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Constant angle surfaces in $\mathbb{S}^2 \times \mathbb{R}$

By

Franki Dillen[1], **Johan Fastenakels**[1],
Joeri Van der Veken[1], and **Luc Vrancken**[2]

[1] Katholieke Universiteit Leuven, Belgium
[2] Université de Valenciennes, France

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Abstract. In this article we study surfaces in $\mathbb{S}^2 \times \mathbb{R}$ for which the unit normal makes a constant angle with the $\mathbb{R}$-direction. We give a complete classification for surfaces satisfying this simple geometric condition.

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1. Introduction

In recent years there has been done some research about surfaces in a 3-dimensional Riemannian product of a surface $\mathbb{M}^2$ and $\mathbb{R}$. This was motivated by the study of minimal surfaces. In particular, Rosenberg and Meeks initiated this in [5] and [6]. This work inspired other geometers, for example in [1], [2], [3] and [4].

In this article we consider a special case of a $\mathbb{M}^2 \times \mathbb{R}$, namely we take $\mathbb{M}^2$ to be the unit 2-sphere $\mathbb{S}^2$. In this space we look at constant angle surfaces. By this we mean a surface for which the unit normal makes a constant angle with the tangent direction to $\mathbb{R}$. We show that this simple geometric condition locally completely determines the surface intrinsically. Furthermore, we prove in the classification theorem that we can construct a constant angle surface starting from an arbitrary curve in $\mathbb{S}^2$.

2. Preliminaries

Let $\mathbb{S}^2 \times \mathbb{R}$ be the Riemannian product of the 2-sphere $\mathbb{S}^2(1)$ and $\mathbb{R}$ with the standard metric $\langle\, ,\, \rangle$ and Levi-Civita connection $\widetilde{\nabla}$. We denote by $\frac{\partial}{\partial t}$ a unit vector field in the tangent bundle $T(\mathbb{S}^2 \times \mathbb{R})$ that is tangent to the $\mathbb{R}$-direction.

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For $p \in (\mathbb{S}^2 \times \mathbb{R})$, the Riemann-Christoffel curvature tensor $\widetilde{R}$ of $\mathbb{S}^2 \times \mathbb{R}$ is given by

$$\langle \widetilde{R}(X,Y)Z,W\rangle = \langle X_{\mathbb{S}^2}, W_{\mathbb{S}^2}\rangle\langle Y_{\mathbb{S}^2}, Z_{\mathbb{S}^2}\rangle - \langle X_{\mathbb{S}^2}, Z_{\mathbb{S}^2}\rangle\langle Y_{\mathbb{S}^2}, W_{\mathbb{S}^2}\rangle$$

where $X,Y,Z,W \in T_p(\mathbb{S}^2 \times \mathbb{R})$ and $X_{\mathbb{S}^2} = X - \langle X, \frac{\partial}{\partial t}\rangle \frac{\partial}{\partial t}$ is the projection of X to the tangent space of $\mathbb{S}^2$.

Let us consider $F : M \to \widetilde{M}$, an isometric immersion of a submanifold M into a Riemannian manifold $\widetilde{M}$ with Levi Civita connection $\widetilde{\nabla}$. Then we have the formulas of Gauss and Weingarten which state that for every X and Y tangent to M and for every N normal to M the equations

$$\widetilde{\nabla}_X Y = \nabla_X Y + h(X,Y), \tag{1}$$

$$\widetilde{\nabla}_X N = -A_N X + \nabla^{\perp}_X N, \tag{2}$$

hold, with ∇ the Levi Civita connection of the submanifold. Here, h is a symmetric (1,2)-tensorfield, taking values in the normal bundle, called the second fundamental form of the submanifold, A_N is a symmetric $(1,1)$-tensorfield, called the shape operator associated to N and $\nabla^{\perp}$ is a connection in the normal bundle. For hypersurfaces, $\nabla^{\perp}$ vanishes, but further on we will need the Weingarten formula also for codimension 2 immersions.

