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Optimal and one-complemented subspaces

By

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Abstract. Let X be a real Banach space and let $V \subset X$ be a closed linear subspace. In [4, Prop. 5] it has been proven that if X is strictly convex, reflexive and smooth and V is an optimal subset of X then V is one-complemented in X. In this note we would like to extend this result to non-smooth Banach spaces. In particular, we show that any existence subspace of $c, c_o$and l_1 is one-complemented. Also some results concerning non-smooth Musielak-Orlicz sequence spaces equipped with the Luxemburg norm will be presented.

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0. Introduction

Let X be a Banach space and let $C \subset X$ be a non-empty set. A continuous mapping $P : X \to C$ is called *a projection onto* C whenever $P|_C = \mathrm{Id}$, that is $P^2 = P$. Setting

$$\mathrm{Min}(C) = \{z \in X : \text{for every } c \in C, x \in X, \text{ if } \|z - c\| \geqslant \|x - c\| \text{ then } x = z\},$$

we say that $C \subset X$ is *optimal* if $\mathrm{Min}(C) = C$. Observe that for any $C \subset X$, $C \subset \mathrm{Min}(C)$. This notion has been introduced by Beauzamy and Maurey in [4], where basic properties concerning optimal sets can be found.

A set $C \subset X$ is called *an existence set of best coapproximation* (*existence set* for brevity), if for any $x \in X$, $R_C(x) \neq \emptyset$, where

$$R_C(x) = \{d \in C : \|d - c\| \leqslant \|x - c\| \text{ for any } c \in C\}.$$

This notion has been introduced in [5]. It is clear that any existence set is an optimal set. The converse, in general, is not true. However, by [4, Prop. 2] if X is one-complemented in X^{**} and strictly convex, then any optimal subset of X is an existence set in X. This, in particular, holds true for strictly convex spaces X, such that $X = Z^*$ for some Banach space Z.

Existence and optimal sets have been studied by many authors from different points of view, mainly in the context of approximation theory (see e.g. [1]–[5], [7]–[11], [13]–[15], [21], [27], [29], [30]).

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Recall that a closed subspace $V$of a Banach space X is called *one-complemented* if there exists a linear projection of norm one from X onto V. Also there is a large number of papers concerning one-complemented subspaces (see e.g. a survey paper [28] and a recent paper [19]). It is obvious that any one-complemented subspace is an existence set. The converse, in general, is not true. By a deep result of Lindenstrauss [23] there exists a Banach space X and a linear subspace V of X, $\operatorname{codim}(V) = 2$, such that:

a) V is one-complemented in any hyperplane Y of X in which it is contained;
b) V is not one-complemented in X.

This fact together with the simple observation stated as Lemma 0.1 below, gives an example of a subspace being an existence set which is not one-complemented.

Lemma 0.1. *Let X be a Banach space and let $V \subset X, V \neq \{0\}$ be a linear subspace. Then V is an existence set in X if and only if for any $x \in X \setminus V$, there exists P_x, a linear projection from Z_x onto V with $\|P_x\| = 1$. Here $Z_x = V \oplus [x]$, where $[x]$ denotes the linear space generated by x.*

Proof. Assume that for any $x \in X \setminus V$ there exists P_x, a linear projection from Z_x onto V with $\|P_x\| = 1$. Fix $z \in Z_x$ and $v \in V$. Note that

$$\|P_x z - v\| = \|P_x(z - v)\| \leqslant \|z - v\|.$$

Hence $P_x z \in R_V(z)$ and so V is an existence set in X. Now assume that V is an existence set in X and fix $x \in X \setminus V$. Take any $d \in R_V(x)$. Since any $z \in Z_x$ can be uniquely expressed as $z = \alpha x + v$ for some $v \in V$ and $\alpha \in \mathbb{R}$, we can define $P_x : Z_x \to V$ by

$$P_x z = \alpha d + v.$$

It is easy to see that P_x is a linear projection from Z_x onto V. To show that $\|P_x\| = 1$, fix $y = \alpha x + v \in Z_x$, with $\alpha \neq 0$. Since $d \in R_V(x)$,

$$\|P_x y\| = \|\alpha d + v\| = |\alpha|\|d + v/\alpha\| \leqslant |\alpha|\|x + v/\alpha\| = \|\alpha x + v\| = \|y\|,$$

which completes the proof. $\square$

However, in [4] (see also [29, p. 162], where the case of contractive retractions onto linear subspaces is considered) the following result has been proven.

Theorem 0.2 (see [4, Prop. 5]). *Let V be a linear subspace of a smooth, reflexive and strictly convex Banach space. If V is an optimal set then V is one-complemented in X. If X is a smooth Banach space, then any subspace of X which is an existence set is one-complemented. Moreover, in both cases a norm-one projection from X onto V is uniquely determined.*

The aim of this paper is to generalize the above result to the case of some non-smooth, real Banach spaces. This can be treated as a partial answer concerning the question from [4, p. 125] about generalizations of [4, Prop. 5] to the non-smooth case. In particular, we show that in c, c_o and l_1 any subspace which is an existence set is one-complemented (see Theorems 2.2, 2.3 and 2.4). Next we demonstrate some results in the case of non-smooth Musielak-Orlicz sequence spaces equipped with the Luxemburg norm.

Now we present some notions and results which will be used in this paper.

In the sequel by $S(X)$ we denote the unit sphere in a Banach space X and by $S(X^*)$ the unit sphere in its dual space. A functional $f \in S(X^*)$ is called a *supporting functional* for $x \in X$, if $f(x) = \|x\|$. Analogously, a point $x \in S(X)$ is called a *norming point* for $f \in X^*$ if $f(x) = \|f\|$. A point $x \in X$ is called a *smooth point* if it has exactly one supporting functional. A Banach space X is called *smooth* if any $x \in S(X)$ is a smooth point.

By $\mathrm{ext}(X)$we denote the set of all extreme points of $S(X)$. A Banach space X is called *strictly convex* if $\mathrm{ext}(X) = S(X)$.

If V is a linear subspace of a Banach space X, by $\mathscr{P}(X, V)$ we will denote the set of all linear, continuous projections from X onto V.

Now we present some introductory facts on Musielak-Orlicz sequence spaces. A function $\phi : \mathbb{R} \to [0, +\infty)$ is said to be an *Orlicz function* if $\phi(0) = 0$, $\phi(t) > 0$ for some $t > 0$, ϕ is even and convex. By ϕ^* we denote its conjugate function in the sense of Young, that is

$$\phi^*(u) = \sup_{v>0}\{|u|v - \phi(v)\},$$

for $u \in \mathbb{R}$ and we notice that ϕ^* is an extended real-valued convex function. If $\phi(u) = (1/p)u^p$, $1 < p < \infty$, then $\phi^*(u) = (1/p')u^{p'}$, where $1/p + 1/p' = 1$. Further, a sequence $\Phi = (\phi_n)$ of Orlicz functions ϕ_n will be called a *Musielak-Orlicz function* whenever $\phi_n(1) = 1$ for every $n \in \mathbb{N}$. By $\Phi^* = (\phi_n^*)$ we will denote its conjugate function.

