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Rigidity Theorem for Hypersurfaces in a Unit Sphere

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Abstract. By investigating hypersurfaces M^n in the unit sphere $S^{n+1}(1)$ with $H_k = 0$ and with two distinct principal curvatures, we give a characterization of torus the $S^1(\sqrt{k/n}) \times S^{n-1}(\sqrt{(n-k)/n})$. We extend recent results of Perdomo [9], Wang [10] and Otsuki [8].

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1. Introduction

Let M be an n-dimensional compact hypersurface in a unit sphere $S^{n+1}(1)$ of dimension $n+1$. Denote by S the square norm of the second fundamental form of M. Perdomo [9] and Wang [10] proved

Theorem 1.1. ([9], [10]) *Let M be an $n(\geqslant 3)$-dimensional compact minimal connected hypersurface in a unit sphere $S^{n+1}(1)$ with two distinct principal curvatures. Let V be the volume of M and assume that one of the principal curvatures of M is simple* (*i.e. multiplicity* 1). *Then the square norm S of the second fundamental form of M satisfies*

$$\int_M S \leqslant nV \tag{1.1}$$

with equality holding if and only if M is a Clifford minimal hypersurface $S^{n-1}\left(\sqrt{\frac{n-1}{n}}\right) \times S^1\left(\sqrt{\frac{1}{n}}\right)$.

In this paper, we consider n-dimensional hypersurfaces with $H_k = 0$ of a unit sphere $S^{n+1}(1)$ and with two distinct principal curvatures. We generalize Theorem 1.1 to hypersurfaces with $H_k = 0$. In fact, we prove the following result

Theorem 1.2. *If M is an n-dimensional compact connected hypersurface $(n \geqslant 3)$ in $S^{n+1}(1)$ with $H_k = 0$ $(k < n)$ and with two distinct principal curvatures. Let V be the volume of M and assume that one of the principal curvatures of M is* *simple (i.e. multiplicity 1). Then the square norm S of the second fundamental form of M satisfies*

$$\int_M S \leqslant \frac{n(k^2-2k+n)}{k(n-k)} V \tag{1.2}$$

with equality holding if and only if M is isometric to the Riemannian product $S^{n-1}(\sqrt{(n-k)/n}) \times S^1(\sqrt{k/n})$, *where* H_k *is the normalized* $k-$*th symmetric function of the principal curvatures of the hypersurface.*

Remark 1.1. When $k=1$, our Theorem 1.2 reduces to Theorem 1.1 of Perdomo and Wang also to Hasanis-Vlachos's result in [4].

2. Preliminaries

Let M be an n-dimensional hypersurface in an $(n+1)$-dimensional unit sphere $S^{n+1}(1)$ with $H_k=0$. Let $\{e_1,\ldots,e_n\}$ be a local orthonormal basis of M with respect to the induced metric, $\omega_1,\ldots,\omega_n$ their dual form. Let e_{n+1} be the local unit normal vector field. In this paper we shall make use of the following convention on the ranges of indices:

$$1 \leqslant A,B,C,\ldots \leqslant n+1, \quad 1 \leqslant i,j,k,\ldots \leqslant n, \quad 1 \leqslant a,b,\ldots \leqslant n-1. \tag{2.1}$$

Then we have the structure equations

$$dx = \sum_i \omega_i e_i, \tag{2.2}$$

$$de_i = \sum_j \omega_{ij} e_j + \sum_j h_{ij}\omega_j e_{n+1} - \omega_i x, \tag{2.3}$$

$$de_{n+1} = -\sum_{i,j} h_{ij}\omega_j e_i, \tag{2.4}$$

where h_{ij} denotes the components of the second fundamental form of M. The Gauss equations are

$$R_{ijkl} = (\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}) + (h_{ik}h_{jl} - h_{il}h_{jk}), \tag{2.5}$$

$$n(n-1)R = n(n-1) + n^2H^2 - S, \tag{2.6}$$

where $R = \frac{\sum_{i,j} R_{ijij}}{n(n-1)}$ is the normalized scalar curvature of M and $S = \sum_{i,j} h_{ij}^2$ is the norm square of the second fundamental form and H is the mean curvature, then we have

$$S = \sum_{i,j} (h_{ij})^2, \qquad H = \frac{1}{n}\sum_k h_{kk}. \tag{2.7}$$

The Codazzi equations are (see [2], [3], [6])

$$h_{ijk} = h_{ikj}, \tag{2.8}$$

where the covariant derivative of h_{ij} is defined by

$$\sum_k h_{ijk}\omega_k = dh_{ij} + \sum_k h_{kj}\omega_{ki} + \sum_k h_{ik}\omega_{kj}. \tag{2.9}$$