Now consider a surface M in $\mathbb{S}^2 \times \mathbb{R}$. Let us denote with ξ a unit normal to M with shape operator A. Then we can decompose $\frac{\partial}{\partial t}$ as

$$\frac{\partial}{\partial t} = T + \cos\theta\, \xi, \tag{3}$$

where T is the projection of $\frac{\partial}{\partial t}$ on the tangent space of M and θ is the angle function defined by

$$\cos\theta(p) = \left\langle \frac{\partial}{\partial t}, \xi \right\rangle \tag{4}$$

for every point $p \in M$.

If we denote by R the curvature tensor of M, then with the previous notation, the equations of Gauss and Codazzi are given by

$$\begin{aligned}\langle R(X,Y)Z,W\rangle = {} & \langle AY,Z\rangle\langle AX,W\rangle - \langle AX,Z\rangle\langle AY,W\rangle + \langle X,W\rangle\langle Y,Z\rangle \\ & - \langle X,Z\rangle\langle Y,W\rangle + \langle Y,T\rangle\langle W,T\rangle\langle X,Z\rangle + \langle X,T\rangle\langle Z,T\rangle\langle Y,W\rangle \\ & - \langle X,T\rangle\langle W,T\rangle\langle Y,Z\rangle - \langle Y,T\rangle\langle Z,T\rangle\langle X,W\rangle \end{aligned} \tag{5}$$

$$\nabla_X AY - \nabla_Y AX - A[X,Y] = \cos\theta\, (\langle Y,T\rangle X - \langle X,T\rangle Y). \tag{6}$$

Furthermore, we have the following proposition.

Proposition 1. *For every $X \in TM$, we have that*

$$\nabla_X T = \cos\theta\, AX, \tag{7}$$

$$X[\cos\theta] = -\langle AX, T\rangle. \tag{8}$$

We can prove this by using that $\frac{\partial}{\partial t}$ is a parallel vector field in $\mathbb{S}^2 \times \mathbb{R}$ and the decomposition (3).

Equations (5), (6), (7) and (8) are called the compatibility equations for $\mathbb{S}^2 \times \mathbb{R}$. In [4], the following theorem was proven.

Theorem 1 (B. Daniel). *Let M be a simply connected Riemannian surface, ds^2 its metric and ∇ its Levi Civita connection. Let A be a field of symmetric operators $A_y : T_y(M) \to T_y(M)$, T a vector field on M and θ a smooth function on M such that $\|T\|^2 = \sin^2\theta$. Assume that (ds^2, A, T, θ) satisfies the compatibility equations for $\mathbb{S}^2 \times \mathbb{R}$. Then there exists an isometric immersion $F : M \to \mathbb{S}^2 \times \mathbb{R}$ such that the shape operator with respect to the unit normal ξ is given by A and such that*

$$\frac{\partial}{\partial t} = T + \cos\theta\, \xi.$$

Moreover the immersion is unique up to global isometries of $\mathbb{S}^2 \times \mathbb{R}$ preserving the orientations of both $\mathbb{S}^2$ and $\mathbb{R}$.

3. Characterizations of constant angle surfaces

In this section we introduce the notion of constant angle surfaces and give some first characterizations.

By a constant angle surface M in $\mathbb{S}^2 \times \mathbb{R}$, we mean a surface for which the angle function θ is constant on M. There are two trivial cases, $\theta = 0$ and $\theta = \frac{\pi}{2}$. The condition $\theta = 0$ means that $\frac{\partial}{\partial t}$ is always normal, so we get a $\mathbb{S}^2 \times \{t_0\}$. In the second case $\frac{\partial}{\partial t}$ is always tangent. This corresponds to the Riemannian product of a curve in $\mathbb{S}^2$ and $\mathbb{R}$.

