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A counterexample on non-archimedean regularity

By

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Abstract. A non-regular inductive sequence of non-archimedean reflexive Fréchet spaces is constructed. On the other hand, it is proved that every inductive sequence of reflexive Banach spaces over a spherically complete field is regular. Also, some applications are given.

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Introduction

A very interesting class of locally convex spaces over non-archimedean valued fields, because of its influence in the applications, are the locally convex inductive limits. We point out the central role that they play in the definition of a p-adic Laplace and Fourier Transform given in [6] and [7] respectively and in the index theory of p-adic differential equations (see e.g. [1]–[4] and [12]). The last of these references shows also the influence of these inductive limits in the study of the p-adic Monsky-Washnitzer cohomology.

Our main goal in this paper is to construct a non-regular inductive sequence of non-archimedean reflexive Fréchet spaces. This is the p-adic counterpart of the classical one of [11], which has a typically archimedean character, forcing us to use a p-adic machinery for our construction. Further, some applications are given.

On the other hand, a well-known classical result assures that every inductive sequence of real or complex reflexive Banach spaces is regular (see e.g. [10]). However, in the non-archimedean case, the validity of this result depends on the ground field. In fact, here we prove that it remains true for LB-spaces over spherically complete fields but fails when the spherical completeness of the ground field is dropped.

1. Preliminaries

Throughout this paper $K := (K, |\cdot|)$ is a non-archimedean non-trivially valued field that is complete with respect to the metric induced by the valuation $|\cdot|$. We assume that K contains the field $\mathbb{Q}_p$ of the p-adic numbers (p a prime number) and

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that the valuation on K extends the p-adic one, $|\cdot|_p$, on $\mathbb{Q}_p$. We denote by $\mathbb{Z}_p$ the set $\{x \in \mathbb{Q}_p : |x|_p \leqslant 1\}$ of the p-adic integers.

For fundamentals on normed and locally convex spaces over K we refer to [13] and [15] respectively. For the notions of (quasi)completeness and sequential completeness (which are the natural translations of the classical ones given in [9]) see e.g. [5]. Recall that completeness $\Longrightarrow$ quasicompleteness $\Longrightarrow$ sequential completeness.

Let $E := (E, \tau)$ be a locally convex space (all the locally convex spaces considered in this paper are over K). For a subset A of E, $E \backslash A := \{x \in E,\ x \notin A\}$, $\overline{A}^{\tau}$ is the closure of A in E, and $\tau | A$ is the restriction of τ to A. The set A is called *absolutely convex* if $0 \in A$ and $x, y \in A$, $\lambda, \mu \in K$, $\max(|\lambda|, |\mu|) \leqslant 1$ implies $\lambda x + \mu y \in A$; *compactoid* if for every zero neighbourhood U in E there is a finite set $B \subset E$ such that $A \subset U + \operatorname{aco} B$, where $\operatorname{aco} B$ is the absolutely convex hull of B. E is *nuclear* if for every continuous seminorm p on E there is a continuous seminorm q on E, $q \geqslant p$, such that the canonical map $E_q \to E_p$ sends the unit ball of E_q into a compactoid in E_p (by E_p and E_q we denote the normed spaces associated to p and q respectively). E is *semi-Montel* if every bounded subset of E is a compactoid. Nuclear spaces are semi-Montel.

A continuous seminorm p on E is called *polar* if $p = \sup\{|v| : v \in E',\ |v| \leqslant p\}$ (E' is the topological dual of E). E is *polar* if its topology is generated by a family of polar seminorms; *of countable type* if for every continuous seminorm p on E the associated normed space E_p is of countable type (recall that a normed space is said to be of countable type if it is the closed linear hull of a countable set). Nuclear spaces are of countable type. If K is spherically complete every locally convex space is polar. For any K, spaces of countable type are polar. The weak topology of a Hausdorff polar space is also Hausdorff. However, there exist Banach spaces over non-spherically complete fields with a trivial topological dual, see [13].

E is *polarly barrelled* if every polar absorbing set is a zero neighbourhood; *polarly bornological* if every K-polar set that absorbs every bounded set is a zero neighbourhood. A set $A \subset E$ is called *polar* (resp. *K-polar*) if for each $x \in E \backslash A$ there exists a v in E' (resp. a v in the algebraic dual E^*) such that $|v(A)| \leqslant 1$ and $|v(x)| > 1$.

