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A variational problem for submanifolds in a sphere*

By

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Abstract. Let $x: M \to S^{n+p}$ be an n-dimensional submanifold in an $(n+p)$-dimensional unit sphere S^{n+p}, M is called a *Willmore submanifold* (see [11], [16]) if it is a critical submanifold to the *Willmore functional* $\int_M (S - nH^2)^{\frac{n}{2}} dv$, where $S = \sum_{\alpha,i,j} (h_{ij}^\alpha)^2$ is the square of the length of the second fundamental form, H is the mean curvature of M. In [11], the second author proved an integral inequality of Simons' type for n-dimensional compact Willmore submanifolds in S^{n+p}. In this paper, we discover that a similar integral inequality of Simons' type still holds for the critical submanifolds of the functional $\int_M (S - nH^2) dv$. Moreover, it has the advantage that the corresponding Euler-Lagrange equation is simpler than the Willmore equation.

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1. Introduction

For brevity, we use the same notations as [11] in this paper. Let M be an n-dimensional compact submanifold of an $(n+p)$-dimensional unit sphere space S^{n+p}. If h_{ij}^α denotes the second fundamental form of M, S denotes the square of the length of the second fundamental form, $\mathbf{H}$ denotes the mean curvature vector and H denotes the mean curvature of M, then we have

$$S = \sum_{\alpha,i,j} (h_{ij}^\alpha)^2, \qquad \mathbf{H} = \sum_\alpha H^\alpha e_\alpha, \qquad H^\alpha = \frac{1}{n} \sum_k h_{kk}^\alpha, \qquad H = |\mathbf{H}|,$$

where e_α $(n+1 \leqslant \alpha \leqslant n+p)$ are orthonormal normal vector fields of M in S^{n+p}.

We define the following non-negative function on M,

$$\rho^2 = S - nH^2, \tag{1.1}$$

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which vanishes exactly at the umbilic points of M. The Willmore functional (see [11], [5], [16]) is

$$W(x) = \int_M \rho^n dv = \int_M (S - nH^2)^{\frac{n}{2}} dv, \tag{1.2}$$

the Euler-Lagrange equation (i.e. Willmore equation) can be found in (1.2) of [11].

In this paper, we consider the following non-negative functional

$$F(x) = \int_M \rho^2 dv = \int_M (S - nH^2) dv, \tag{1.3}$$

which vanishes if and only if M is a totally umbilical submanifold, so the function $F(x)$ measures how derivation $x(M)$ is from totally umbilical submanifold.

Remark 1.1. When $n = 2$, $F(x)$ reduces to the well-known Willmore functional $W(x)$, and its critical points are called *Willmore* surfaces. The Willmore surfaces in a sphere were studied by Thomsen [20], Willmore [22], Bryant [3], Pinkall [17], Weiner [21], Montiel [15], Li [7], Li and Simon [12], Li and Vrancken [13] and many others (also see Blaschke [2]).

In this paper, we first calculate the Euler-Lagrangian equation of $F(x)$ given by (1.3).

Theorem 1.1. *Let $x\colon M \to S^{n+p}$ be an n-dimensional submanifold in an $(n+p)$-dimensional unit sphere S^{n+p}. Then M is a extremal submanifold of $F(x)$ if and only if for $n+1 \leqslant \alpha \leqslant n+p$*

$$(n-1)\Delta^{\perp} H^{\alpha} + \sum_{\beta,i,j,k} h^{\alpha}_{ij} h^{\beta}_{ik} h^{\beta}_{kj} - \sum_{\beta,i,j} H^{\beta} h^{\beta}_{ij} h^{\alpha}_{ij} - \frac{n}{2}\rho^2 H^{\alpha} = 0, \tag{1.4}$$

where $\Delta^{\perp} H^{\alpha} = \sum_i H^{\alpha}_{,ii}$ (for notations here, see [11]).

We call $x\colon M \to S^{n+p}$ an extremal submanifold if it satisfies Euler-Lagrange equation (1.4).

Remark 1.2. When $n = 2$, Theorem 1.1 was proved by Weiner in [21]. In this case (1.4) reduces to the following well-known equation of Willmore surfaces (see [21] or [7])

$$\Delta^{\perp} H^{\alpha} + \sum_{\beta,i,j} h^{\alpha}_{ij} h^{\beta}_{ij} H^{\beta} - 2H^2 H^{\alpha} = 0, \quad 3 \leqslant \alpha \leqslant 2+p. \tag{1.5}$$

Remark 1.3. It is remarkable that when $n \geqslant 3$, the Euler-Lagrange equation (1.4) of the functional $F(x)$ is much simpler that the Willmore equation (1.2) of [11].