Let l_o denote the space of all real-valued sequences. With each Musielak-Orlicz function Φ we can associate a mapping $\rho_\Phi : l_o \to [0, +\infty]$ defined by

$$\rho_\Phi(x) = \sum_{n=1}^{\infty} \phi_n(|x_n|),$$

where $x = (x_n) \in l_o$. Given a Musielak-Orlicz function Φ, let l_Φ denote the corresponding *Musielak-Orlicz space*, that is

$$l_\Phi = \{x \in l_o : \lim_{\lambda \to 0} \rho_\Phi(\lambda x) = 0\}. \tag{0.1}$$

If a sequence $\Phi = (\phi_n)$ is constant, that is $\phi_n = \phi$ for every $n \in \mathbb{N}$, then l_Φ is an *Orlicz sequence space* and further it will be denoted by l_ϕ. The space l_Φ equipped with the Luxemburg norm

$$\|x\| = \|x\|_\Phi := \inf\{\lambda > 0 : \rho_\Phi(x/\lambda) \leqslant 1\} \tag{0.2}$$

is a Banach space.

Observe that the assumption $\phi_n(1) = 1$ for every $n \in \mathbb{N}$ is not a real restriction on Musielak-Orlicz function Φ. In fact, for every sequence $\Phi = (\phi_n)$, where ϕ_n are Orlicz functions, there exists a function $\Psi = (\psi_n)$ with $\psi_n(1) = 1$ and such that l_Φ is isometric to l_Ψ. It is enough to take $\psi_n(t) = \phi_n(a_n t)$, where $\phi_n(a_n) = 1$ for every $n \in \mathbb{N}$.

We will also consider here the finite dimensional spaces $l_\Phi^{(m)}$, defined on $\mathbb{R}^m$ analogously as l_Φ. The space $l_\Phi^{(m)}$ can be identified with the subspace of l_Φ consisting of all $x = (x_n) \in l_\Phi$ such that $x_n = 0$ for all $n \geqslant m+1$.

An important subspace of l_Φ, called the *subspace of finite elements* and denoted by h_Φ is defined as

$$h_\Phi = \{x \in l_\Phi : \rho_\Phi(\lambda x) < \infty \text{ for any} \lambda > 0\}. \tag{0.3}$$

It is well known that h_Φ is a closed separable subspace of l_Φ with the Schauder basis consisting of the standard unit vectors $e_i = (0, \ldots, 1_i, 0, \ldots)$. It is easy to see that for every $x \in h_\Phi$, $\|x\| = 1$ if and only if $\rho_\Phi(x) = 1$. Moreover, $h_\Phi = l_\Phi$ if and only if either the dimension of l_Φ is finite or Φ satisfies a growth condition called δ_2 [22, 25], that is there exist $K, \delta > 0$ and a nonnegative sequence $(c_n) \subset l_1$such that for every $n \in \mathbb{N}$ and every $t \geqslant 0$

$$\phi_n(2t) \leqslant K\phi_n(t) + c_n, \tag{0.4}$$

whenever $\phi_n(t) \leqslant \delta$.

Recall that for every $y \in l_{\Phi^*}$, the functional

$$f_y(x) = \sum_{n=1}^{\infty} x_n y_n, \quad x = (x_n) \in l_\Phi,$$

is bounded on $(l_\Phi, \| \ \|_\Phi)$and is called a *regular functional*. We denote by R_Φ the set of all regular functionals on l_Φ. The spaces R_Φ and l_{Φ^*} are order isomorphic [see e.g. 31] and so by usual identification we often write $f_y = y$. More information on Musielak-Orlicz spaces can be found in [6], [16–18], [20], [22], [25], [26], [31], [32]. The following description of supporting functionals can be deduced from [17, Lemma 1.7 and Theorem 1.9]. Set for any $i \in \mathbb{N}$, and $x \in l_\Phi$

$$\Delta\Phi_i(x) = [\phi_i^-(x_i), \phi_i^+(x_i)], \tag{0.5}$$

if $x_i \geqslant 0$ and

$$\Delta\Phi_i(x) = [\phi_i^+(x_i), \phi_i^-(x_i)], \tag{0.6}$$

if $x_i < 0$. Here for any $i \in \mathbb{N}$ and $x \in \mathbb{R}$ we donote by $\phi_i^+(x)$ ($\phi_i^-(x)$, resp.) the right-hand side (the left hand-side, resp.) derivative of ϕ_i at x. Also for any sequence $x = (x_1, x_2, \ldots)$ define

$$\operatorname{supp}(x) = \{i \in \mathbb{N} : x_i \neq 0\}.$$

Theorem 0.3. *Let $\Phi = (\phi_n)$ be a Musielak-Orlicz function and let $x = (x_n) \in h_\Phi$, $\|x\|_\Phi = 1$. Then any supporting functional f of x in l_Φ (with respect to $\| \cdot \|_\Phi$) is a regular functional. Moreover, a regular functional $f = f_z$ determined by $z \in l_{\Phi*}$ is a supporting functional for x if and only if*

a) $\sup\{\rho_\Phi(y) : \|y\|_\Phi \leqslant 1, \operatorname{supp}(y) \subset \operatorname{supp}(z)\} = 1$;
b) *for any* $i \in \operatorname{supp}(z)$

$$z_i = d_i \Big/ \Bigg(\sum_{j \in \operatorname{supp}(z)} d_j x_j \Bigg),$$

where $d_i \in \Delta\Phi_i(x)$ *for any* $i \in \operatorname{supp}(z)$ *and* $\sum_{j \in \operatorname{supp}(z)} d_j x_j < \infty$.

From [17, Theorem 3.1] we can easily obtain

Theorem 0.4. *$(h_\Phi, \|\cdot\|_\Phi)$ is smooth if and only if ϕ_n is differentiable in $(-1,1)$ for any $n\in\mathbb{N}$. Since ϕ_n is an even and convex function for any $n\in\mathbb{N}$, this implies that $(\phi_n)'(0)=0$ for any $n\in\mathbb{N}$.*

Also we need the following

Theorem 0.5. [*22, p. 148*] *$(l_\Phi, \|\cdot\|_\Phi)$ is reflexive if and only if Φ and Φ^* satisfy the δ_2 condition, that is if and only if $l_\Phi = h_\Phi$ and $l_{\Phi^*} = h_{\Phi^*}$.*

In the sequel we will apply the following results. The next one can be easily deduced from [20, Theorem 3] (see also [18, Theorem 2.3] in the case of Orlicz sequence spaces h_ϕ determined by ϕ which does not satisfy the δ_2 condition). Recall that a convex function $f:\mathbb{R}\to\mathbb{R}$ is called *strictly convex* in $[a,b]$ if for any $x,y\in[a,b]$, $x\neq y$ and $\alpha\in(0,1)$,

$$f(\alpha x+(1-\alpha)y) < \alpha f(x)+(1-\alpha)f(y).$$

Theorem 0.6. *Set for any $n\in\mathbb{N}$,*

$$\begin{aligned} E_n &= \{x\in[0,1), \phi_n(ax+(1-a)y) = a\phi_n(x)+(1-a)\phi_n(y) \\ &\qquad \text{for some } y>x \text{ and } 0<a<1\}\cup\{1\}, \\ F_n &= \{x\in[0,1] : \phi_n(x)\leqslant 1/2\} \end{aligned}$$

and

$$a_n = inf\{\phi_n(x) : x\in E_n\}.$$

Then $(h_\Phi, \|\cdot\|_\Phi)$ is strictly convex if and only if

a) *there exists at most one $n\in\mathbb{N}$ such that ϕ_n is not strictly convex on F_n; and*
b) *for any $m\in\mathbb{N}$, $m\neq n$, ϕ_m is strictly convex on $\{x\in\mathbb{R}\colon \phi_m(x)\leqslant 1-a_n\}$.*

In particular, if all functions ϕ_n are strictly convex in $[0,\phi_n^{-1}([0,1])]=[0,1]$ then $(h_\Phi, \|\cdot\|_\Phi)$ and $(l_\Phi^{(m)}, \|\cdot\|_\Phi)$ are strictly convex.
An Orlicz space $(h_\phi, \|\cdot\|_\phi)$ or $(l_\phi^{(m)}, \|\cdot\|_\phi)$ for $m\geqslant 3$ is strictly convex if and only if ϕ is strictly convex in the interval $[0,u_o]$, where $\phi(u_o)=1/2$.