By E'_b we denote the *strong dual* of E i.e. the space E' equipped with the so called *strong topology*, which is the one of uniform convergence on the bounded subsets of E. E'' is the *bidual* of E, that is, $(E'_b)'_b$. E is *semi-reflexive* if the natural map $j_E : E \to E''$ is surjective; *reflexive* if j_E is a surjective homeomorphism (or equivalently, if E is Hausdorff, polar, polarly barrelled and weakly quasicomplete, [15], Theorem 9.6). The term "weakly" refers to the weak topology. E is *Montel* if it is semi-Montel and reflexive. Every Fréchet nuclear space is Montel. As usual, a Fréchet space is a complete metrizable locally convex space.

A very interesting class of locally convex spaces (because of its influence in the applications, see the Introduction), to which is devoted the present paper, is formed by the locally convex inductive limits. An *inductive sequence* is an increasing sequence $E_1 \subset E_2 \subset \ldots$ of locally convex spaces E_n in such a way that each inclusion $E_n \to E_{n+1}$ is continuous. The *inductive limit* of this sequence is the space $E := \cup_n E_n$ equipped with the strongest locally convex topology τ_{ind} for which all the inclusions $E_n \to E$ are continuous (usually called *inductive topology*). $(E_n)_n$ is said to be *Hausdorff* if its inductive limit is Hausdorff; *regular* if for each bounded set B in E there exists an n such that $B \subset E_n$ and B is bounded in E_n. Every regular inductive sequence of Hausdorff spaces is Hausdorff. When the steps E_n of an inductive sequence are Banach (resp. Fréchet) spaces, $(E_n)_n$ is called an LB (resp. LF)-space.

By reversing the arrows we arrive at the following dual concept. A *projective sequence* is a decreasing sequence $F_1 \supset F_2 \supset \dots$ of locally convex spaces F_n in such a way that each inclusion $F_{n+1} \to F_n$ is continuous. The *projective limit* of this sequence is the space $F := \cap_n F_n$, equipped with the weakest locally convex topology τ_{proj} for which all the inclusions $F \to F_n$ are continuous (usually called *projective topology*). The name "dual" is justified by the following: If $(E_n)_n$ is an inductive sequence with inductive limit E and such that each E_n is dense in E_{n+1}, then the adjoint of each inclusion $E_n \to E_{n+1}$ is a continuous injective linear map $(E_{n+1})'_b \to (E_n)'_b$, that sends each $v \in E'_{n+1}$ to its restriction to E_n. Thus, identifying $(E'_{n+1})_b$ with its image under this adjoint, we obtain a projective sequence $((E_n)'_b)_n$, whose projective limit F is algebraic isomorphic to E'_b. Even more, there is a continuous bijective linear map

$$\Psi : E'_b \longrightarrow F, \quad v \longmapsto \Psi(v), \quad (\Psi(v))(x) = v(x), \quad x \in E_n, \quad n \in \mathbb{N}. \qquad (1)$$

If, in addition, $(E_n)_n$ is regular then Ψ is a homeomorphism ([5], Theorem 1.3.5). Also, the dual of a projective limit "is" an inductive limit, see Theorem 1.3.7 of [5] for details.

We devote the end of these Preliminaries to some spaces of differentiable functions that will be ones of the key ingredients of Counterexample 2.4. First we recall the definition of a $C^r(C^\infty)$-function (see [14]). Let X be a non-empty open subset of $\mathbb{Q}_p$. For $s \in \mathbb{N}$ set

$$\triangledown^s X := \{(x_1, \dots, x_s) \in X^s : \text{if } i \neq j \text{ then } x_i \neq x_j\}$$