In order to state our main result, we recall the following important examples

Example 1 (see [4] or [6]). The Clifford torus

$$C_{m,m} = S^m\left(\sqrt{\frac{1}{2}}\right) \times S^m\left(\sqrt{\frac{1}{2}}\right), \quad n = 2m, \tag{1.6}$$

is an extremal hypersurface in S^{n+1}. In fact, the principal curvatures $k_1, \dots, k_n$ of $C_{m,m}$ are

$$k_1 = \cdots = k_m = 1, \quad k_{m+1} = \cdots = k_n = -1, \quad n = 2m. \tag{1.7}$$

We have from (1.7)

$$H = 0, \quad S = n, \quad \sum_i k_i^3 = 0. \tag{1.8}$$

Thus we easily check that (1.4) holds, i.e., $C_{m,m}$ is an extremal hypersurface. In particular, we note that ρ^2 of $C_{m,m}$ satisfy

$$\rho^2 = n. \tag{1.9}$$

Example 2 (see [4] or [7], [11]). The *Veronese surface* satisfies (1.5) and $\rho^2 = \frac{4}{3}$.

Example 3. If x: $M \to S^{n+p}$ is a minimal surface or an n-dimensional $(n \geqslant 3)$ Einstein and minimal submanifold, then it must be an extremal submanifold. It can be checked directly that in this case (1.4) is satisfied by use of Gauss equation and minimal condition $H^\alpha = 0$.

In [10], [11], the second author proved the following integral inequality of Simons' type:

Theorem 1.2 ([10], [11]). *Let M be an n-dimensional $(n \geqslant 2)$ compact Willmore submanifold in $(n+p)$-dimensional unit sphere S^{n+p}. Then we have*

$$\int_M \rho^n \left(\frac{n}{2 - 1/p} - \rho^2 \right) dv \leqslant 0. \tag{1.10}$$

In particular, if

$$0 \leqslant \rho^2 \leqslant \frac{n}{2 - 1/p}, \tag{1.11}$$

then either $\rho^2 \equiv 0$ and M is totally umbilical, or $\rho^2 \equiv \frac{n}{2-1/p}$. In the latter case, either $p = 1$ and M is a Willmore torus $W_{m,n-m}$, or $n = 2$, $p = 2$ and M is the Veronese surface.

In this paper we discover that a similar integral inequality of Simons' type still holds for compact extremal submanifolds in S^{n+p}.

Theorem 1.3. *Let M be an n-dimensional $(n \geqslant 2)$ compact extremal submanifold in $(n+p)$-dimensional unit sphere S^{n+p}. Then we have*

$$\int_M \rho^2 \left(\frac{n}{2 - 1/p} - \rho^2 \right) dv \leqslant 0. \tag{1.12}$$

In particular, if

$$0 \leqslant \rho^2 \leqslant \frac{n}{2 - 1/p}, \tag{1.13}$$

then either $\rho^2 \equiv 0$ and M is totally umbilic, or $\rho^2 \equiv \frac{n}{2-1/p}$. In the latter case, either $p = 1$, $n = 2m$ and M is a Clifford torus $C_{m,m}$ defined by (1.6); *or $n = 2$, $p = 2$ and M is the Veronese surface.*

Remark 1.4. When $n=2$, Theorem 1.3 was proved by the second author in [7] (also see Li and Simon [12]).

2. Proof of Theorem 1.1

We use the same notations as in [11]. Let x_0: $M \longrightarrow S^{n+p}$ be an n-dimensional compact submanifold. Now we calculate the first variation of the functional $F(x_0)$

