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Complete Hypersurfaces with Constant Mean Curvature in a Unit Sphere

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Abstract. By investigating hypersurfaces M^n in the unit sphere $S^{n+1}(1)$ with constant mean curvature and with two distinct principal curvatures, we give a characterization of the torus $S^1(a) \times S^{n-1}(\sqrt{1-a^2})$, where $a^2 = \frac{2+nH^2 \pm \sqrt{n^2H^4+4(n-1)H^2}}{2n(1+H^2)}$. We extend recent results of Hasanis et al. [5] and Otsuki [10].

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1. Introduction

Let M be an n-dimensional hypersurface in a unit sphere $S^{n+1}(1)$ of dimension $n+1$. Denote by S the square norm of the second fundamental form of M. In [5], Hasanis et al. proved

Theorem 1.1 ([5]). *Let M be an n-dimensional connected, complete and minimal hypersurface with at most two principal curvatures in $S^{n+1}(1)$. If $S \geqslant n$, then $S = n$ and M is the minimal Clifford torus $S^m(\sqrt{m/n}) \times S^{n-m}(\sqrt{(n-m)/n})$, $1 \leqslant m \leqslant n-1$.*

In this paper, we consider n-dimensional hypersurfaces with constant mean curvature and with two distinct principal curvatures in a unit sphere $S^{n+1}(1)$. In fact, we prove the following results.

Theorem 1.2. *Let M be an n-dimensional $(n \geqslant 3)$ connected, complete hypersurface with constant mean curvature H and with two distinct principal curvatures such that the multiplicity of one of the principal curvatures is $n-1$ in $S^{n+1}(1)$. If*

$$S \geqslant n + \frac{n^3H^2}{2(n-1)} + \frac{n(n-2)}{2(n-1)}\sqrt{n^2H^4+4(n-1)H^2}, \tag{1.1}$$

then $S = n + \frac{n^3H^2}{2(n-1)} + \frac{n(n-2)}{2(n-1)}\sqrt{n^2H^4+4(n-1)H^2}$ and M is isometric to the Riemannian product $S^1(a) \times S^{n-1}(\sqrt{1-a^2})$, where $a^2 = \frac{2+nH^2-\sqrt{n^2H^4+4(n-1)H^2}}{2n(1+H^2)}$.

Remark 1.1. When $H\equiv 0$, our Theorem 1.2 and Proposition 2.1 reduce to Theorem 1.1 of Hasanis et al. in [5].

Theorem 1.3. *Let M be an n-dimensional $(n\geqslant 3)$ connected, complete hypersurface with constant mean curvature H and with two distinct principal curvatures such that the multiplicity of one of the principal curvatures is $n-1$ in $S^{n+1}(1)$. If*

$$S\leqslant n+\frac{n^3H^2}{2(n-1)}-\frac{n(n-2)}{2(n-1)}\sqrt{n^2H^4+4(n-1)H^2}, \tag{1.2}$$

then $S=n+\frac{n^3H^2}{2(n-1)}-\frac{n(n-2)}{2(n-1)}\sqrt{n^2H^4+4(n-1)H^2}$ and M is isometric to the Riemannian product $S^1(a)\times S^{n-1}(\sqrt{1-a^2})$, where $a^2=\frac{2+nH^2+\sqrt{n^2H^4+4(n-1)H^2}}{2n(1+H^2)}$.

Theorem 1.4. *Let M be an n-dimensional $(n\geqslant 3)$ connected, complete and minimal hypersurface with two distinct principal curvatures. If*

$$S\leqslant n,$$

then $S=n$ and M is isometric to the Riemannian product $S^m(\sqrt{m/n})\times S^{n-m}(\sqrt{(n-m)/n})$, $1\leqslant m\leqslant n-1$.