(notice that $\nabla^1 X = X$). For $r \in \mathbb{N} \cup \{0\}$ and $f : X \to K$ let us define the rth order difference quotient $\Phi_r f : \triangledown^{r+1} X \to K$ inductively by $\Phi_0 f := f$ and, for $r \in \mathbb{N}$, $(x_1, \dots, x_{r+1}) \in \triangledown^{r+1} X$,

$$\Phi_r f(x_1, \dots, x_{r+1}) := \frac{\Phi_{r-1} f(x_1, x_3, \dots, x_{r+1}) - \Phi_{r-1} f(x_2, x_3, \dots, x_{r+1})}{x_1 - x_2}.$$

f is C^r *at a point* $\alpha \in X$ if the limit

$$\lim_{\nu \to a} \Phi_r f(\nu) \qquad (a := (\alpha, \dots, \alpha) \in X^{r+1}, \quad \nu \in \triangledown^{r+1} X)$$

exists. f is a C^r*-function* (*on* X) if f is C^r at each $\alpha \in X$, or equivalently ([14], Theorem 29.9), if $\Phi_r f$ can be extended to a continuous function $\overline{\Phi}_r f : X^{r+1} \to K$ (observe that this extension is unique since $\triangledown^{r+1} X$ is dense in X^{r+1}). We denote by $C^r(X)$ the vector space of all C^r-functions $X \to K$. Also, the elements of $C^\infty(X) := \cap_r C^r(X)$ are the C$^\infty$*-functions* (*on* X). Clearly, for each r, every $f \in C^r(X)$ is continuous on X.

We equip $C^\infty(X)$ with the topology τ_c^∞ of uniform convergence of $\overline{\Phi}_r$ on compact subsets of X^{r+1} for all r, which can be described as follows (see [15], Example 2.3). For each $m \in \{0, 1, \dots\}$, set $X_m := \{x \in \mathbb{Q}_p : |x|_p \leqslant p^m,\ B(x, p^{-m}) \subset X\}$ (for

$x \in \mathbb{Q}_p$ and $R > 0$, $B(x,R) := \{y \in \mathbb{Q}_p : |y-x|_p \leqslant R\}$). Each X_m is compact and open in $\mathbb{Q}_p$, $X_0 \subset X_1 \subset X_2 \subset \ldots$, $\cup_m X_m = X$. Then the topology τ_c^∞ is defined by the seminorms q_m ($m \in \{0,1,\ldots\}$) where, for each $f \in C^\infty(X)$,

$$q_m(f) = \max_{0 \leqslant r \leqslant m} \|\overline{\Phi}_r(f|X_m)\|_\infty$$

(for $Y \subset X$ and $f : X \to K$, $f|Y$ is the restriction of f to Y; $\| \ \|_\infty$ is the canonical supremum norm for continuous functions).

2. The counterexample

In Theorem 4 of [10] it was proved that any real or complex LB-space with reflexive steps is regular. However, as a simple application of some known p-adic facts, we prove in the next theorem that, for non-archimedean LB-spaces, this result remains true when K is spherically complete but fails for non-spherically complete K, revealing this failure a sharp contrast with the classical situation.

Theorem 2.1.

(i) *If K is spherically complete every LB-space with reflexive steps is regular.*

(ii) *If K is not spherically complete there exist Hausdorff LB-spaces with reflexive steps that are not regular.*

Proof. (i) Let K be spherically complete and let $(E_n)_n$ be an LB-space with reflexive steps. Every E_n is finite-dimensional ([13], Theorem 4.16). So, for each n, the topology on E_n is the one induced by E_{n+1} and E_n is closed in E_{n+1}. Then regularity follows from Theorem 1.4.13 (i) of [5].

(ii) Let K be not spherically complete. Let $(c_0(\mathbb{N}, 1/b^k))_k$ be the LB-space of Example 3.2.14 of [5]. It was proved there that this inductive sequence is not regular. Also, by Theorem 3.2.6 of [5], its inductive limit is Hausdorff. On the other hand, the steps of this inductive sequence are Banach spaces of countable type, hence reflexive ([13], Corollary 4.18). □

The distinction that we have made in Theorem 2.1 between spherically and non-spherically complete ground fields, for LB-spaces, does not make sense for LF-spaces. Indeed, for any K, there exist Hausdorff LF-spaces with reflexive steps that are not regular, as we show in Counterexample 2.4. This is the non-archimedean version of the classical one given in [11], which has a typically archimedean character, forcing us to use a non-archimedean machinery for the construction of our LF-space. We start with two basic lemmas.