Let x: $M\times R \longrightarrow S^{n+p}$ be a smooth variation of x_0 such that $x(\cdot,t)=x_0$ on the boundary. Along x: $M\times R\longrightarrow S^{n+p}$, we choose a local orthonormal basis $\{e_A\}$ for TS^{n+p} with dual basis $\{\omega_A\}$, such that $\{e_i(\cdot,t)\}$ forms a local orthonormal basis for x_t: $M\times\{t\}\longrightarrow S^{n+p}$. Since $T^*(M\times R)=T^*M\oplus T^*R$, the pullback of $\{\omega_A\}$ and $\{\omega_{AB}\}$ on S^{n+p} through x: $M\times R\longrightarrow S^{n+p}$ have the decomposition

$$x^*\omega_\alpha = a_\alpha dt, \qquad x^*\omega_i=\theta_i+a_idt, \tag{2.1}$$

$$x^*\omega_{ij}=\theta_{ij}+a_{ij}dt, \qquad x^*\omega_{i\alpha}=\theta_{i\alpha}+a_{i\alpha}dt, \qquad x^*\omega_{\alpha\beta}=\theta_{\alpha\beta}+a_{\alpha\beta}dt, \tag{2.2}$$

where $\{a_i,\ a_\alpha,\ a_{ij},\ a_{i\alpha},\ a_{\alpha\beta}\}$ are local functions on $M\times R$ with $a_{ij}=-a_{ji}$, $a_{\alpha\beta}=-a_{\beta\alpha}$ and

$$V=\frac{d}{dt}\bigg|_{t=0}x_t=\sum_i a_idx_0(e_i)+\sum_\alpha a_\alpha e_\alpha, \tag{2.3}$$

is the variation vector field of x_t: $M\longrightarrow S^{n+p}$. We note that the one forms $\{\theta_i,\ \theta_{ij},\ \theta_{i\alpha},\ \theta_{\alpha\beta}\}$ are defined on $M\times\{t\}$, for $t=0$, they reduce to the forms with the same notation on M.

We denote by d_M the differential operator on T^*M, then we have $d=d_M+dt\frac{\partial}{\partial t}$ on $T^*(M\times R)$.

Lemma 2.1. *Under the above notations, we have*

$$\frac{\partial\theta_i}{\partial t}=\sum_j(a_{i,j}+a_{ij})\theta_j-\sum_{j,\alpha}h^\alpha_{ij}a_\alpha\theta_j, \tag{2.4}$$

$$a_{i\alpha}=a_{\alpha,i}+\sum_j h^\alpha_{ij}a_j, \tag{2.5}$$

$$\frac{\partial\theta_{i\alpha}}{\partial t}=\sum_j\left(a_{i\alpha,j}+\sum_k a_{ik}h^\alpha_{jk}-\sum_\beta a_{\beta\alpha}h^\beta_{ij}+a_\alpha\delta_{ij}\right)\theta_j, \tag{2.6}$$

where h^α_{ij} and the covariant derivatives $a_{i,j}$, $a_{\alpha,i}$ and $a_{i\alpha,j}$ are defined on $M\times\{t\}$ by

$$\theta_{i\alpha}=\sum_j h^\alpha_{ij}\theta_j, \tag{2.7}$$

$$\sum_j a_{i,j}\theta_j=d_Ma_i+\sum_j a_j\theta_{ji}, \tag{2.8}$$

$$\sum_i a_{\alpha,i}\theta_i=d_Ma_\alpha+\sum_\beta a_\beta\theta_{\beta\alpha}, \tag{2.9}$$

$$\sum_j a_{i\alpha,j}\theta_j=d_Ma_{i\alpha}+\sum_j a_{j\alpha}\theta_{ji}+\sum_\beta a_{i\beta}\theta_{\beta\alpha}. \tag{2.10}$$

Proof. These are direct calculations. In fact, substituting (2.1) and (2.2) into the following equations, respectively,

$$d(x^*\omega_i) = x^*(d\omega_i) = x^*\left(\sum_j \omega_{ij}\wedge\omega_j + \sum_\alpha \omega_{i\alpha}\wedge\omega_\alpha\right),$$
$$d(x^*\omega_\alpha) = x^*(d\omega_\alpha) = x^*\left(\sum_j \omega_{\alpha j}\wedge\omega_j + \sum_\beta \omega_{\alpha\beta}\wedge\omega_\beta\right),$$
$$d(x^*\omega_{i\alpha}) = x^*(d\omega_{i\alpha}) = x^*\left(\sum_j \omega_{ij}\wedge\omega_{j\alpha} + \sum_\beta \omega_{i\beta}\wedge\omega_{\beta\alpha} - \omega_i\wedge\omega_\alpha\right),$$

and comparing the terms in $T^*M\wedge dt$ for each equation on the both sides, we can get (2.4), (2.5) and (2.6), respectively.