Theorem 0.7 (The Šmulian theorem) [see e.g. 12, p. 243]. *Assume X is a Banach space and let $x\in S(X)$ be a smooth point. If $f_n, g_n\in S(X^*)$ are such that $f_n(x)\to 1$ and $g_n(x)\to 1$, then $f_n-g_n\to 0$ weakly* in X^*.*

Theorem 0.8 (The Mazur theorem) [see e.g. 12, p. 248]. *Let X be a separable Banach space. Then the set of all smooth points of X is a dense G_δ subset of X.*

1. General results

We start with

Lemma 1.1. *Let X be a Banach space. For $n\in\mathbb{N}$, let $x_n\in X$ and $\|x_n-x\|\to 0$ for some $x\in X$. For each $n\in\mathbb{N}$ fix $f_n\in S(X^*)$ with $f_n(x_n)=\|x_n\|$. Set for $n\in\mathbb{N}$*

$$A_n = cl(\{f_k : k\geqslant n\}),$$

where the closure is taken with respect to the weak-* topology in X^*. Let $f \in A(\{f_n\}) = \bigcap_{n=1}^{\infty} A_n$. Then $f(x) = \|x\|$ and moreover $\|f\| = 1$ for $x \neq 0$.

Proof. If $x = 0$, then obviously $f(x) = \|x\|$ for any $f \in A(\{f_n\})$. If $x \neq 0$, then $x_n/\|x_n\| \to x/\|x\|$. Hence without loss of generality we can assume that $\|x_n\| = \|x\| = 1$. Let $f \in A(\{f_n\})$. By the Banach-Alaoglu Theorem $\|f\| \leqslant 1$. Fix $\epsilon > 0$ and $n_o \in \mathbb{N}$ with $\|x_n - x\| \leqslant \epsilon/2$ for $n \geqslant n_o$. Also there exists $k \geqslant n_o$ with $|(f_k - f)x| \leqslant \epsilon/2$. Note that

$$|1 - f(x)| \leqslant |f_k(x_k - x)| + |(f_k - f)x| \leqslant \|f_k\|\|x_k - x\| + |(f_k - f)x| \leqslant \epsilon.$$

Consequently, $f(x) = \|x\| = 1 = \|f\|$, as required. □

Lemma 1.2. *Let $V \subset X$ be a linear subspace of a Banach space X. Take $x, y \in X$ satisfying*

$$\|y - v\| \leqslant \|x - v\|$$

for any $v \in V$. Then for any $v \in V$ there exists $f_v \in S(X^*)$, $f_v(v) = \|v\|$ and $f_v(x - y) = 0$.

Proof. Fix $v \in V$. Without loss of generality, we can assume that $v \neq 0$. Since V is a linear subspace, for any $k \in \mathbb{R}$

$$\|y - kv\| \leqslant \|x - kv\|.$$

Hence for any $k \neq 0$,

$$\|y/k - v\| \leqslant \|x/k - v\|.$$

Let $k \in \mathbb{N} \setminus \{0\}$. Choose $f_k \in S(X^*)$ such that

$$f_k(x/k - v) = \|x/k - v\|.$$

Note that

$$f_k(y/k - v) \leqslant \|y/k - v\| \leqslant \|x/k - v\|.$$

Hence $f_k(x/k - y/k) \geqslant 0$, which gives

$$f_k(y - x) \leqslant 0$$

for any $k \in \mathbb{N} \setminus \{0\}$. Observe that $\|x/k - v - (-v)\| \to 0$. By Lemma 1.1 and the Banach-Alaoglu Theorem there exists $f_+ \in A(\{f_k\})$ such that $f_+(-v) = \|v\|$. Since $f_+ \in A(\{f_k\})$, $f_+(y - x) \leqslant 0$. For $k \in \mathbb{Z} \setminus \mathbb{N}$, by the above reasoning, $f_k(x/k - y/k) \geqslant 0$, and consequently

$$f_k(y - x) \geqslant 0.$$

Again, by Lemma 1.1 and the Banach-Alaoglu Theorem, there exists $f_- \in S(X^*)$, $f_-(-v) = \|v\|$ such that

$$f_-(y - x) \geqslant 0.$$

If $f_-(y - x) = 0$ or $f_+(y - x) = 0$, the lemma is proved. In the opposite case, there exists $a \in (0, 1)$ such that

$$af_-(x - y) + (1 - a)f_+(x - y) = 0$$

Set

$$f_v = -(af_- + (1-a)f_+).$$

Obviously, $\|f_v\| = f_v(v/\|v\|) = 1$ and $f_v(y-x) = 0$. The lemma is proved. $\square$

Theorem 1.3. *Let X be a real Banach space and let $V \subset X$ be a linear subspace. Assume that V is an existence set and $V \neq \{0\}$. Put*

$$G_V = \{v \in V \setminus \{0\} : \textit{there exists exactly one } f \in S(X^*) : f(v) = \|v\|\}. \tag{1.1}$$

Assume that the norm closure of G_V in X is equal to V. Then there exists exactly one projection $P \in \mathscr{P}(X,V)$ such that $\|P\| = 1$.

Proof. Fix $x \in X$. Since V is an existence set, there exists $y \in V$ such that

$$\|y - v\| \leqslant \|x - v\|$$

for any $v \in V$. By Lemma 1.2 applied to x and y, for any $v \in V$ there exists $f_v \in S(X^*)$, $f_v(v) = \|v\|$, such that

$$f_v(x-y) = 0. \tag{1.2}$$

Now we show that there exists exactly one $y \in V$ satisfying (1.2) for any $v \in G_V$. Assume, on the contrary, that there exist $y_1, y_2 \in V$, $y_1 \neq y_2$ satisfying (1.2) for any $v \in G_V$. Since $cl(G_V) = V$, there exists $\{z_n\} \subset G_V$ with

$$\|z_n - (y_1 - y_2)\| \to 0. \tag{1.3}$$

By (1.2), for any $n \in \mathbb{N}$ there exists $f_n^1 \in S(X^*)$ and $f_n^2 \in S(X^*)$, with $f_n^i(z_n) = \|z_n\|$ for $i = 1,2$ such that $f_n^i(x - y_i) = 0$. Since $z_n \in G_V$, $f_n^1 = f_n^2$. Hence for any $n \in \mathbb{N}$

$$f_n^1(y_1 - y_2) = 0. \tag{1.4}$$

Since $\|z_n - (y_1 - y_2)\| \to 0$, and $f_n^1(z_n) = \|z_n\|$, by Lemma 1.1, for any $f \in A(\{f_n^1\})$

$$f(y_1 - y_2) = \|y_1 - y_2\|. \tag{1.5}$$

By (1.4), $f(y_1 - y_2) = 0$, which gives, $y_1 = y_2$; a contradiction.