2. Preliminaries

Let M be an n-dimensional hypersurface in an $(n+1)$-dimensional unit sphere $S^{n+1}(1)$ with constant mean curvature H. Let $\{e_1,\dots,e_n\}$ be a local orthonormal basis of M with respect to the induced metric, $\omega_1,\dots,\omega_n$ their dual form. Let e_{n+1} be the local unit normal vector field. In this paper, we shall make use of the following convention on the ranges of indices:

$$1\leqslant i,j,k,\dots\leqslant n,\qquad 1\leqslant a,b,\dots\leqslant m,\qquad m+1\leqslant \alpha,\beta,\dots\leqslant n. \tag{2.1}$$

Then we have the structure equations

$$dx=\sum_i \omega_i e_i, \tag{2.2}$$

$$de_i=\sum_j \omega_{ij}e_j+\sum_j h_{ij}\omega_j e_{n+1}-\omega_i x, \tag{2.3}$$

$$de_{n+1}=-\sum_{i,j} h_{ij}\omega_j e_i, \tag{2.4}$$

where h_{ij} denotes the components of the second fundamental form of M. The Gauss equations are

$$R_{ijkl}=(\delta_{ik}\delta_{jl}-\delta_{il}\delta_{jk})+(h_{ik}h_{jl}-h_{il}h_{jk}), \tag{2.5}$$

$$n(n-1)r=n(n-1)+n^2H^2-S \tag{2.6}$$

where r is the normalized scalar curvature of M and $S=\sum_{i,j}h_{ij}^2$ is the norm square of the second fundamental form and H is the mean curvature, then we have

$$S=\sum_{i,j}(h_{ij})^2, \qquad H=\frac{1}{n}\sum_k h_{kk}. \tag{2.7}$$

The Codazzi equations are

$$h_{ijk}=h_{ikj}, \tag{2.8}$$

where the covariant derivative of h_{ij} is defined by

$$\sum_k h_{ijk}\omega_k=dh_{ij}+\sum_k h_{kj}\omega_{ki}+\sum_k h_{ik}\omega_{kj}. \tag{2.9}$$

The second covariant derivative of h_{ij} is defined by

$$\sum_l h_{ijkl}\omega_l=dh_{ijk}+\sum_l h_{ljk}\omega_{li}+\sum_l h_{ilk}\omega_{lj}+\sum_l h_{ijl}\omega_{lk}. \tag{2.10}$$

By exterior differentiation of (2.6), we have the following Ricci identities

$$h_{ijkl}-h_{ijlk}=\sum_m h_{mj}R_{mikl}+\sum_m h_{im}R_{mjkl}. \tag{2.11}$$

We may choose a frame field $\{e_1,\dots,e_{n+1}\}$ such that

$$\omega_{in+1}=\lambda_i\omega_i, \quad \text{that is } h_{ij}=\lambda_i\delta_{ij},\ i=1,2,\dots,n. \tag{2.12}$$

where λ_i are principal curvatures.

Now we assume that M has two distinct principal curvature λ and μ, that is,

$$\lambda_1=\lambda_2=\cdots=\lambda_m=\lambda; \qquad \lambda_{m+1}=\cdots=\lambda_n=\mu. \tag{2.13}$$

We have

$$m\lambda+(n-m)\mu=nH, \quad \text{that is } m(\lambda-H)+(n-m)(\mu-H)=0, \tag{2.14}$$

$$\lambda-\mu=\frac{n}{n-m}(\lambda-H), \qquad \lambda\mu=\frac{(nH-m\lambda)\lambda}{n-m}. \tag{2.15}$$

Example. $M_{1,n-1}=S^1(a)\times S^{n-1}(\sqrt{1-a^2})$. Then $M_{1,n-1}$ has two distinct constant principal curvatures

$$\lambda_1=\lambda=\frac{\sqrt{1-a^2}}{a}, \qquad \lambda_2=\cdots=\lambda_n=\mu=-\frac{a}{\sqrt{1-a^2}}$$

and constant mean curvature $H=\frac{1}{n}\sum_{i=1}^n\lambda_i=\frac{1-na^2}{na\sqrt{1-a^2}}$.

The square norm of the second fundamental form of $M_{1,n-1}$ satisfies

$$S=\sum_{i=1}^n\lambda_i^2=\frac{(1-a^2)}{a^2}+\frac{(n-1)a^2}{1-a^2}.$$

By a straightforward computation for $M_{1,n-1}=S^1(a)\times S^{n-1}(\sqrt{1-a^2})$, we obtain

$$a^2=\frac{(2+nH^2)\pm\sqrt{n^2H^4+4(n-1)H^2}}{2n(1+H^2)},$$

and

$$S = n + \frac{n^3H^2}{2(n-1)} \mp \frac{n(n-2)}{2(n-1)} \sqrt{n^2H^4 + 4(n-1)H^2}.$$

Now we have to consider two cases.

