume that F has $\{k, m, n, l\}$-bounded index in G.

Suppose first that F is central. In this case, $|G : Z(G)|$ is $\{k, m, n, l\}$-bounded and Schur's Theorem [8, p. 102] guarantees that so is $\left|G'\right|$. Since G can be generated by m elements of order dividing l, it follows that $|G|$ is $\{k, m, n, l\}$-bounded.

If F is not central, consider the subgroup

$$N = \langle [g_1, F], [g_2, F], \ldots, [g_m, F] \rangle .$$

It is easy to see that N is normal in G. Applying the results of the previous paragraph to the quotient G/N, it follows that $|G : N|$ is $\{k, m, n, l\}$-bounded. We will show that $|N|$, and therefore $|G|$, is $\{k, m, n, l\}$-bounded.

We know that N can be generated by a $\{k, m, n, l\}$-bounded number of elements. Let d be the minimal number of generators of N. Denote by $\pi(N)$ the set of prime divisors of $|N|$. Since N is nilpotent, $\pi(N)$ consists of prime divisors of l. Thus, it is sufficient to bound the order of the Sylow p-subgroup of N for every prime $p \in \pi(N)$. Let P be the Sylow p-subgroup of N and write $N = P \times O_{p'}(N)$. If $y_1, y_2, \dots$ is the list of all elements of the form $[g_i, y]$, where $1 \leq i \leq m$ and $y \in F$, we write $b_1, b_2, \dots$ for the corresponding projections of y_j in P. Then $P = \langle b_1, b_2, \dots \rangle$. Since P is a finite d-generated p-group, the Burnside Basis Theorem [4, III.3.15] shows that P is actually generated by d elements in the list $b_1, b_2, \dots$. By the hypothesis, the order of each of them divides l. Let q be the maximal power of p dividing n. Since $s(n, k) \geq t(q, k)$, by Proposition 3.2 we conclude that P has $\{k, m, n, l\}$-bounded order. The proof is complete. □

We are now ready to prove Theorem 1.5.

Let $\mathfrak{X}$ denote the class of all groups with the identity $w_s^n \equiv 1$ and having the verbal subgroup corresponding to the kth Engel word locally finite. It is easy to see that the class $\mathfrak{X}$ is closed to taking subgroups and quotients of its members. Hence, we only need to show that if D is a cartesian product of groups from $\mathfrak{X}$, then $D \in \mathfrak{X}$. Obviously, the identity $w_s^n \equiv 1$ holds in D so it remains only to show that the verbal subgroup R of D corresponding to the kth Engel word is locally finite. Let S be any finite subset of R. Clearly one can find finitely many k-Engel values $h_1, h_2, \dots, h_m \in D$ such that $S \leq \langle h_1, h_2, \dots, h_m \rangle = H$. Thus it is sufficient to prove that the subgroup H is finite. The order of each h_i divides n. Moreover, if $h \in \{h_1, h_2, \dots, h_m\}$ and $x \in H$, then each commutator of the form $[h, x]$ is a product of n k-Engel values. It is clear from the choice of s that $s(n, k) \geq n$ for any $n \geq 2$. So the order of each of the commutators divides n. Note that R is residually locally finite. If Q is any locally finite quotient of R, by Proposition 3.3 the order of the image of H in Q is finite and $\{k, m, n\}$-bounded, so it follows that this order actually does not depend on Q. We conclude that H is finite, as required. □

We record one immediate corollary of Theorem 1.5 that is related to the results obtained in [10].

Corollary 3.4 *Let n and k be positive integers and let s be as in Theorem 1.5. If G is a residually finite group satisfying the identity $([x_1, {}_k y_1] \cdots [x_s, {}_k y_s])^n \equiv 1$, then the verbal subgroup of G corresponding to the kth Engel word is locally finite.*

Proof Let $\mathfrak{X}$ have the same meaning as above. Then any finite quotient of G belongs to the variety $\mathfrak{X}$. However, it is clear that if a group residually belongs to a certain variety, then it actually belongs to the variety. Thus, it follows that the verbal subgroup of G corresponding to the kth Engel word is locally finite. □

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