The second covariant derivative of h_{ij} is defined by (see [2], [6])

$$\sum_l h_{ijkl}\omega_l = dh_{ijk} + \sum_l h_{ljk}\omega_{li} + \sum_l h_{ilk}\omega_{lj} + \sum_l h_{ijl}\omega_{lk}. \tag{2.10}$$

By exterior differentiation of (2.9), we have the following Ricci identities

$$h_{ijkl} - h_{ijlk} = \sum_m h_{mj}R_{mikl} + \sum_m h_{im}R_{mjkl}. \tag{2.11}$$

We may choose proper frame field $\{e_1, \ldots, e_{n+1}\}$ such that

$$\omega_{in+1} = \lambda_i\omega_i, \quad \text{that is } h_{ij} = \lambda_i\delta_{ij}, \;\; i = 1, 2, \ldots, n, \tag{2.12}$$

where λ_i are principal curvatures.

Now we assume that M has two distinct principal curvatures λ (multiplicity $n-1$) and μ (multiplicity 1), that is,

$$\lambda_1 = \lambda_2 = \cdots = \lambda_{n-1} = \lambda; \quad \lambda_n = \mu. \tag{2.13}$$

From (2.6), we have

$$n(n-1)R = n(n-1) + (n-1)\lambda[(n-2)\lambda + 2\mu]. \tag{2.14}$$

Let H_k be the normalized $k-$th symmetric function of the principal curvatures of an hypersurface:

$$C_n^k H_k = \sum_{1 \leqslant i_1 < i_2 < \cdots < i_k \leqslant n} \lambda_{i_1} \cdots \lambda_{i_k},$$

where $C_n^k = \frac{n!}{k!(n-k)!}$.

From (2.13) and $H_k \equiv 0$, We deduce that

$$C_n^k H_k = C_{n-1}^k \lambda^k + C_{n-1}^{k-1}\lambda^{k-1}\mu \equiv 0. \tag{2.15}$$

and it follows that

$$\lambda^{k-1}[(n-k)\lambda + k\mu] \equiv 0. \tag{2.16}$$

If $\lambda = 0$ at some point p, we can deduce from (2.16) that $\lambda \equiv 0$ on M. In fact, let $N = \{x \,|\, x \in M, \lambda(x) \neq 0\}$, $Q = \{y \,|\, y \in M, (n-k)\lambda(y) + k\mu(y) = 0\}$. Since these principal curvatures λ and μ are continuous on M, we know that N is an open set, Q is a closed set and $N \neq M$ (since $\lambda(p) = 0$). Next we prove $N = Q$. On the one hand, if $x \in N$, then $\lambda(x) \neq 0$. By (2.16), we obtain $(n-k)\lambda(x) + k\mu(x) = 0$, that is, $x \in Q$. Hence $N \subseteq Q$. On the other hand, if $y \in Q$, then $(n-k)\lambda(y) + k\mu(y) = 0$. Since λ and μ are two distinct principal curvatures of M, we have $\lambda(y) \neq \mu(y)$. We see from $(n-k)\lambda(y) + k\mu(y) = 0$ that $\lambda(y) \neq 0$ (If $\lambda(y) = 0$, then $\mu(y) = 0 = \lambda(y)$. This is a contradiction.) That is $y \in N$, then we have

$Q \subseteq N$. Therefore $N = Q$. We see that N is not only an open set but also a closed set. Combining M connected with $N \neq M$, we obtain N is an empty set. It follows that $\lambda \equiv 0$ on M.