Lemma 2.2. *Let Y be a subset of $\{\frac{1}{p^j} : j \in \{0,1,\ldots\}\}$. Then $X := \mathbb{Q}_p \backslash Y$ is an open dense subset of $\mathbb{Q}_p$.*

Proof. For $j \neq j'$, $|\frac{1}{p^j} - \frac{1}{p^{j'}}|_p = p^{\max(j,j')} \geqslant 1$, hence Y is closed and so X is open. For the density it suffices to prove that, for each $s \in \{0,1,\ldots\}$, $\frac{1}{p^s} \in \overline{X}$. For that, let us take such a s. Since $\lim_i p^i = 0$ in $\mathbb{Q}_p$, we have

$$\frac{1}{p^s} = \lim_i \left(\frac{1}{p^s} + p^i \right),$$

and it is easily seen that $\frac{1}{p^s} + p^i \neq \frac{1}{p^j}$ for all $j \in \{0, 1, \ldots\}$, $i \in \mathbb{N}$ (this is clear when $j = s$; for $j \neq s$, note that $|\frac{1}{p^s} + p^i|_p = p^s \neq p^j = |\frac{1}{p^j}|_p$). Therefore, for each $i \in \mathbb{N}$, $\frac{1}{p^s} + p^i \in X$, so that $\frac{1}{p^s} \in \overline{X}$, and we are done. □

Lemma 2.3. *For each $\alpha \in \mathbb{Q}_p$ there exists a continuous function $\varphi_\alpha : \mathbb{Q}_p \to \mathbb{Q}_p$ such that $\varphi_\alpha \in C^\infty(\mathbb{Q}_p \backslash \{\alpha\})$ but φ_α is not C^1 at α.*

Proof. In [14], proof of Example 26.6 of page 75 and of Remark 1 of page 77 it was proved the existence of a continuous function $g : \mathbb{Z}_p \to \mathbb{Q}_p$ that is locally constant on $\mathbb{Z}_p \backslash \{0\}$ (hence is a C^∞-function on $Z_p \backslash \{0\}$, [14], Corollary 29.10) but that is not C^1 at 0. Then a straightforward verification shows that the function $\varphi_\alpha : \mathbb{Q}_p \to \mathbb{Q}_p$ defined by

$$\varphi_\alpha(x) = \begin{cases} g(x-\alpha) & \text{if } x - \alpha \in \mathbb{Z}_p \\ 0 & \text{otherwise} \end{cases}$$

has the required properties. □

Now we have all the material to construct the announced non-regular LF-space. Notice that its steps are Fréchet and nuclear, hence Montel and so reflexive.

Counterexample 2.4. *There exists a non-regular Hausdorff LF-space whose steps are Fréchet nuclear spaces.*

Proof. Let $n \in \mathbb{N}$, $D_n := \mathbb{Q}_p \backslash \{1, \frac{1}{p}, \ldots, \frac{1}{p^{n-1}}\}$, $E_n := (C^\infty(D_n), \tau_c^\infty)$. Clearly D_n is open in $\mathbb{Q}_p$. Also, E_n is a Fréchet nuclear space ([15], Example 2.3). Its topology τ_c^∞ is defined by the increasing sequence of seminorms q_m^n, $m \in \{0, 1, \ldots\}$, where

$$q_m^n(f) = \max_{0 \leqslant r \leqslant m} \|\overline{\Phi}_r(f|X_m^n)\|_\infty \quad (f \in E_n),$$

with

$$X_m^n := \left\{ x \in \mathbb{Q}_p : |x|_p \leqslant p^m, \left| x - \frac{1}{p^j} \right|_p > \frac{1}{p^m} \text{ for all } j \in \{0, 1, \ldots, n-1\} \right\}$$

(see the Preliminaries). Put $B_m^n := \{f \in E_n : q_m^n(f) \leqslant 1\}$.

For each n, the linear map $i_n : E_n \to E_{n+1}, f \mapsto f|D_{n+1}$ is continuous (because $D_{n+1} \subset D_n$) and injective (because D_{n+1} is dense in D_n, by Lemma 2.2). We identify each E_n with its image $i_n(E_n)$ and then $(E_n)_n$ is an inductive sequence. Let $E := (E, \tau_{\text{ind}})$ be its inductive limit.

To see that E is Hausdorff, consider the (open, by Lemma 2.2) set $D := \mathbb{Q}_p \backslash \{\frac{1}{p^j} : j \in \{0, 1, \ldots\}\}$. The linear map $E \to (C^\infty(D), \tau_c^\infty), f \mapsto f|D$ is continuous (because, as $D \subset D_n$ for all n, the restriction of this map to each E_n is continuous) and injective (because, for all n, D is dense in D_n by Lemma 2.2). Thus, since $(C^\infty(D), \tau_c^\infty)$ is Hausdorff, also so is E.