Lemma 2.2.

$$\frac{\partial h^\alpha_{ij}}{\partial t} = a_{\alpha,ij} + \sum_k\left(a_{ik}h^\alpha_{kj} + a_{jk}h^\alpha_{ki} + h^\alpha_{ijk}a_k\right) + \sum_\beta a_{\alpha\beta}h^\beta_{ij} + \delta_{ij}a_\alpha + \sum_{k,\beta} h^\alpha_{ik}h^\beta_{kj}a_\beta. \tag{2.11}$$

Remark 2.1. When $p = 1$, Lemma 2.2 was proved by Barbosa and Colares [1] (see Lemma 6.1 of [1]). We also note that the sign of the second fundamental form here is different from theirs.

Proof. Differentiating (2.7) with respect to t and using (2.4) and (2.6), we get

$$\frac{\partial h^\alpha_{ij}}{\partial t} = a_{i\alpha,j} + \sum_k a_{ik}h^\alpha_{jk} - \sum_\beta a_{\beta\alpha}h^\beta_{ij} + a_\alpha\delta_{ij} - \sum_k(h^\alpha_{ik}a_{k,j} + h^\alpha_{ik}a_{kj}) + \sum_{k,\beta} h^\alpha_{ik}h^\beta_{kj}a_\beta.$$

Covariant differentiating (2.5) over $M\times\{t\}$ and using the Codazzi equation for $x_t\colon M\to S^{n+p}$, we get

$$\begin{aligned} a_{i\alpha,j} &= a_{\alpha,ij} + \sum_k(a_{k,j}h^\alpha_{ik} + a_k h^\alpha_{ikj}) \\ &= a_{\alpha,ij} + \sum_k(a_{k,j}h^\alpha_{ik} + a_k h^\alpha_{ijk}). \end{aligned}$$

Combining the above two equations, we prove Lemma 2.2.

Set $i = j$ in (2.11) and making summation over i with using $\sum_{i,k} a_{ik}h^\alpha_{ki} = 0$, we get

$$\frac{\partial H^\alpha}{\partial t} = \frac{1}{n}\Delta^\perp a_a + \sum_k H^\alpha_{,k}a_k + \sum_\beta a_{\alpha\beta}H^\beta + \frac{1}{n}\sum_{i,k,\beta} h^\alpha_{ik}h^\beta_{ki}a_\beta + a_\alpha. \tag{2.12}$$

From (2.11) and the fact that

$$S = \sum_{i,j,\alpha}(h^\alpha_{ij})^2, \qquad \sum_{i,j,\alpha,\beta} a_{\alpha\beta}h^\beta_{ij}h^\alpha_{ij} = 0, \qquad \sum_{i,j,k,\alpha} a_{jk}h^\alpha_{ki}h^\alpha_{ij} = 0,$$

we obtain

$$\frac{1}{2}\frac{\partial S}{\partial t} = \sum_{i,j,\alpha} h_{ij}^{\alpha} a_{\alpha,ij} + \frac{1}{2}\sum_{k} S_{,k} a_k + nH^{\alpha} a_{\alpha} + \sum_{i,j,k,\alpha,\beta} h_{ij}^{\alpha} h_{ik}^{\alpha} h_{kj}^{\beta} a_{\beta}. \tag{2.13}$$

From (2.12) and $\sum_{\alpha,\beta} a_{\alpha\beta} H^{\alpha} H^{\beta} = 0$, we obtain

$$\frac{n}{2}\frac{\partial H^2}{\partial t} = \sum_{\alpha} H^{\alpha} \Delta^{\perp} a_{\alpha} + \frac{n}{2}\sum_{k} (H^2)_{,k} a_k + \sum_{i,j,\alpha,\beta} H^{\alpha} h_{ij}^{\alpha} h_{ij}^{\beta} a_{\beta} + n\sum_{\alpha} H^{\alpha} a_{\alpha}. \tag{2.14}$$

For x_t: $M \to S^{n+p}$, we consider the functional

$$F(x_t) = \int_M \rho^2 dv = \int_M (S - nH^2)\theta_1 \wedge \cdots \wedge \theta_n. \tag{2.15}$$