Now suppose $\theta \notin \{0, \frac{\pi}{2}\}$. From (8) we immediately see that as θ is a constant,

$$\langle AX, T\rangle = \langle AT, X\rangle = 0 \tag{9}$$

for every $X \in T_p(M)$. This implies that T is a principal direction with principal curvature 0.

Thus if we take an orthonormal basis $\{e_1, e_2\}$ with $e_1 = \frac{T}{\|T\|}$ and e_2 a unit vector field perpendicular to e_1, the shape operator A takes the following form:

$$A = \begin{pmatrix} 0 & 0 \\ 0 & \lambda \end{pmatrix} \tag{10}$$

for a function λ on M.

Combining this with Gauss' equation (5) we find for the Gaussian curvature K

$$K = \langle R(e_1, e_2)e_2, e_1\rangle = \cos^2\theta. \tag{11}$$

We can summarize this in the following proposition.

Proposition 2. *If M is a constant angle surface in $\mathbb{S}^2 \times \mathbb{R}$ with constant angle θ, then M has constant Gaussian curvature $K = \cos^2\theta$ and the projection T of $\frac{\partial}{\partial t}$ is a principal direction.*

Remark that with Proposition 2 the intrinsic geometry of constant angle surfaces is locally completely determined.

4. Classification theorem

In this section we completely describe the constant angle surfaces. We look at $\mathbb{S}^2 \times \mathbb{R}$ as a hypersurface in $\mathbb{E}^4$ and denote $\frac{\partial}{\partial t}$ by $(0,0,0,1)$. We then prove the following classification theorem.

Theorem 2. *A surface M immersed in $\mathbb{S}^2 \times \mathbb{R}$ is a constant angle surface if and only if the immersion F is (up to isometries of $\mathbb{S}^2 \times \mathbb{R}$) locally given by $F : M \to \mathbb{S}^2 \times \mathbb{R} : (u,v) \mapsto F(u,v)$, where*

$$F(u,v) = (\cos(u\cos\theta)f(v) + \sin(u\cos\theta)f(v) \times f'(v), u\sin\theta), \tag{12}$$

$f : I \to \mathbb{S}^2$ is a unit speed curve in $\mathbb{S}^2$ and $\theta \in [0,\pi]$ is the constant angle.

Proof. First we prove that the given immersion (12) is a constant angle surface in $\mathbb{S}^2 \times \mathbb{R}$. To see this we first calculate the tangent vectors

$$\begin{aligned} F_u &= (\cos\theta(-\sin(u\cos\theta)f(v) + \cos(u\cos\theta)f(v) \times f'(v)), \sin\theta) \\ F_v &= (\cos(u\cos\theta)f'(v) + \sin(u\cos\theta)f(v) \times f''(v), 0) \\ &= ((\cos(u\cos\theta) + \sin(u\cos\theta)\tau(v))f'(v), 0) \end{aligned}$$

for some function τ on M. We know that $f \times f''$ is a scalar multiple of f' since f is a unit speed curve in $\mathbb{S}^2$.

The normal $\tilde{\xi}$ of $\mathbb{S}^2 \times \mathbb{R}$ in $\mathbb{E}^4$ is nothing but the position vector where we take the last component to be 0, thus

$$\tilde{\xi} = (\cos(u\cos\theta)f(v) + \sin(u\cos\theta)f(v) \times f'(v), 0).$$

So we find that the unit normal ξ on M in $\mathbb{S}^2 \times \mathbb{R}$ is given by

$$\xi = (-\sin\theta(-\sin(u\cos\theta)f(v) + \cos(u\cos\theta)f(v) \times f'(v)), \cos\theta),$$

and thus we see that

$$\left\langle \xi, \frac{\partial}{\partial t} \right\rangle = \cos\theta$$

is a constant.