From Gauss Eq. (2.14), we know that $R = 1$. By (2.5), we obtain that the sectional curvature of M is not less than 1. Hence M is compact by use of the Bonnet-Myers Theorem. According to Theorem 2 in [2] due to Cheng and Yau, we know that M is a totally umbilical hypersurface. As a result, we get $\lambda \neq 0$ and

$$(n-k)\lambda + k\mu \equiv 0. \tag{2.17}$$

Example. $M_{k,n-k} = S^1(\sqrt{k/n}) \times S^{n-1}(\sqrt{(n-k)/n}), \quad 1 \leqslant k \leqslant n-1$. Then $M_{k,n-k}$ has two distinct constant principal curvatures

$$\lambda_1 = \cdots = \lambda_{n-1} = \sqrt{k/(n-k)}, \quad \lambda_n = -\sqrt{(n-k)/k}. \tag{2.18}$$

Hence, $H_k \equiv 0$ and the square norm of the second fundamental form of $M_{k,n-k}$ satisfies

$$S = \frac{n(k^2 - 2k + n)}{k(n-k)}. \tag{2.19}$$

In [8], Otsuki proved the following

Lemma 2.1. (see p. 150 of [8]) *Let M be an n-dimensional compact hypersurface in a unit sphere $S^{n+1}(1)$ such that the multiplicities of principal curvatures are all constant. Then the distribution of the space of principal vectors corresponding to each principal curvature is completely integrable. In particular, if the multiplicity of a principal curvature is greater than* 1, *then this principal curvature is constant on each integral submanifold of the corresponding distribution of the space of principal vectors.*

By Lemma 2.1 and (2.17), we have

$$\lambda_{,1} = \cdots = \lambda_{,n-1} = 0, \qquad \mu_{,1} = \cdots = \mu_{,n-1} = 0. \tag{2.20}$$

By means of (2.9) and (2.13), we obtain

$$h_{ijk}\omega_k = \delta_{ij} d\lambda_j + (\lambda_i - \lambda_j)\omega_{ij}. \tag{2.21}$$

Summarizing the arguments above, we obtain

$$h_{ijk} = 0, \quad \text{if } i \neq j,\ \lambda_i = \lambda_j, \tag{2.22}$$

$$h_{aab} = 0, \quad h_{aan} = \lambda_{,n}, \tag{2.23}$$

$$h_{nna} = 0, \quad h_{nnn} = \mu_{,n}. \tag{2.24}$$

By making use of methods similar to those in [8], we can prove the following

Proposition 2.1. *If M is an n-dimensional connected hypersurface $(n \geqslant 3)$ in $S^{n+1}(1)$ with $H_k = 0$ $(k < n)$ and with two distinct principal curvatures λ and μ with multiplicities $(n-1)$ and* 1, *respectively. Then M is a locus of the moving*

$(n-1)$-dimensional submanifold $M_1^{n-1}(s)$ along which the principal curvature λ of multiplicity $n-1$ is constant and which is locally isometric to an $(n-1)$-dimensional sphere $S^{n-1}(c(s)) = E^n(s) \cap S^{n+1}(1)$ of constant curvature and λ satisfies the ordinary differential equation of order 2

$$\frac{d^2\lambda}{ds^2} = \frac{n+k}{n\lambda}\left(\frac{d\lambda}{ds}\right)^2 - \frac{n(n-k)\lambda^3}{k^2} + \frac{n\lambda}{k}, \tag{2.25}$$

where $E^n(s)$ is an n-dimensional linear subspace in the Euclidean space R^{n+2} which is parallel to a fixed E^n.

Remark 2.1. When $H \equiv 0$, our Proposition 2.1 reduces to Theorem 4 of Otsuki in [8].

3. Proof of Theorem 1.2

We first prove the following key lemma

Lemma 3.1. *Let M be an n-dimensional $(n \geqslant 3)$ hypersurface in $S^{n+1}(1)$ with $H_k = 0$ $(k < n)$ and with two distinct principal curvatures and assume that one of the principal curvatures of M is simple. Then we have*

$$\frac{1}{S}\sum_k (S_{,k})^2 = \frac{4n(k^2-2k+n)}{(3n-2)k^2-2nk+n^2}\sum_{i,j,k} h_{ijk}^2. \tag{3.1}$$

Proof. Let $\lambda_1 = \cdots = \lambda_{n-1} = \lambda, \quad \lambda_n = \mu$, then we have $(n-k)\lambda + k\mu = 0$. A direct calculation then gives

$$S = (n-1)\lambda^2 + \mu^2 = \frac{n(k^2-2k+n)\lambda^2}{k^2}, \tag{3.2}$$

$$S_{,i} = \frac{2n(k^2-2k+n)}{k^2}\lambda\lambda_{,i}. \tag{3.3}$$

By use of (3.2), (3.3) and (2.20), we have

$$\frac{1}{S}\sum_k (S_{,k})^2 = \frac{1}{S}(S_{,n})^2 = \frac{4n(k^2-2k+n)}{k^2}(\lambda_{,n})^2. \tag{3.4}$$