Now, for any $x \in X$, let Px denote the only element $y \in V$ satisfying (1.2) for any $v \in G_V$. We show that P is a linear mapping. To do this, fix $x_1, x_2 \in X$ and $v \in G_V$. Note that $f_v(x_1 - Px_1) = 0$ and $f_v(x_2 - Px_2) = 0$, where f_v denotes the only supporting functional for v in X^*. Consequently, for any $v \in G_V$, $a_1, a_2 \in \mathbb{R}$,

$$f_v(a_1x_1 + a_2x_2 - (a_1Px_1 + a_2Px_2)) = 0. \tag{1.6}$$

Since for any $x \in X$ there exists exactly one element satisfying (1.2) for any $v \in G_V$, by (1.6),

$$P(a_1x_1 + a_2x_2) = a_1P(x_1) + a_2P(x_2),$$

which shows that P is a linear mapping. Taking $v = 0$ we get $\|Px\| \leqslant \|x\|$. Since $Pv = v$ for any $v \in V$ and $V \neq \{0\}$, we get $\|P\| = 1$. By the above proof there exists exactly one projection $P \in \mathscr{P}(X,V)$ of norm one. The proof is complete. $\square$

Corollary 1.4. *Assume X is a smooth space. If a linear subspace $V \subset X, V \neq \{0\}$, is an existence set then V is one-complemented in X.*

Proof. Note that in our case $G_V = V \setminus \{0\}$. Hence the statement follows immediately from Theorem 1.3. □

Corollary 1.5. *Assume that X is a strictly convex Banach space such that X is one-complemented in X^{**}. (In particular, X can be a reflexive space or $X = Z^*$ for some Banach space Z.) If a linear subspace $V \subset X$, $V \neq \{0\}$, is an optimal set satisfying the requirements of Theorem 1.3, then V is one-complemented.*

Proof. By [4, Prop. 2], V is an existence set in X. By Theorem 1.3, V is one-complemented in X. □

Now we present another class of subspaces, which has been introduced in [33], satisfying the assumptions of Theorem 1.3.

Definition 1.6. [33] Let X be a Banach space and let $V \subset X$ be a linear subspace. V is called weakly separating if any $g \in \mathrm{ext}(V^*)$ has exactly one Hahn-Banach extension $f \in S(X^*)$.

Let us define

$$G_{V,1} = \{v \in V \setminus \{0\} : \text{there exists exactly one } g \in S(V^*), g(v) = \|v\|\}. \tag{1.7}$$

Theorem 1.7. *Let $V \subset X$, $V \neq \{0\}$ be a separable, weakly separating subspace. Then the norm closure of G_V is equal to V.*

Proof. Since V is separable, by Theorem 0.8 applied to V, the set $G_{V,1}$ is dense in V. To finish the proof, it is enough to show that $G_{V,1} = G_V$. Fix $v \in G_{V,1}$ and $g \in S(V^*)$ satisfying $g(v) = \|v\|$. Since $v \in G_{V,1}$, $g \in \mathrm{ext}(V^*)$. Now we show that there exists exactly one $f \in S(X^*)$ such that $f(v) = \|v\|$, which means that $v \in G_V$. Indeed, assume that there exist $f_1, f_2 \in S(X^*)$, $f_1 \neq f_2$ such that $f_i(v) = \|v\|$ for $i = 1,2$. Hence $f_i|_V(v) = \|v\|$for $i = 1,2$. Since $v \in G_{V,1}$, $f_1|_V = f_2|_V = g$. Since $g \in \mathrm{ext}(V^*)$, and V is weakly separating, $f_1 = f_2$; a contradiction. It is obvious, that $G_V \subset G_{V,1}$. The proof is complete. □

Now, after [24], we present some examples of weakly separating subspaces of $C(E)$, where E is a compact set and $C(E)$ is the space of continuous, real-valued functions defined on E equipped with the supremum norm. For $t \in E$, define $\hat{t} \in (C(E))^*$ by:

$$\hat{t}(f) = f(t) \tag{1.8}$$

for $f \in C(E)$. We need the following

Theorem 1.8. [24] *A linear subspace $V \subset C(E)$ is weakly separating if and only if for any $t_1, t_2 \in \sigma(V)$, $t_1 \neq t_2$, there exist $v_1, v_2 \in V$ such that*

$$v_1(t_1) \neq v_1(t_2)$$

and

$$v_2(t_1) \neq -v_2(t_2).$$

Here

$$\sigma(V) = \{t \in E : \hat{t}|_V \in \text{ext}(V^*)\}.$$

Applying Theorem 1.8 one can easily show

Example 1.9 [24]. Let $V \subset C(E)$ be a linear subspace such that $1 \in V$ and $\hat{t_1}|_V \neq \hat{t_2}|_V$ for any $t_1, t_2 \in E$, $t_1 \neq t_2$. Then V is weakly separating.

Recall that n-dimensional subspace V of $C(E)$ is called a Haar subspace if and only if for any $t_1, \ldots, t_n \in E$, $t_i \neq t_j$ for $i \neq j$ the set $\{\hat{t_1}|_V, \ldots, \hat{t_n}|_V\}$ is linearly independent in V^*.

Example 1.10 [24]. Let $V \subset C(E)$ be a Haar subspace of dimension $\geqslant 2$. Then V is weakly separating. Also any subspace $W \subset C(E)$ containing two-dimensional Haar subspace is weakly separating.

2. Particular cases

Now applying Theorem 1.3, we show that any linear subspace $V \neq \{0\}$ of c and c_o which is an existence set must be one-complemented. To do this, we need a well known

Lemma 2.1. *Let X and Y be two Banach spaces and let $T : X \to Y$ be a linear isometry. Let $V \subset X$. Then V is an existence set in X if and only if $T(V)$ is an existence set in $T(X)$. If $V \subset X$, $V \neq \{0\}$, is a linear subspace then V is one-complemented in X if and only if $T(V)$ is one-complemented in $T(X)$.*

Proof. Since V is an existence set for any $x \in X$ there exists $Qx \in V$ such that for any $v \in V$

$$\|v - Qx\| \leqslant \|v - x\|.$$

Hence

$$\|T(v) - T(Qx)\| \leqslant \|T(v) - T(x)\|$$

for any $v \in V$. This means that $T(V)$ is an existence set in $T(X)$. Applying the above reasoning to $T(X)$ and T^{-1} we get the first claim of our lemma.