Suppose now that we have a surface M in $\mathbb{S}^2 \times \mathbb{R}$ with constant angle function θ. If M is one of the trivial cases, M can be parameterized by (12) as can easily be seen. Suppose from now on that $\theta \notin \{0, \frac{\pi}{2}\}$. Then we can take an orthonormal basis of the tangent space $e_1 = \frac{T}{\|T\|}$ and e_2 perpendicular to e_1. As we saw earlier, the shape operator A corresponding to the unit normal ξ with respect to e_1 and e_2 is then given by

$$A = \begin{pmatrix} 0 & 0 \\ 0 & \lambda \end{pmatrix} \tag{13}$$

for a function λ on M.

Using (7), one can calculate that the Levi-Civita connection ∇ of M satisfies

$$\nabla_{e_1} e_1 = 0, \tag{14}$$

$$\nabla_{e_1} e_2 = 0, \tag{15}$$

$$\nabla_{e_2} e_1 = \lambda \cot\theta \; e_2, \tag{16}$$

$$\nabla_{e_2} e_2 = -\lambda \cot\theta \; e_1. \tag{17}$$

Now take coordinates (u, v) on M with $\frac{\partial}{\partial u} = \alpha e_1$ and $\frac{\partial}{\partial v} = \beta e_2$. From the condition $\left[\frac{\partial}{\partial u}, \frac{\partial}{\partial v}\right] = 0$ we find, using (15) and (16):

$$\alpha_v = 0, \tag{18}$$

$$\beta_u = \alpha\beta\lambda \cot\theta. \tag{19}$$

Equation (18) implies that, after a change of the u-coordinate, we can assume $\alpha = 1$ and thus the metric takes the form

$$ds^2 = du^2 + \beta^2(u, v)\; dv^2 \tag{20}$$

and the Eqs. (14), (15), (16) and (17) become

$$\nabla_{\frac{\partial}{\partial u}} \frac{\partial}{\partial u} = 0, \tag{21}$$

$$\nabla_{\frac{\partial}{\partial u}} \frac{\partial}{\partial v} = \lambda \cot\theta \; \frac{\partial}{\partial v}, \tag{22}$$

$$\nabla_{\frac{\partial}{\partial v}} \frac{\partial}{\partial v} = -\beta\beta_u \; \frac{\partial}{\partial u} + \frac{\beta_v}{\beta} \; \frac{\partial}{\partial v}. \tag{23}$$

Furthermore we find from Codazzi's equation (6) that λ must satisfy

$$\lambda_u = -\cos\theta \sin\theta - \lambda^2 \cot\theta. \tag{24}$$

Solving (19) and (24) we find

$$\lambda(u, v) = -\sin\theta \; \tan(u\cos\theta + C(v)), \tag{25}$$

$$\beta(u, v) = D(v) \; \cos(u\cos\theta + C(v)) \tag{26}$$

for some functions C and D on M.

Now let us consider our surface M as a codimension 2 immersed surface in $\mathbb{E}^4$ and denote with D the Euclidean connection and with $\nabla^\perp$ the normal connection. Then we have two unit normals: $\xi = (\xi_1, \xi_2, \xi_3, \cos\theta)$ tangent to $\mathbb{S}^2 \times \mathbb{R}$ and $\tilde{\xi} = (F_1, F_2, F_3, 0)$ normal to $\mathbb{S}^2 \times \mathbb{R}$ with shape operator A respectively $\widetilde{A}$. We have for every $X = (X_1, X_2, X_3, X_4) \in T_p(M)$,

$$\begin{aligned} \nabla^\perp_X \tilde{\xi} &= \langle D_X \tilde{\xi}, \xi \rangle \xi \\ &= \langle (X_1, X_2, X_3, 0), \xi \rangle \xi \\ &= -\cos\theta \langle X, T \rangle \xi \end{aligned} \tag{27}$$

and hence

$$\nabla^\perp_X \xi = \cos\theta \langle X, T \rangle \tilde{\xi}. \tag{28}$$

From (27) and the formula of Weingarten (2) we get

$$\widetilde{A}\left(\frac{\partial}{\partial u}\right) = -((F_1)_u, (F_2)_u, (F_3)_u, 0) - \cos\theta\sin\theta(\xi_1, \xi_2, \xi_3, \cos\theta) \quad (29)$$

$$\widetilde{A}\left(\frac{\partial}{\partial v}\right) = -((F_1)_v, (F_2)_v, (F_3)_v, 0). \quad (30)$$