On the other hand, by use of (2.22), (2.23) and (2.24), we have

$$\begin{aligned}\sum_{i,j,k} h_{ijk}^2 &= \sum_{a,b,c} h_{abc}^2 + 3\sum_{a,b} h_{abn}^2 + 3\sum_a h_{ann}^2 + h_{nnn}^2 \\ &= 3\sum_a h_{naa}^2 + h_{nnn}^2 = 3(n-1)(\lambda_{,n})^2 + (\mu_{,n})^2 \\ &= \frac{(3n-2)k^2-2nk+n^2}{k^2}(\lambda_{,n})^2.\end{aligned} \tag{3.5}$$

This completes the proof.

Lemma 3.2. ([7]) *Let M be an $n(\geqslant 2)$-dimensional hypersurface in S^{n+1}, then we have*

$$\frac{1}{2}\Delta S = \sum_{i,j,k} h_{ijk}^2 + \sum_i \lambda_i (nH)_{,ii} + \frac{1}{2}\sum_{i,j} R_{ijij}(\lambda_i - \lambda_j)^2, \tag{3.6}$$

where λ_i are principal curvatures of M, $(.)_{,ij}$ is the covariant derivative relative to the induced metric.

Proof of Theorem 1.2. First, we compute

$$\begin{aligned}\frac{1}{2}\Delta(\ln S) &= \frac{1}{2}\sum_k (\ln S)_{,kk} = \frac{1}{2}\sum_k \left(\frac{S_{,k}}{S}\right)_{,k} \\ &= \frac{1}{2}\frac{\Delta S}{S} - \frac{1}{2}\frac{\sum_k (S_{,k})^2}{S^2}.\end{aligned} \tag{3.7}$$

By use of Lemma 3.2 and the Gauss equation $R_{anan} = 1 + \lambda\mu$, we obtain

$$\begin{aligned}\frac{1}{2}\Delta S &= \sum_{i,j,k} h_{ijk}^2 + \frac{1}{2}\sum_{i,j} R_{ijij}(\lambda_i - \lambda_j)^2 + \sum_i \lambda_i (nH)_{,ii} \\ &= \sum_{i,j,k} h_{ijk}^2 + \sum_a R_{anan}(\lambda - \mu)^2 + \sum_i \lambda_i (nH)_{,ii} \\ &= \sum_{i,j,k} h_{ijk}^2 + (n-1)(1+\lambda\mu)(\lambda-\mu)^2 + \sum_i \lambda_i (nH)_{,ii} \\ &= \sum_{i,j,k} h_{ijk}^2 + \frac{(n-1)n^2}{k^2}\left[1 - \frac{n-k}{k}\lambda^2\right]\lambda^2 + \sum_i \lambda_i (nH)_{,ii}.\end{aligned} \tag{3.8}$$

From (2.10) and (2.12), we have

$$\lambda_{,ij}\omega_j = d\lambda_{,i} + \lambda_{,j}\omega_{ji}. \tag{3.9}$$

From (2.21), (2.22), (2.23) and (2.24), we obtain

$$\omega_{an} = \frac{\lambda_{,n}}{\lambda - \mu}\omega_a.$$

Therefore, we have $d\omega_n = \sum_i \omega_{ni} \wedge \omega_i = 0$, which shows that we may put

$$\omega_n = ds.$$

Then we have

$$\omega_{an} = \frac{k\lambda_{,n}}{n\lambda}\omega_a = (\log \lambda^{k/n})'\omega_a,$$

where the prime denotes the derivative with respect to s.

Let $i = a$ in (3.9), we see from (2.20), (2.22), (2.23) and (2.24) that

$$\begin{aligned}\lambda_{,aj}\omega_j &= d\lambda_{,a} + \lambda_{,j}\omega_{ja} = \lambda_{,n}\omega_{na} \\ &= \lambda_{,n}\frac{\lambda_{,n}}{\mu - \lambda}\omega_a = -\frac{k}{n\lambda}(\lambda_{,n})^2\omega_a.\end{aligned} \tag{3.10}$$

It follows that

$$\lambda_{,aa} = -\frac{k}{n\lambda}(\lambda_{,n})^2. \tag{3.11}$$

Let $i = n$ in (3.9), we know from (2.20) and (2.25) that

$$\begin{aligned}\lambda_{,nj}\omega_j &= d\lambda_{,n} + \lambda_{,j}\omega_{jn} = d\lambda_{,n}\\ &= \left\{\frac{n+k}{n\lambda}(\lambda_{,n})^2 - \frac{n(n-k)\lambda^3}{k^2} + \frac{n\lambda}{k}\right\}\omega_n.\end{aligned} \tag{3.12}$$