Now suppose that V is one-complemented subspace of X. Take a projection $P_o \in \mathscr{P}(X, V)$, $\|P_o\| = 1$. Set $P_1 = T \circ P_o \circ T^{-1}$. Then obviously $P_1 \in \mathscr{P}(T(X), T(V))$, and $\|P_1\| = 1$. The converse is obvious. $\square$

Theorem 2.2. *Let $V \subset c$, $V \neq \{0\}$, be a linear subspace which is an existence set. Then V is one-complemented in c.*

Proof. Since by the Hahn-Banach theorem any one-dimensional subspace of c is one-complemented, we can assume that $\dim(V) \geqslant 2$. For each $i \in \mathbb{N}$ set

$$C_{i,1} = \{j \in \mathbb{N}, j \neq i,\ v_i = v_j \quad \text{for any } v \in V\},$$
$$C_{i,2} = \{j \in \mathbb{N}, j \neq i,\ v_i = -v_j \quad \text{for any } v \in V\}$$

and

$$C_i = \{i\} \cup C_{i,1} \cup C_{i,2}.$$

Observe that for any $i, j \in \mathbb{N}$,

$$C_i \cap C_j = \emptyset \quad \text{or} \ \ C_i = C_j. \tag{2.1}$$

Moreover it is clear that

$$\bigcup_{i=1}^{\infty} C_i = \mathbb{N}.$$

First assume that for any $n \in \mathbb{N}$

$$\bigcup_{i=1}^{n} C_i \neq \mathbb{N}. \tag{2.2}$$

Define

$$i_1 = 1 = \min(C_1),$$
$$i_2 = \min(\mathbb{N} \setminus C_1),$$

and

$$i_n = \min\left(\mathbb{N} \setminus \bigcup_{j=1}^{n-1} C_{i_j}\right).$$

Note that, by (2.2), $\mathbb{N} \setminus \bigcup_{j=1}^{n-1} C_{i_j} \neq \emptyset$, so the above definition is correct. Put

$$N_1 = \{n \in \mathbb{N} : \operatorname{card}(C_{i_n}) < \infty\}.$$

Since $V \subset c$, $N_1 = \mathbb{N}$ or there exists exactly one $n_o \in \mathbb{N}$ such that

$$N_1 = \mathbb{N} \setminus \{n_o\}. \tag{2.3}$$

Set

$$V_1 = \{x \in c : \text{there exists } v \in V, \ x_j = v_{i_j} \quad \text{for } j \in N_1\}. \tag{2.4}$$

Note that for any $x \in V_1$ there exists exactly one $v \in V$ such that (2.4) is satisfied. Let us denote it by Lx. Observe that L is a linear isometry between V_1 and V equal to the identity mapping if $C_i = \{i\}$ for any $i \in \mathbb{N}$.

Now we show that V_1 is an existence set in c. First we assume that $N_1 = \mathbb{N}$. Take $x \in c$. Define a sequence Tx by

$$T(x)_k = \begin{cases} x_j, & \text{if } \ k \in C_{i_j,1} \cup \{i_j\}, \\ -x_j & \text{if } \ k \in C_{i_j,2}. \end{cases} \tag{2.5}$$

By (2.1), $T(x)$ is properly defined. If $N_1 = \mathbb{N} \setminus \{n_o\}$ (see (2.3)), we modify our definition for $k \in C_{i_{n_o},i}$, if card $(C_{i_{n_o},i}) = \infty$, for $i = 1$ or $i = 2$ setting

$$T(x)_k = \lim_n x_n. \tag{2.6}$$

By (2.2), (2.5) and (2.6) T is a linear isometry going from c into c. Observe that $T|_{V_1} = L$. Since V is an existence set in c, $V = T(V_1) \subset T(c)$ is an existence set in $T(c)$. By Lemma 2.1, $V_1 = T^{-1}(V)$ is an existence set in c. $\square$

Now we consider two cases.

Case I. For any $i \in \mathbb{N}$ there exists $v \in V_1$ and $w \in V_1$ with $v_i \neq lim_n v_n$ and $w_i \neq -lim_n w_n$. We show that V_1 is a weakly separating subspace of c. Note that c is isometric to $C(E)$, where $E = \{0, 1/n : n \in \mathbb{N}\}$ and an isometry $I : c \to C(E)$ is given by:

$$(Ix)(0) = \lim_n(x) \quad \text{and} \qquad (Ix)(1/n) = x_n.$$

Hence it is enough to show that $I(V_1)$ is a weakly separating subspace of $C(E)$. To do this, we apply Theorem 1.8. Take any $t_1, t_2 \in \sigma(V_1)$, (see Theorem 1.8) $t_1 \neq t_2$. If $t_i \neq 0$ for $i = 1, 2$, by the construction of V_1, there exists $v_1 \in V_1$, such that

$$I(v_1)(t_1) \neq I(v_1)(t_2)$$

and $v_2 \in V_1$ with

$$I(v_2)(t_1) \neq -I(v_2)(t_2).$$

If $t_1 = 0$ or $t_2 = 0$, we can find $v_1 \in V_1$ and $v_2 \in V_1$ satisfying the above conditions, because we consider the Case I. By Theorem 1.8, $I(V_1)$ is a weakly separating subspace of $C(E)$ and consequently V_1 is a weakly separating subspace of c. By Theorems 1.3, 1.7 and Lemma 2.1, $V = T(V_1)$ is one-complemented in $T(c)$. Let $P_1 \in \mathscr{P}(T(c), V)$ be a projection of norm one. Set

$$Q_1 = P_1 \circ T \circ R,$$

where

$$Rx = (x_{i_1}, x_{i_2}, \ldots)$$

for $x \in c$ and $i_1, i_2, \ldots \in N_1$. It is clear that $\|Q_1\| \leqslant 1$. Since $R|_V = T^{-1}|_V = L^{-1}$ and $T|_{V_1} = L$, Q_1 is a norm-one projection belonging to $\mathscr{P}(c, V)$, which completes our proof in this case.

Case II. There exists $i \in \mathbb{N}$ such that for any $w \in V_1$ $w_i = \lim_n w_n$ or there exists $i \in \mathbb{N}$ such that for any $w \in V_1$ $w_i = -\lim_n w_n$. By definition of V_1, there exists exactly one $i \in \mathbb{N}$ satisfying the above condition. Without loss of generality, we can assume that $i = 1$ and that for any $w \in V$, $w_1 = \lim_n w_n$. Let $S : c \to c$ be given by:

$$S(x) = (\lim_n x_n, x). \tag{2.7}$$

Note that $V_1 \subset S(c)$. Set

$$V_2 = \{(v_2, \ldots) : v = (v_1, v_2, \ldots) \in V_1\}.$$

It is clear that $V_1 = S(V_2)$. Since V_1 is an existence set in c, and $V_1 \subset S(c)$, V_1 is an existence set in $S(c)$. By Lemma 2.1, V_2 is an existence set in c. Reasoning as in the Case I, we get that V_2 is a weakly separating subspace of c. By Theorems 1.3 and 1.7 there exists $P_2 : c \to V_2$, a linear projection of norm one. Define $R_1 : c \to V_1$ by

$$R_1(x) = (S \circ P_2)(x_2, \ldots).$$

It is clear that $\|R_1\| = 1$, $R_1(c) \subset V_1$ and $R_1|_{V_1} = id_{V_1}$. Hence V_1 is one-complemented in c. Reasoning as in the Case I, we get that V is one-complemented in c. This completes the proof of our theorem under assumption (2.2).