From (2.4), we have

$$\begin{aligned}\frac{\partial}{\partial t}(\theta_1 \wedge \cdots \wedge \theta_n) &= \sum_i \theta_1 \wedge \cdots \wedge \frac{\partial \theta_i}{\partial t} \wedge \cdots \wedge \theta_n \\ &= \sum_i (a_{i,i} + a_{ii} - h_{ii}^{\alpha} a_{\alpha})\theta_1 \wedge \cdots \wedge \theta_n \\ &= \left(\sum_i a_{i,i} - n\sum_{\alpha} H^{\alpha} a_{\alpha}\right)\theta_1 \wedge \cdots \wedge \theta_n. \end{aligned} \tag{2.16}$$

Differentiating (2.15) with respect to t, we get by use of (2.13), (2.14) and (2.16)

$$\begin{aligned}\frac{\partial F(x_t)}{\partial t} = \int_M \Bigg\{ &\left[2\sum_{i,j,\alpha} h_{ij}^{\alpha} a_{\alpha,ij} - 2\sum_{\alpha} H^{\alpha} \Delta^{\perp} a_{\alpha} + \sum_k (\rho^2)_{,k} a_k + \rho^2 \sum_k a_{k,k}\right] \\ &+ 2\sum_{\alpha}\left[\sum_{i,j,k,\beta} h_{ij}^{\beta} h_{ik}^{\beta} h_{kj}^{\alpha} - \sum_{i,j,\beta} H^{\beta} h_{ij}^{\beta} h_{ij}^{\alpha} - \frac{n}{2}\rho^2 H^{\alpha}\right] a_{\alpha}\Bigg\} dv \end{aligned} \tag{2.17}$$

We note that

$$\sum_k (\rho^2)_{,k} a_k + \rho^2 \sum_k a_{k,k} = \sum_k \left(\rho^2 a_k\right)_{,k}, \tag{2.18}$$

and M is compact (without boundary), also noting

$$\sum_j h_{ijj}^{\alpha} = nH_{,i}^{\alpha}, \qquad \sum_{i,j} h_{ijji}^{\alpha} = n\Delta^{\perp} H^{\alpha}, \tag{2.19}$$

it follows from (2.17), (2.18) and Green's formula that

$$\frac{\partial F(x_t)}{\partial t} = 2\int_M \sum_{\alpha}\left[\sum_{i,j,k,\beta} h_{ij}^{\beta} h_{ik}^{\beta} h_{kj}^{\alpha} - \sum_{i,j,\beta} H^{\beta} h_{ij}^{\beta} h_{ij}^{\alpha} - \frac{n}{2}\rho^2 H^{\alpha} + (n-1)\Delta^{\perp} H^{\alpha}\right] a_{\alpha} dv \tag{2.20}$$

From (2.3) and (2.20) with restriction to $t = 0$, we have proved Theorem 1.1.

3. The Lemmas and Proof of Theorem 1.3

Define tensors

$$\tilde{h}^{\alpha}_{ij} = h^{\alpha}_{ij} - H^{\alpha}\delta_{ij}, \tag{3.1}$$

$$\tilde{\sigma}_{\alpha\beta} = \sum_{i,j} \tilde{h}^{\alpha}_{ij}\tilde{h}^{\beta}_{ij}, \qquad \sigma_{\alpha\beta} = \sum_{i,j} h^{\alpha}_{ij}h^{\beta}_{ij}. \tag{3.2}$$

By use of Theorem 1.1, (3.1) and (3.2), we can get

Lemma 3.1. *Let M be an n-dimensional submanifold in an $(n+p)$-dimensional unit sphere S^{n+p}. Then M is an extremal submanifold if and only if it satisfies for $n+1 \leqslant \alpha \leqslant n+p$*

$$\sum_{\beta,i,j,k} \tilde{h}^{\alpha}_{ij}\tilde{h}^{\beta}_{ik}\tilde{h}^{\beta}_{kj} = -(n-1)\Delta^{\perp}H^{\alpha} - \sum_{\beta} H^{\beta}\tilde{\sigma}_{\alpha\beta} - H^{\alpha}\rho^2 + \frac{n}{2}H^{\alpha}\rho^2. \tag{3.3}$$