It follows that

$$\lambda_{,nn} = \frac{n+k}{n\lambda}(\lambda_{,n})^2 - \frac{n(n-k)\lambda^3}{k^2} + \frac{n\lambda}{k}. \tag{3.13}$$

Putting (3.11) and (3.13) into (3.8), we have

$$\begin{aligned}\frac{1}{2}\Delta S &= \sum_{i,j,k} h_{ijk}^2 + \frac{(n-1)n^2}{k^2}\left[1 - \frac{n-k}{k}\lambda^2\right]\lambda^2 + \sum_i \lambda_i (nH)_{,ii}\\ &= \sum_{i,j,k} h_{ijk}^2 + \frac{(n-1)n^2}{k^2}\left[1 - \frac{n-k}{k}\lambda^2\right]\lambda^2 + (n-1)\lambda\frac{n(k-1)}{k}\lambda_{,aa}\\ &\quad + \mu\frac{n(k-1)}{k}\lambda_{,nn}\\ &= \sum_{i,j,k} h_{ijk}^2 + \frac{(n-1)n^2}{k^2}\left[1 - \frac{n-k}{k}\lambda^2\right]\lambda^2 + (n-1)(1-k)(\lambda_{,n})^2\\ &\quad - \frac{n(n-k)(k-1)}{k^2}\left[\frac{n+k}{n}(\lambda_{,n})^2 - \frac{n(n-k)}{k^2}\lambda^4 + \frac{n}{k}\lambda^2\right]\\ &= \left\{1 - \frac{(k-1)[(n-2)k^2+n^2]}{(3n-2)k^2-2nk+n^2}\right\}\left(\sum_{i,j,k} h_{ijk}^2\right)\\ &\quad + \frac{n^2(k^2-2k+n)}{k^4}\lambda^2\{k-(n-k)\lambda^2\}.\end{aligned} \tag{3.14}$$

Putting (3.14), (3.1) and (3.2) into (3.7), we have

$$\begin{aligned}\frac{1}{2}\Delta(\ln S) &= \frac{1}{2}\frac{\Delta S}{S} - \frac{1}{2}\frac{\sum_k (S_{,k})^2}{S^2}.\\ &= \frac{1}{2S}\left\{\left[1 - \frac{(k-1)((n-2)k^2+n^2)}{(3n-2)k^2-2nk+n^2}\right]\left(\sum_{i,j,k} h_{ijk}^2\right)\right.\\ &\quad \left. + \frac{n^2(k^2-2k+n)}{k^4}\lambda^2[k-(n-k)\lambda^2]\right\} - \frac{1}{2}\frac{\sum_k (S_{,k})^2}{S^2}\\ &= \left\{\frac{(3n-2)k^2-2nk+n^2-(k-1)[(n-2)k^2+n^2]}{4n(k^2-2k+n)} - \frac{1}{2}\right\}\left(\frac{\sum_k (S_{,k})^2}{S^2}\right)\end{aligned}$$

$$
\begin{aligned}
&+\frac{n}{k}\left\{1-\frac{k(n-k)}{n(k^2-2k+n)}S\right\}\\
&=-\frac{k(n-2)(k-1)^2}{4n(k^2-2k+n)}\left\{\frac{\sum_k (S_{,k})^2}{S^2}\right\}+\frac{n}{k}\left\{1-\frac{k(n-k)}{n(k^2-2k+n)}S\right\}\\
&\leqslant \frac{n}{k}\left\{1-\frac{k(n-k)}{n(k^2-2k+n)}S\right\}. \qquad (3.15)
\end{aligned}
$$

Integrating (3.12) over M, we get

$$\int_M S \leqslant \frac{n(k^2-2k+n)}{k(n-k)}V. \qquad (3.16)$$

Thus we get (1.2). If equality holds in (1.2), then we see from (3.15) that $S=\frac{n(k^2-2k+n)}{k(n-k)}$. It follows that λ is constant. Then we have from Example and a result of Cartan [1] that M is isometric to the Riemannian product $S^{n-1}(\sqrt{(n-k)/n})\times S^1(\sqrt{k/n})$. This completes the proof of Theorem 1.2.

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