If (2.2) is not satisfied, without loss of generality, we can assume that

$$\mathbb{N} = \bigcup_{j=1}^{n} C_{i_j},$$

where $1 = i_1 < i_2 <, \dots, < i_n$ and $C_{i_j} \cap C_{i_k} = \emptyset$ for $j \neq k$. Let $T : l_\infty^{(n)} \to c$ be defined by (2.5) and (2.6). Note that $V \subset T(l_\infty^{(n)})$. Let $V_1 = T^{-1}(V)$. Reasoning as in the first part of the proof, we get that V_1 is an existence set in $l_\infty^{(n)}$ and V_1 is a weakly separating subspace of $l_\infty^{(n)}$. By Theorems 1.3 and 1.7 there exists a norm-one projection $P_3 \in \mathscr{P}(l_\infty^{(n)}, V_1)$. Set

$$Q_2 = T \circ P_3 \circ R.$$

Here $R : c \to l_\infty^{(n)}$ is given by

$$Rx = (x_{i_1}, \dots, x_{i_n}).$$

Since $R|_V = T^{-1}|_V$, $Q_2 \in \mathscr{P}(c, V)$ and $\|Q_2\| = 1$. The proof is complete. □

In an analogous way we can prove

Theorem 2.3. *Let $V \subset c_o$, $V \neq \{0\}$, be a linear subspace which is an existence set. Then V is one-complemented in c_o.*

Now we consider the case $X = l_1$.

Theorem 2.4. *Let $V \subset l_1$, $V \neq \{0\}$, be a linear subspace, which is an existence set. Then V is one-complemented in l_1.*

Proof. Set

$$\operatorname{supp}(V) = \bigcup_{v \in V} \operatorname{supp}(v). \tag{2.8}$$

Without loss of generality we can assume that $\operatorname{supp}(V) = \mathbb{N}$. We show that $G_{V,1} = G_V$ (see (1.1) and (1.8)). Since $(l_1)^* = l_\infty$, $v \in G_{V,1}$ if and only if $\operatorname{supp}(v) = \operatorname{supp}(V) = \mathbb{N}$. Hence $v \in G_V$. By Theorems 0.8, 1.3 and 1.7, V is one-complemented in l_1. □

Now we consider the case of Musielak-Orlicz sequence spaces equipped with the Luxemburg norm (see (0.2)). For a linear subspace $V \subset l_\Phi$ (see (0.1)), $n \in \mathbb{N}$ and $v \in V$ set

$$I_n(v) = v_n. \tag{2.9}$$

Notice that $I_n \in V^*$ for any $n \in \mathbb{N}$. We start with

Lemma 2.5. *Let $V \subset h_\Phi$ (see (0.3)) be a linear subspace. Assume that $\operatorname{supp}(V) = \mathbb{N}$. Let $v = (v_1, v_2, \dots) \in G_{V,1}$, (see (1.8)), $\|v\| = 1$. If $v \in G_{V,1} \setminus G_V$, then there exist $n_o \in \mathbb{N}$ such that v is a norming point for a functional $J_{n_o} = I_{n_o}/\|I_{n_o}\|$ and ϕ_{n_o} is not differentiable at $v_{n_o} = \|I_{n_o}\|$.*

Proof. Take $v \in G_{V,1} \setminus G_V$ with $\|v\| = 1$. Hence there exists $f_1, f_2 \in (l_\Phi)^*$, supporting functionals for v, such that $f_1|_{h_\Phi} \neq f_2|_{h_\Phi}$. Since the standard unit vectors e_i form a Schauder basis of h_Φ, $f_1(e_{n_o}) \neq f_2(e_{n_o})$ for some $n_o \in \mathbb{N}$. By Theorem 0.3, f_i is a regular functional for $i = 1, 2$. Moreover $f_1 = f_{z^1}$ and $f_2 = f_{z^2}$, where for $i = 1, 2$ and any $j \in \operatorname{supp}(z^i)$

$$z^i_j = d^i_j \Big/ \Big(\sum_{k \in \operatorname{supp}(z^i)} d^i_k x_k \Big).$$

Here $d^i_j \in \Delta\Phi_j(v)$ for any $j \in \operatorname{supp}(z^i)$ is so chosen that

$$\sum_{k \in \operatorname{supp}(z^i)} d_k x_k < \infty.$$

Since $f_1(e_{n_o}) \neq f_2(e_{n_o})$, $n_o \in \operatorname{supp}(z^1) \cup \operatorname{supp}(z^2)$. Replacing f_1 by $(f_1 + f_2)/2$, if necessary, we can assume that $n_o \in \operatorname{supp}(z^1) \cap \operatorname{supp}(z^2)$ and $d^1_{n_o} \neq d^2_{n_o}$. Hence ϕ_{n_o} is not differentiable at v_{n_o}, as required. Note that for any $y \in l_\Phi$,

$$f_1(y) = \Big(\sum_{n \in \operatorname{supp}(z^1), n \neq n_o} d^1_n y_n + d^1_{n_o} y_{n_o} \Big) \Big/ \Big(\sum_{n \in \operatorname{supp}(z^1)} d^1_n v_n \Big).$$

Now, define for $y \in l_\Phi$,

$$g_2(y) = \Big(\sum_{n \in \operatorname{supp}(z^1), n \neq n_o} d^1_n y_n + d^2_{n_o} y_{n_o} \Big) \Big/ \Big(\sum_{n \in \operatorname{supp}(z^1), n \neq n_o} d^1_n v_n + d^2_{n_o} v_{n_o} \Big),$$

By Theorem 0.3, g_2 is a supporting functional for v. Since $v \in G_{V,1}, f_1|_V = g_2|_V$. Hence $n_o \in \operatorname{supp}(v)$. If not, since $\operatorname{supp}(V) = \mathbb{N}$, there exists $w \in V$ such that $w_{n_o} \neq 0$. Hence

$$f_1(w) - g_2(w) = (d^1_{n_o} - d^2_{n_o}) w_{n_o} \Big/ \Big(\sum_{n \in \operatorname{supp}(z^1), n \neq n_o} d^1_n v_n \Big) \neq 0;$$

a contradiction.

Now we show that $f_1|_V = J_{n_o}$. This is obvious if $\operatorname{supp}(v) = \{n_o\}$. So assume that $\operatorname{supp}(v) \neq \{n_o\}$. If $f_1|_V \neq J_{n_o}$, then there exists $w \in V$ such that $f_1(w) \neq 0$ and $I_{n_o}(w) = 0$. Hence

$$f_1(w) = \Big(\sum_{n \in \operatorname{supp}(z^1), n \neq n_o} d^1_n w_n \Big) \Big/ \Big(\sum_{n \in \operatorname{supp}(z^1)} d^1_n v_n \Big),$$

and

$$g_2(w) = \Big(\sum_{n \in \operatorname{supp}(z^1), n \neq n_o} d^1_n w_n \Big) \Big/ \Big(\sum_{n \in \operatorname{supp}(z^1), n \neq n_o} d^1_n v_n + d^2_{n_o} v_{n_o} \Big).$$

Since $v_{n_o} \neq 0$ and $d^1_{n_o} \neq d^2_{n_o}$, $f_1(w) \neq g_2(w)$; a contradiction. To end the proof, note that

$$1 = f_1(v) = J_{n_o}(v) = v_{n_o} / \|I_{n_o}\|,$$

which gives $v_{n_o} = \|I_{n_o}\|$. The proof is complete. □

We also need the following very simple

Lemma 2.6. *h_Φ is contained in c_o.*

Proof. Assume that h_Φ is not a subset of c_o. Fix $z = (z_1, z_2, \ldots) \in h_\Phi \setminus c_o$. Hence we can find $d > 0$ and a subsequence n_k such that $|z_{n_k}| \geqslant d$. Set $\lambda = 1/d$. Since $\phi_n(1) = 1$ for any $n \in \mathbb{N}$,

$$\rho_\Phi(\lambda z) = \sum_{n=1}^{\infty} \phi_n(\lambda |z_n|) \geqslant \sum_{k=1}^{\infty} \phi_{n_k}(\lambda |z_{n_k}|) \geqslant \sum_{k=1}^{\infty} \phi_{n_k}(1) = +\infty.$$