The proof of non-regularity of $(E_n)_n$ is more involved. We will construct a τ_{ind}-bounded subset B of E that is contained in E_n for no n.

Let $n \in \mathbb{N}$. Applying Lemma 2.3 for $\alpha := \frac{1}{p^{n-1}}$ we obtain a continuous function $\varphi_n : \mathbb{Q}_p \to \mathbb{Q}_p$ that is in $C^\infty(\mathbb{Q}_p \backslash \{\frac{1}{p^{n-1}}\})$ but that is not C^1 at $\frac{1}{p^{n-1}}$. Set

$$Y_n := \{x \in \mathbb{Q}_p : |x|_p < p^{n-1}\}.$$

Y_n is compact and open in $\mathbb{Q}_p$ and is contained in $\mathbb{Q}_p\backslash\{\frac{1}{p^{n-1}}\}$, where φ_n is a C^∞-function. So, $\|\overline{\Phi}_r(\varphi_n|Y_n)\|_\infty < \infty$ for all $r \in \{0, 1, \ldots\}$. Multiplying by an adequate scalar we may assume that

$$\max_{0 \leqslant r \leqslant n} \|\overline{\Phi}_r(\varphi_n|Y_n)\|_\infty \leqslant 1. \tag{2}$$

Let $f_n := \varphi_n|D_n$. Then $f_n \in E_n$ (because $D_n \subset \mathbb{Q}_p\backslash\{\frac{1}{p^{n-1}}\}$). But $f_{n+1} \notin E_n$. In fact, if this last were not true then there would exist a $g_n \in C^\infty(D_n)$ such that $g_n = \varphi_{n+1}$ on D_{n+1}. Thus, by continuity of g_n and of φ_{n+1} on D_n and by density of D_{n+1} in D_n (Lemma 2.2), $g_n = \varphi_{n+1}$ on D_n, from which we would deduce that $\varphi_{n+1}|D_n \in C^\infty(D_n)$. In particular, φ_{n+1} is C^1 at $\frac{1}{p^n} \in D_n$, a contradiction.

Now let $B := \{f_n : n \in \mathbb{N}\}$. The above tells us that B is not contained in any E_n. It remains to show that B is τ_{ind}-bounded in E. For that, let U be an absolutely convex zero neighbourhood in E. The inclusion $E_1 \to E$ is continuous, hence there exist $m \in \mathbb{N}$ and $\lambda \in K$ such that $B_m^1 \subset \lambda U$. Fix this m and take $n > m + 1$. Since the inclusion $E_n \to E$ is continuous there exist $k \in \mathbb{N}$ and $\mu \in K$, $|\mu| \geqslant |\lambda|$, such that

$$B_k^n \subset \mu U, \tag{3}$$

and then also

$$B_m^1 \subset \mu U. \tag{4}$$

The formula

$$\phi_n(x) = \begin{cases} \varphi_n(x) & \text{if } x \in \mathbb{Q}_p\backslash B(\frac{1}{p^{n-1}}, \frac{1}{p^k}) \\ 0 & \text{otherwise} \end{cases}$$

defines a function $\phi_n : \mathbb{Q}_p \to \mathbb{Q}_p$ that is in $C^\infty(\mathbb{Q}_p)$ (because $\varphi_n \in C^\infty(\mathbb{Q}_p\backslash\{\frac{1}{p^{n-1}}\})$ and $B(\frac{1}{p^{n-1}}, \frac{1}{p^k})$ is clopen).

Now let $h_n := \phi_n|D_n$. Clearly $h_n \in E_n$. Also, the following holds.

(i) $q_k^n(f_n - h_n) = 0$. In fact, $X_k^n \subset \mathbb{Q}_p\backslash B(\frac{1}{p^{n-1}}, \frac{1}{p^k})$, so $\varphi_n = \phi_n$ on X_k^n i.e. $f_n = h_n$ on X_k^n, and we are done.