Since $\frac{\partial}{\partial u} = e_1 = \frac{T}{\|T\|}$ and $\frac{\partial}{\partial v} = \beta e_2$ with e_2 normal to e_1 we find that

$$(F_4)_u = \left\langle F_u, \frac{\partial}{\partial t}\right\rangle = \sin\theta, \quad (31)$$

$$(F_4)_v = \left\langle F_v, \frac{\partial}{\partial t}\right\rangle = 0. \quad (32)$$

Thus we can take $F_4 = u\sin\theta$, since translations in the direction of $(0,0,0,1)$ are isometries of $\mathbb{S}^2 \times \mathbb{R}$.

By looking at (30) and at the fourth component of (29) we see that the shape operator $\widetilde{A}$ with respect to $\frac{\partial}{\partial u}$ and $\frac{\partial}{\partial v}$ is of the following form:

$$\widetilde{A} = \begin{pmatrix} -\cos^2\theta & 0 \\ 0 & -1 \end{pmatrix}. \quad (33)$$

Comparing the other components of (29) we get

$$\xi_j = -\tan\theta(F_j)_u \quad (34)$$

for $j = 1,2,3$.

Now applying the formula of Gauss (1), using (21), (22), (23), (13), (33) and (34) we find

$$(F_j)_{uu} = -\cos^2\theta F_j, \quad (35)$$

$$(F_j)_{uv} = \lambda\cot\theta(F_j)_v, \quad (36)$$

$$(F_j)_{vv} = -\beta\beta_u(F_j)_u + \frac{\beta_v}{\beta}(F_j)_v - \lambda\beta^2\tan\theta(F_j)_u - \beta^2 F_j \quad (37)$$

for $j = 1,2,3$.

From (36) we find that

$$(F_j)_v = \cos(u\cos\theta + C(v))H_j(v) \quad (38)$$

and hence

$$F_j = \int_{v_0}^{v} \cos(u\cos\theta + C(y))H_j(y)\,dy + I_j(u) \quad (39)$$

for $j = 1,2,3$ and with H_j and I_j arbitrary functions on M.

From (35) we find that the function I_j from (39) also must satisfy

$$I_j(u) = K_j\cos(u\cos\theta) + L_j\sin(u\cos\theta), \quad (40)$$

where K_j and L_j are constants.

To summarize, we see that our immersion F is of the following form:

$$F = \Bigg(\Bigg(K_1 + \int_{v_0}^{v} \cos(C(y))H_1(y)\,dy\Bigg)\cos(u\cos\theta) + \Bigg(L_1 - \int_{v_0}^{v} \sin(C(y))H_1(y)\,dy\Bigg)\sin(u\cos\theta), \ldots, u\sin\theta\Bigg). \quad (41)$$

Now define the functions

$$f_j(v) = K_j + \int_{v_0}^{v} \cos(C(y))H_j(y)\,dy, \quad (42)$$

$$g_j(v) = L_j - \int_{v_0}^{v} \sin(C(y))H_j(y)\,dy. \quad (43)$$

Moreover we have the following conditions

$$\langle F_u, F_u\rangle = 1,\ \langle F_v, F_v\rangle = \beta^2(u,v),\ \langle F_u, F_v\rangle = 0,$$
$$\langle F_u, \xi\rangle = 0,\ \langle F_v, \xi\rangle = 0,\ \langle \xi, \xi\rangle = 1,$$
$$\langle F_u, \tilde{\xi}\rangle = 0,\ \langle F_v, \tilde{\xi}\rangle = 0,\ \langle \tilde{\xi}, \tilde{\xi}\rangle = 1,$$
$$\langle \xi, \tilde{\xi}\rangle = 0,$$

which are equivalent to

$$\sum_{j=1}^{3} f_j^2 = 1, \quad (44)$$

$$\sum_{j=1}^{3} g_j^2 = 1, \quad (45)$$

$$\sum_{j=1}^{3} f_j g_j = 0, \quad (46)$$

$$\sum_{j=1}^{3} f_j' g_j = 0, \quad (47)$$

$$\sum_{j=1}^{3} H_j^2 = \sum_{j=1}^{3} (f_j')^2 + (g_j')^2 = D(v)^2. \quad (48)$$