Consequently, $z \notin h_\Phi$: a contradiction. $\square$

Theorem 2.7. *Let Φ be a Musielak – Orlicz function. Let $V \subset h_\Phi$, $V \neq \{0\}$, be a linear subspace which is an existence set. Set*

$$N_1 = \{n \in \mathbb{N} : I_n \neq 0, \phi_n \text{ is not differentiable at } \|I_n\|\} \tag{2.10}$$

and for any $n \in N_1$

$$Z_n = \{v \in S(V) : J_n(v) = (I_n/\|I_n\|)(v) = 1\}. \tag{2.11}$$

Assume that for any $n \in N_1$, $\mathrm{int}(Z_n)$ with respect to $S(V)$ is empty where for any $D \subset S(V)$, $\mathrm{int}(D)$ denotes the interior of D with respect to $S(V)$. Here $S(V)$ is considered with the topology induced by the norm topology from h_Φ. If $\liminf_n \|I_n\| > 0$ or if there exist $a > 0$ and $n_o \in \mathbb{N}$ such that ϕ_n are differentiable in $(0, a]$, for $n \geqslant n_o$, then V is one-complemented in h_Φ.

Proof. Note that, by the Hahn-Banach theorem, any one-dimensional subspace of h_Φ is one-complemented. Hence we can assume that $\dim V \geqslant 2$. We will apply Theorem 1.3. To do this, it is necessary to show that $\mathrm{cl}(G_V) = V$.

First we assume that $\mathrm{supp}(V) = \mathbb{N}$. Since h_Φ is separable, by Theorem 0.8, $\mathrm{cl}(G_{V,1}) = V$. To end the proof, we demonstrate that any $v \in G_{V,1}$, can be approximated by elements belonging to G_V. Fix $v = (v_1, v_2, \ldots) \in G_{V,1} \setminus G_V$, $\|v\| = 1$. By Lemma 2.5, there exists $n_o \in \mathbb{N}$ with $I_{n_o} \neq 0$, such that ϕ_{n_o} is not differentiable at v_{n_o} and v is a norming point for J_{n_o}. Set

$$S_k = \{w \in S(V) : \|w - v\| \leqslant 1/k\}. \tag{2.12}$$

Now we show that for any $k \in \mathbb{N}$ there exists $w^k \in G_{V,1} \cap S_k$, such that

$$w^k \notin \bigcup_{j \in N_1, j \leqslant k} (Z_j). \tag{2.13}$$

Indeed, if there exists $k \in \mathbb{N}$ such that (2.13) is not satisfied for any $w \in G_{V,1} \cap S_k$, by Theorem 0.8,

$$S_k = \mathrm{cl}(G_{V,1} \cap S_k) = \bigcup_{j \in N_1, j \leqslant k} (Z_j \cap S_k). \tag{2.14}$$

Since h_Φ is a Banach space, and V is closed, S_k is a complete, metric space (with respect to the norm topology). By the Baire Property and (2.14), $\mathrm{int}(Z_j)$ with respect to $S(V)$ is nonempty for some $j \in N_1$; a contradiction.

Now we show that for k sufficiently large there exists $w^k \in S_k \cap G_V$ satisfying (2.13).

First assume that there exist $a > 0$ and $n_o \in \mathbb{N}$ such that ϕ_n is differentiable in $(0, a]$ for any $n \geqslant n_o$. Fix, for any $k \in \mathbb{N}$, $w^k \in (G_{V,1} \cap S_k) \setminus \bigcup_{j \in N_1, j \leqslant k}(Z_j)$. If there exists a subsequence $\{n_k\}$ such that $w^{n_k} \notin G_V$, by Lemma 2.5, w^{n_k} is a norming point for some J_{m_k}. By (2.13), $m_k \geqslant n_k$ and consequently $m_k \to +\infty$. Note that

$$|1 - J_{m_k}(v)| = |J_{m_k}(w^{n_k} - v)| \leqslant \|w_{n_k} - v\| \to 0.$$

Since $v \in G_{V,1}$ and v is a norming point for J_{n_o}, by Theorem 0.7 applied to $f_k = J_{m_k}$, $g_k = J_{n_o}$, and V,

$$I_{m_k}/\|I_{m_k}\| = J_{m_k} \to J_{n_o} = I_{n_o}/\|I_{n_o}\|. \tag{2.15}$$

weakly* in V^*. Hence $J_{m_k}(v) \to J_{n_o}(v) = 1$. Consequently, since $\|I_{m_k}\| \leqslant 1$,

$$I_{m_k}(v) - \|I_{m_k}\| = v_{m_k} - \|I_{m_k}\| \to 0.$$

By Lemma 2.6, $\lim_n v_n = 0$ which gives that $\|I_{m_k}\| \to 0$. Since there exist $a > 0$ and $n_o \in \mathbb{N}$ such that ϕ_n is differentiable in $(0, a]$ for any $n \geqslant n_o$, N_1 is a finite set. Hence for $k \geqslant k_o \phi_{m_k}$ is differentiable at $\|I_m\|$ for $m \geqslant n_{k_o}$. By (2.13) and Lemma 2.5, $w^k \in G_V$ for $k \geqslant k_o$: a contradiction.

If $\liminf \|I_n\| > 0$, we proceed in the same way as above. Hence we have proved that G_V is a dense subset of V. By Theorem 1.3, V is one-complemented in h_Φ, which completes the proof in the case when $\mathrm{supp}(V) = \mathbb{N}$.

If $\mathrm{supp}(V) \neq \mathbb{N}$, set

$$X_1 = \{x \in h_\Phi : x_i = 0 \quad \text{for } i \notin \mathrm{supp}(V)\}.$$

Notice that $V \subset X_1$. Since V is an existence set in h_Φ, V is an existence set in X_1. Reasoning as in the the first part of the proof, we can show that V is one-complemented in X_1. Also X_1 is one-complemented in h_ϕ. Indeed, a mapping $P : h_\Phi \to X_1$, defined by

$$(Px)_k = \begin{cases} x_k, & \text{if } k \in \mathrm{supp}(V) \\ 0 & \text{if } k \notin \mathrm{supp}(V) \end{cases}$$

for $x \in h_\Phi$ is a norm-one projection from h_Φ onto X_1. Consequently, V is one-complemented in h_Φ. The proof is complete. □

Theorem 2.8. *Let $V \subset l_\Phi^{(m)}$, $V \neq \{0\}$, be a linear subspace which is an existence set. Assume that for any $n \in N_1$, (see* (2.10)) $\mathrm{int}(Z_n)$ *with respect to $S(V)$ is empty. Then V is one-complemented in $l_\Phi^{(m)}$.*

Proof. Goes in the same way as the proof of Theorem 2.7. □

Now we present some applications of Theorems 2.7 and 2.8.