(ii) $h_n \in E_1$ and $q_m^1(h_n) \leqslant 1$. That $h_n \in E_1$ is obvious because $h_n = (\phi_n|D_1)|D_n$ and $\phi_n|D_1 \in E_1$. Now let us see that $q_m^1(\phi_n|D_1) \leqslant 1$. For that, firstly note that from the definitions of X_m^1 and Y_n and since $m < n - 1$, we obtain

$$X_m^1 \subset Y_n. \tag{5}$$

Also, one verifies that $x \in Y_n \implies |x|_p < p^{n-1} = |\frac{1}{p^{n-1}}|_p \implies |x - \frac{1}{p^{n-1}}|_p = p^{n-1} > 1 > \frac{1}{p^k}$, from which we have

$$Y_n \subset \mathbb{Q}_p\backslash B\left(\frac{1}{p^{n-1}}, \frac{1}{p^k}\right),$$

so that

$$\phi_n = \varphi_n \text{ on } Y_n. \tag{6}$$

Applying (5), (6) and (2) we arrive at

$$\begin{aligned} q_m^1(\phi_n|D_1) &= \max_{0 \leqslant r \leqslant m} \|\overline{\Phi}_r(\phi_n|X_m^1)\|_\infty \leqslant \max_{0 \leqslant r \leqslant m} \|\overline{\Phi}_r(\phi_n|Y_n)\|_\infty \\ &= \max_{0 \leqslant r \leqslant m} \|\overline{\Phi}_r(\varphi_n|Y_n)\|_\infty \leqslant \max_{0 \leqslant r \leqslant n} \|\overline{\Phi}_r(\varphi_n|Y_n)\|_\infty \leqslant 1, \end{aligned}$$

and the proof of (ii) is finished.

Next, by using (i) and (ii) we deduce that, for all $n > m+1$,

$$f_n - h_n \in B_k^n, \quad h_n \in B_m^1, \tag{7}$$

and taking into account (3), (4) and (7) we have

$$f_n = (f_n - h_n) + h_n \in \mu U + \mu U = \mu U.$$

Therefore, we have found $m \in \mathbb{N}$ and $\mu \in K$ such that $\{f_n : n > m+1\} \subset \mu U$. Also, obviously there is a $\rho \in K$ with $|\rho| \geqslant |\mu|$ such that $\{f_1, \ldots, f_{m+1}\} \subset \rho U$. Thus, finally $B = \{f_n : n \in \mathbb{N}\} \subset \rho U$ and τ_{ind}-boundedness of B is proved. □

Remark 2.5. The natural version of Counterexample 2.4 for LB-spaces does not hold. In fact, every LB-space $(E_n)_n$ with semi-Montel steps is regular. To prove this last assertion, note that, for each n, the unit ball of E_n is a compactoid and hence E_n is finite-dimensional ([13], Theorem 4.37). Then regularity of $(E_n)_n$ follows with the same reasoning as in Theorem 2.1 (i).

We finish by giving some topological properties of the above inductive limit and some applications.

Theorem 2.6. *Let* $(E_n)_n$ *be the inductive sequence of Counterexample* 2.4, *let* E *be its inductive limit. Then we have the following.*

(i) *E is Hausdorff, nuclear* (*hence semi-Montel and of countable type*), *polarly barrelled and polarly bornological.*

(ii) *E is not* (*weakly*) *sequentially complete* (*hence neither* (*weakly*) (*quasi*)-*complete*).

(iii) *E is not (semi-)reflexive (hence neither Montel).*

(iv) $((E_n)'_b)_n$ *is a projective sequence such that, for its projective limit F, the continuous bijective linear map*

$$\Psi : E'_b \longrightarrow F, \quad v \longmapsto \Psi(v), \quad (\Psi(v))(f) = v(f), \quad f \in E_n, \quad n \in \mathbb{N} \tag{8}$$

(*see* (1)) *is not a homeomorphism.*

Proof. (i) That E is Hausdorff was already proved in Counterexample 2.4. Nuclearity (resp. polar barrelledness) follows from [8], Proposition 3.5 (resp. from [5], Proposition 1.1.10 (ii)).

Now let us see that E is polarly bornological. Let A be a K-polar subset of E that absorbs every bounded set in E. Then A is absolutely convex and, for each n, $A \cap E_n$ is a K-polar subset of E_n that absorbs every bounded subset of the (polarly bornological, [15], Proposition 6.9) space E_n. Hence $A \cap E_n$ is a zero neighbourhood in each E_n i.e. A is a zero neighbourhood in E ([5], Proposition 1.1.6 (ii)).