From (44) and (45) we see that $f(v) = (f_1(v), f_2(v), f_3(v))$ and $g(v) = (g_1(v), g_2(v), g_3(v))$ are curves in $\mathbb{S}^2$. Moreover if we change the v-coordinate such that f becomes a unit speed curve, which corresponds to setting $D(v)^2 = \sec^2(C(v))$, we see from (46) and (47) that g is a unit vector perpendicular to the unit vectors f and f'. Thus $g = \pm f \times f'$ and we can choose $g = f \times f'$. Then the immersion $F : M \to \mathbb{S}^2 \times \mathbb{R}$ is given by

$$F(u,v) = (\cos(u\cos\theta)f(v) + \sin(u\cos\theta)f(v) \times f'(v), u\sin\theta) \quad (49)$$

as we wished to prove. □

5. Final remarks

We see that Eq. (37) is also satisfied. After a straightforward computation, we see that (37) expresses that $f_j^2 + g_j^2 + \left(\frac{H_j}{D}\right)^2$ must be a constant for every j. This is the case since $\frac{1}{D}(H_1, H_2, H_3)$ is a unit vector in the direction of f' and thus f, g and $\frac{1}{D}(H_1, H_2, H_3)$ form an orthonormal basis. Also the equations from the formula of Weingarten are satisfied.

Remark also that the two trivial cases are included in the parametrization (12). If $\theta = 0$, (12) becomes

$$F(u, v) = (\cos(u)f(v) + \sin(u)f(v) \times f'(v), 0) \tag{50}$$

which gives us $\mathbb{S}^2 \times \{0\}$.
For $\theta = \frac{\pi}{2}$, (12) becomes

$$F(u, v) = (f(v), u). \tag{51}$$

This clearly gives the Riemannian product of a curve in $\mathbb{S}^2$ and $\mathbb{R}$.

Finally we want to give a non-trivial example of a constant angle surface. In fact we can construct many examples since we know from Theorem 2 that there is a constant angle surface for every curve in $\mathbb{S}^2$. We want to give one special case explicitly. Therefore look at the immersion $F : M \to \mathbb{S}^2 \times \mathbb{R} \subset \mathbb{E}^4$ given by

$$F(u, v) = (\cos u \, \cos v, \, \cos u \, \sin v, \, \sin u, u \tan \theta) \tag{52}$$

where $\theta \in]0, \frac{\pi}{2}[$ is a constant. This is a reparametrization of (12) if f is a great circle. We can see geometrically that this is a constant angle surface. If we take $v = 0$, then we get a curve in $\mathbb{S}^1 \times \mathbb{R}$. This curve is nothing but a helix which has the property that the tangent vector makes a constant angle with $\frac{\partial}{\partial t}$. Now we get the surface (52) by rotating this curve.

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Authors' addresses: Franki Dillen, Johan Fastenakels, and Joeri Van der Veken, Departement Wiskunde, Katholieke Universiteit Leuven, Celestijnenlaan 200 B, B-3001 Leuven, Belgium, e-mail: franki.dillen@wis.kuleuven.be, johan.fastenakels@wis.kuleuven.be, joeri.vanderveken@wis.kuleuven.be; Luc Vrancken, Lamath, ISTV2, Université de Valenciennes, Campus du Mont Houy, 59313 Valenciennes, Cedex 9, France, e-mail: luc.vrancken@univ-valenciennes.fr