Theorem 2.9. *Assume that $(h_\Phi, \|\cdot\|_\Phi)$ is a strictly convex space (compare with Theorem* 0.6). *If there exist $a > 0$ and $n_o \in \mathbb{N}$ such that ϕ_n are differentiable in $(0, a]$ for $n \geqslant n_o$ then any subspace $V \subset h_\Phi$, $V \neq \{0\}$, which is an existence set is one-complemented in h_Φ. If $(l_\Phi^{(m)}, \|\cdot\|_\Phi)$ is strictly convex, then any subspace $V \subset h_\Phi$, $V \neq \{0\}$, which is an existence set is one-complemented in $l_\Phi^{(m)}$.*

Proof. By Theorem 2.7, it is enough to show that $\mathrm{int}(Z_n)$ with respect to $S(V)$ is empty for any $n \in N_1$ (see (2.11)). But it follows immediately from the strict convexity of h_Φ. The case of $l_\Phi^{(m)}$ follows from Theorem 2.8. □

Corollary 2.10. *Assume that an Orlicz space* $(h_\phi, \|\cdot\|_\Phi)$ *is a strictly convex space* (*compare with Theorem* 0.6). *If there exists* $a>0$ *such that* ϕ *is differentiable in* $(0,a]$ *then any subspace* $V\subset h_\phi$, $V\neq\{0\}$, *which is an existence set is one-complemented in* h_Φ.

Corollary 2.11. *Assume that* $(l_\Phi, \|\cdot\|_\Phi)$ *is a strictly convex, reflexive space* (*compare with Theorem* 0.5 *and* 0.6). *If there exists* $a>0$ *and* $n_o\in\mathbb{N}$ *such that* ϕ_n *are differentiable in* $(0,a]$ *for* $n\geqslant n_o$ *then any optimal subspace* $V\subset l_\Phi$, $V\neq\{0\}$, *is one-complemented in* l_Φ.

Proof. Since l_Φ is reflexive and strictly convex, by [4, Prop. 2], any optimal subspace of l_Φ is an existence set. Also by Theorem 0.5, $l_\Phi = h_\Phi$. By Theorem 2.7, V is one-complemented in l_Φ. □

Now we present other applications of Theorem 2.7.

Theorem 2.12. *Let* Φ *be a Musielak-Orlicz function such that* ϕ_k *are strictly convex for* $k>k_o$. *Set for any* $F\in S((l_\Phi)^*)$,

$$Z_F = \{x\in S(h_\Phi) : F(x) = 1\}. \tag{2.16}$$

Fix $x\in Z_F$. *Then* $\dim(\mathrm{span}(Z_F - x)) < k_o$. *The same result holds true in* $l_\Phi^{(m)}$.

Proof. Assume on the contrary, that $\dim(\mathrm{span}(Z_F - x)) \geqslant k_o$. Hence there exists $x^1,\dots,x^{k_o+1}\in Z_F$ such that $y^i = x^{i+1} - x^1$, $i=1,\dots,k_o$ are linearly independent. Note that by the definition of Z_F,

$$\begin{aligned}
1 &= F\left(\sum_{j=1}^{k_o+1} x^j/(k_o+1)\right)\\
&= \left\|\sum_{j=1}^{k_o+1} x^j/(k_o+1))\right\|_\Phi = \sum_{l=1}^{\infty}\phi_l\left(\sum_{j=1}^{k_o+1}(x^j/(k_o+1))_l\right)\\
&\leqslant (1/(k_o+1))\sum_{j=1}^{k_o+1}\left(\sum_{l=1}^{\infty}\phi_l((x^j)_l\right)\\
&\leqslant (1/(k_o+1))\sum_{j=1}^{k_o+1}\rho_\Phi(x^j) = 1.
\end{aligned}\tag{2.17}$$

Since for $k>k_o$ ϕ_k are strictly convex, by (2.17), $(x^j)_l = (x^1)_l$ for $l>k_o$ and $j=1,\dots,k_o+1$. Hence for $j=1,\dots,k_o$,

$$y^j = x^{j+1} - x^1 = ((x^j - x^1)_1,\dots,(x^j-x^1)_{k_o},0,\dots,).$$

Since y^j, $j=1,\dots,k_o$ are linearly independent,

$$\det((y^j)_k)_{j,k=1,\dots,k_o}\neq 0. \tag{2.18}$$

Because of (2.18) for any $j=2,\dots,k_o$ there exists $l\in\{2,\dots,k_o\}$ with $(x^j)_l\neq (x^1)_L$. By (2.17), for any $l=1,\dots,k_o$, ϕ_l is an affine function in some interval

$E_l = [c_l, d_l]$, $c_l < d_l$ containing all the coordinates $(x^j)_l$, $j = 1, \ldots, k_o + 1$. Assume that for $l = 1, \ldots, k_o + 1$ and $x \in E_l$,

$$\phi_l(x) = a_l x + b_l.$$

By (2.17) for any $j = 1, \ldots, k_o + 1$,

$$\sum_{l=1}^{k_o} a_l (x_j)_l + b_l = 1 - \sum_{l=k_o+1}^{\infty} \phi_l((x_1)_l). \tag{2.19}$$

Since all the functions ϕ_l are even and strictly increasing in $[0, +\infty)$, $a_l \neq 0$ for $l = 1, \ldots, k_o$. By (2.19) we get for $j = 1, \ldots, k_o$,

$$\sum_{l=1}^{k_o} a_l (y^j)_l = 0,$$

which contradicts (2.18).

In the case of $l_\Phi^{(m)}$ the proof goes in the same manner, so we omit it. $\square$

An easy consequence of Theorem 2.12 is

Corollary 2.13. *Let Φ satisfy the assumptions of Theorem 2.12 with some $k_o \geqslant 1$. Then for any subspace $V \subset h_\Phi$ $\dim(V) \geqslant k_o + 1$ and for any $n \in \mathbb{N}$ $\operatorname{int}(Z_n)$ with respect to $S(V)$ is empty. In particular if there exist $a > 0$ and $n_o \in \mathbb{N}$ such that ϕ_n are differentiable in $(0, a]$ for $n \geqslant n_o$, then any subspace of h_Φ of dimension $\geqslant k_o + 1$ which is an existence set is one-complemented in l_Φ. Analogously, any subspace $V \subset l_\Phi^{(m)}$, $\dim(V) \geqslant k_o + 1$, which is an existence set in $l_\Phi^{(m)}$ is one-complemented in $l_\Phi^{(m)}$.*

Proof. Fix $V \subset h_\Phi$ with $\dim(V) \geqslant k_o + 1$. By Theorem 2.12, for any $n \in \mathbb{N}$, and $v \in Z_n$, $\dim(\operatorname{span}(Z_n) - v) < k_o$. This means that for any $n \in \mathbb{N}$ int (Z_n) with respect to $S(V)$ is empty. By Theorem 2.7, V is one-complemented in h_Φ. The case of $l_\Phi^{(m)}$ can be proved in the same manner. $\square$

Remark 2.14. Note that c_o, c and l_1 are non-smooth spaces. In general, Musielak-Orlicz or Orlicz spaces h_Φ and $l_\Phi^{(m)}$ satisfying the requirements of Theorem 2.7 and Theorem 2.8 are not smooth too (compare with Theorem 0.4). Moreover, the spaces h_Φ and $l_\Phi^{(m)}$ considered in Corallary 2.13 are in general neither strictly convex nor smooth (compare with Theorem 0.4 and 0.5). This shows that Theorem 1.3, which proof is very similar to that of [4, Prop. 5], can be applied in the non-smooth case in many concrete situations. Also, by Theorem 1.3, there exists unique projection of norm one onto any subspace $V \neq \{0\}$ which is an existence set satisfying the requirements of Theorems 2.7, 2.8 or the requirements of Corollary 2.13.

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