(ii) $(E_n)_n$ is not regular, hence E is not sequentially complete ([5], Propositions 2.3.2 and 2.3.3). Further, E is Hausdorff and of countable type by (i), so the weakly convergent sequences of E coincide with the convergent ones ([15], Theorem 4.4 and Proposition 4.11). Thus, E cannot be weakly sequentially complete.

(iii) By (ii), E is not weakly quasicomplete, hence it is not reflexive. Also, by (i), E is Hausdorff, polar and polarly barrelled. It follows from Lemmas 9.2 and 9.3 of [15] that reflexivity and semireflexivity of E are equivalent properties. Therefore, E is not semireflexive.

(iv) Firstly we see that, for each n, E_n is dense in E_{n+1}. Let $f \in E_{n+1}$, $m \in \{0, 1, \ldots\}$. Define $g : D_n \to K$ by

$$g(x) = \begin{cases} f(x) & \text{if } x \in X_m^{n+1} \\ 0 & \text{otherwise} \end{cases}$$

(we follow the same notations as in the proof of Counterexample 2.4). Then $g \in E_n$ (because $f|X_m^{n+1} \in C^\infty(X_m^{n+1})$ and X_m^{n+1} is clopen in $\mathbb{Q}_p$) and $f = g$ on X_m^{n+1}, so $q_m^{n+1}(g|D_{n+1} - f) = 0$ and we are done.

Thus, $((E_n')_b)_n$ is a projective sequence (see the Preliminaries). Let F be its projective limit and let $\Psi : E_b' \to F$ be as in (8). Assume Ψ is a homeomorphism; we derive a contradiction. First we prove that this assumption implies that $(E_n)_n$ satisfies the following property:

$$\begin{aligned} &\text{For every } \tau_{\text{ind}}\text{-bounded set } B \subset E \text{ there exists an } n \in \mathbb{N} \quad \text{and} \\ &\text{a } \tau_n\text{-bounded set } B_n \subset E_n \text{ such that } B \subset \overline{B_n}^{\tau_{\text{ind}}} \end{aligned} \tag{9}$$

(τ_{ind} is the inductive topology of E and, for each n, τ_n is the topology of E_n).

In order to get (9), let $B \subset E$ be τ_{ind}-bounded. Then $B^o := \{v \in E' : |v(B)| \leqslant 1\}$ is a zero neighbourhood in E_b'. Since Ψ is homeomorphic, there is an n and an absolutely convex τ_n-bounded set $A_n \subset E_n$ such that $\Psi(B^o) \supset \{w \in F : |w(A_n)| \leqslant 1\}$, which implies that $B^o \supset A_n^o := \{v \in E' : |v(A_n)| \leqslant 1\}$. Thus, $B^{oo} \subset A_n^{oo}$ (the two bipolars are with respect to the duality (E, E') and are defined in the usual way). As E is of countable type by (i), we have that $A_n^{oo} \subset \cap\{\lambda \overline{A_n}^{\tau_{\text{ind}}} : \lambda \in K, |\lambda| > 1\}$ ([15], Theorem 4.4 and comments after Proposition 4.10). Therefore, by taking $\lambda \in K$ with $|\lambda| > 1$, we obtain that $B \subset B^{oo} \subset A_n^{oo} \subset \lambda \overline{A_n}^{\tau_{\text{ind}}}$. So, $B_n := \lambda A_n$ meets the requirements of (9).

Now we use (9) to prove that $(E_n)_n$ is regular (then we arrive at the desired contradiction and the proof of (iv) is finished). For that, let $B \subset E$ be τ_{ind}-bounded, let n and B_n be as in (9). We may assume that B_n is absolutely convex. As E_n is a Fréchet semi-Montel space, $\overline{B_n}^{\tau_n}$ is metrizable, compactoid and complete in E_n. Also, since $\tau_{\text{ind}}|E_n$ is Hausdorff and weaker than τ_n, we have $\tau_{\text{ind}}|\overline{B_n}^{\tau_n} = \tau_n|\overline{B_n}^{\tau_n}$ ([16], Proposition 9.1). Thus, $\overline{B_n}^{\tau_n}$ is τ_{ind}-complete, hence τ_{ind}-closed, so that $\overline{B_n}^{\tau_{\text{ind}}} = \overline{B_n}^{\tau_n}$. By (9), $B \subset \overline{B_n}^{\tau_n}$, from which it follows that B is contained and bounded in E_n. Then we get regularity of $(E_n)_n$. □