Lemma 3.2 (see Lemma 4.5 of [11]). *Let x: $M \to S^{n+p}$ be an n-dimensional submanifold in S^{n+p}. Then*

$$\begin{aligned}\frac{1}{2}\Delta\rho^2 \geqslant{} & |\nabla h|^2 - n^2|\nabla^{\perp}\mathbf{H}|^2 + \sum_{\alpha,i,j,k}(h^{\alpha}_{ij}h^{\alpha}_{kki})_j + n\sum_{\alpha,\beta,i,j,m} H^{\beta}\tilde{h}^{\beta}_{mj}\tilde{h}^{\alpha}_{ij}\tilde{h}^{\alpha}_{im} \\ & + n\rho^2 + nH^2\rho^2 - \left(2-\frac{1}{p}\right)\rho^4 - \frac{1}{2}\Delta(nH^2).\end{aligned} \tag{3.4}$$

From (3.3), we have

$$\begin{aligned} n\sum_{\alpha,\beta,i,j,k} H^{\beta}\tilde{h}^{\beta}_{mj}\tilde{h}^{\alpha}_{ij}\tilde{h}^{\alpha}_{im} ={} & -n(n-1)\sum \Delta^{\perp}H^{\beta}\cdot H^{\beta} - n\sum_{\alpha,\beta} H^{\alpha}H^{\beta}\tilde{\sigma}_{\alpha\beta} - n|\vec{H}|^2\rho^2 \\ & + \frac{n^2}{2}|\vec{H}|^2\rho^2. \end{aligned} \tag{3.5}$$

Integrating (3.4) over M and using Stokes' formula, we have by use of (3.5),

$$\begin{aligned} 0 &\geqslant \int_M \bigg[|\nabla h|^2 - n^2|\nabla^{\perp}\mathbf{H}|^2 - n(n-1)\sum H^{\alpha}\Delta^{\perp}H^{\alpha} - n\sum H^{\alpha}H^{\beta}\tilde{\sigma}_{\alpha\beta} + n\rho^2 \\ &\qquad + \frac{n^2}{2}H^2\rho^2 - \left(2-\frac{1}{p}\right)\rho^4\bigg]dv \\ &\geqslant \int_M \left[n(\rho^2H^2 - H^{\alpha}H^{\beta}\tilde{\sigma}_{\alpha\beta}) + \rho^2\left(n - \left(2-\frac{1}{p}\right)\rho^2\right) + \frac{n(n-2)}{2}H^2\rho^2\right]dv \\ &\geqslant \int_M \rho^2\left[n - \left(2-\frac{1}{p}\right)\rho^2\right]dv, \end{aligned} \tag{3.6}$$

where we used Lemma 4.2 of [11] and $\tilde{\sigma}_{\alpha\beta} = \tilde{\sigma}_{\alpha}\delta_{\alpha\beta}$ (see (4.9) of [11])

$$\sum_{\alpha,\beta} H^{\alpha}H^{\beta}\tilde{\sigma}_{\alpha\beta} = \sum_{\alpha}(H^{\alpha})^2\tilde{\sigma}_{\alpha} \leqslant \sum_{\alpha}(H^{\alpha})^2\cdot\sum_{\beta}\tilde{\sigma}_{\beta} = H^2\rho^2. \tag{3.7}$$

Thus we reach the integral inequality (1.12) in Theorem 1.3.

If (1.12) holds, then we conclude from (5.4) that either $\rho^2 \equiv 0$, or $\rho^2 \equiv n/\left(2 - \frac{1}{p}\right)$. In the first case, we know that $S \equiv nH^2$, i.e. M is totally umbilic; in the latter case, i.e.,

$$\rho^2 = \sum_{\alpha,i,j} (\tilde{h}_{ij}^{\alpha})^2 \equiv n/\left(2 - \frac{1}{p}\right), \tag{3.8}$$

(3.6) becomes an equality, we conclude that either $H = 0$ if $n \geqslant 3$, or $n = 2$.

If $n \geqslant 3$ and $H = 0$, from Main Theorem of Chern, Do Carmo and Kobayashi [4] we get $p = 1$ and

$$M = S^m\left(\sqrt{\frac{m}{n}}\right) \times S^{n-m}\left(\sqrt{\frac{n-m}{n}}\right), \tag{3.9}$$

combining (3.9) with (1.4), we conclude that $n = 2m$ and $M = C_{m,m}$, thus we prove Theorem 1.3. If $n = 2$, our Theorem 1.3 comes from Theorem 3 of [7] (also see [12]). We complete the proof of Theorem 1.3.

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