Finally we give some applications of our previous theory. As a direct consequence of (iv) of Theorem 2.6 we obtain that

Application 2.7. *The continuous linear bijection of the strong dual of an inductive limit onto the projective limit of the strong duals of its steps* (*see* (1)) *may fail to be a homeomorphism.*

Next, recall that to be of countable type, polarly barrelled (resp. nuclear, resp. polarly bornological) are stable properties by taking inductive limits, see [5], Proposition 1.1.10 (resp. [8], Proposition 3.5, resp. proof of (i) of Theorem 2.6). However, there are other properties for which that stability does not hold. This is the case for polarity and for metrizability, see Example 1.4.22 and Theorem 2.1.4 of [5] respectively. Also, Counterexample 2.4 leads to the following.

Application 2.8. (*Weak*) (*Quasi*)*completeness*, (*weak*) *sequential completeness*, (*semi-*)*reflexivity and Montelness are not always stable by taking inductive limits*.

Proof. Let $(E_n)_n$ be as in Counterexample 2.4, let E be its inductive limit. Each E_n is a Fréchet nuclear space, so it has all the properties mentioned in the statement. On the other hand, (ii) and (iii) of Theorem 2.6 tell us that all of these properties fail for E. □

Remark 2.9. In [5] one can find conditions under which inductive limits preserve the above properties.

But the following is unknown.

Problem. Is semi-Montelness stable for taking inductive limits?

References

[1] Christol G, Mebkhout Z (1993) Sur le théorème de l'indice des équations différentielles p-adiques I. Ann Inst Fourier **43**: 1545–1574
[2] Christol G, Mebkhout Z (1997) Sur le théorème de l'indice des équations différentielles p-adiques II. Ann Math **146**: 345–410
[3] Christol G, Mebkhout Z (2000) Sur le théorème de l'indice des équations différentielles p-adiques III. Ann Math **151**: 385–457
[4] Christol G, Mebkhout Z (2001) Sur le théorème de l'indice des équations différentielles p-adiques IV. Invent Math **143**: 629–672
[5] De Grande-De Kimpe N, Kąkol J, Perez-Garcia C, Schikhof WH (1997) p-adic locally convex inductive limits. In: Schikhof WH, Perez-Garcia C, Kąkol J (eds) p-adic Functional Analysis. Lect Notes Pure Appl Math **192**, pp 159–222. New York: Dekker
[6] De Grande-De Kimpe N, Khrennikov A (1996) The non-archimedean Laplace Transform. Bull Belg Math Soc Simon Stevin **3**: 225–237
[7] De Grande-De Kimpe N, Khrennikov A, van Hamme L (1999) The Fourier Transform for p-adic tempered distributions. In: Kąkol J, De Grande-De Kimpe N, Perez Garcia C (eds) p-adic Functional Analysis. Lect Notes Pure Appl Math **207**, pp 97–112. New York: Dekker
[8] De Grande-De Kimpe N, Perez-Garcia C (1993) p-adic semi-Montel spaces and polar inductive limits. Results Math **24**: 66–75
[9] Jarchow H (1981) Locally Convex Spaces. Stuttgart: Teubner
[10] Kucera J, McKennon K (1980) Dieudonné-Schwartz Theorem on bounded sets in inductive limits. Proc Amer Math Soc **78**: 366–368
[11] Kucera J, McKennon K (1985) Köthe's example of an incomplete LB-space. Proc Amer Math Soc **93**: 79–80
[12] Robba P, Christol G (1994) Équations Différentielles p-adiques. Paris: Hermann
[13] van Rooij ACM (1978) Non-Archimedean Functional Analysis. New York: Dekker
[14] Schikhof WH (1984) Ultrametric Calculus. An Introduction to p-adic Analysis. Cambridge: Univ Press
[15] Schikhof WH (1986) Locally convex spaces over nonspherically complete valued fields I-II. Bull Soc Math Belg Sér B **38**: 187–224
[16] Schikhof WH (1995) A perfect duality between p-adic Banach spaces and compactoids. Indag Math (N.S.) **6**: 325–339

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