Case 1: $2 \leqslant m \leqslant n-2$. In [10], Otsuki proved the following

Lemma 2.1. (Theorem 2 and Corollary of [10]). *Let M be an n-dimensional hypersurface in a unit sphere $S^{n+1}(1)$ such that the multiplicities of principal curvatures are all constant. Then the distribution of the space of principal vectors corresponding to each principal curvature is completely integrable. In particular, if the multiplicity of a principal curvature is greater than* 1, *then this principal curvature is constant on each integral submanifold of the corresponding distribution of the space of principal vectors.*

From Lemma 2.1. we can easily obtain the following

Proposition 2.1. *Let M be an n-dimensional hypersurface in a unit sphere $S^{n+1}(1)$ with constant mean curvature H and with two distinct principal curvatures. If the multiplicities of these two distinct principal curvatures are greater than* 1, *then M is isometric to the Riemannian product $S^m(a) \times S^{n-m}(\sqrt{1-a^2})$*, $2 \leqslant m \leqslant n-2$.

Case 2: $m = n-1$. In this case, $m = n-1 \geqslant 2$.

$$\lambda_1 = \cdots = \lambda_{n-1} = \lambda \neq \mu = \lambda_n, \lambda - \mu = n(\lambda - H), \lambda\mu = nH\lambda - (n-1)\lambda^2, \quad (2.16)$$

where λ_i are the principal curvature of M.

By Lemma 2.1 and $(n-1)\lambda + \mu = nH$, we have

$$\lambda_{,1} = \cdots = \lambda_{,n-1} = 0, \qquad \mu_{,1} = \cdots = \mu_{,n-1} = 0. \quad (2.17)$$

By means of (2.9) and (2.12), we obtain

$$h_{ijk}\omega_k = \delta_{ij} d\lambda_j + (\lambda_i - \lambda_j)\omega_{ij}. \quad (2.18)$$

From these and (2.16), we obtain

$$h_{ijk} = 0, \quad \text{if } i \neq j, \qquad \lambda_i = \lambda_j, \quad (2.19)$$

$$h_{aab} = 0, \qquad h_{aan} = \lambda_{,n}, \quad (2.20)$$

$$h_{nna} = 0, \qquad h_{nnn} = \mu_{,n}. \quad (2.21)$$

Let us define a positive function $w(s)$ over $s \in (-\infty, +\infty)$ by

$$w = \begin{cases} (\lambda - H)^{-1/n}, & \text{for } \lambda - H > 0 \\ (H - \lambda)^{-1/n}, & \text{for } \lambda - H < 0 \end{cases}$$

By making use of the similar methods in [10], we can prove the following

Proposition 2.2. *Let M be an n-dimensional connected hypersurface $(n \geqslant 3)$ in $S^{n+1}(1)$ with constant mean curvature H and with two distinct principal* *curvatures* λ *and* μ *with multiplicities* $(n-1)$ *and* 1, *respectively. Then* M *is a locus of the moving* $(n-1)$*-dimensional submanifold* $M_1^{n-1}(s)$ *along which the principal curvature* λ *of multiplicity* $n-1$ *is constant and which is locally isometric to an* $(n-1)$*-dimensional sphere* $S^{n-1}(c(s)) = E^n(s) \cap S^{n+1}(1)$ *of constant curvature and* $w = |\lambda - H|^{-1/n}$ *satisfies the ordinary differential equation of order* 2

$$\frac{d^2w}{ds^2} + w[1 + H^2 + (2-n)Hw^{-n} + (1-n)w^{-2n}] = 0, \quad \text{for } \lambda - H > 0, \quad (2.22)^+$$

or

$$\frac{d^2w}{ds^2} + w[1 + H^2 + (n-2)Hw^{-n} + (1-n)w^{-2n}] = 0, \quad \text{for } \lambda - H < 0, \quad (2.22)^-$$

where $E^n(s)$ *is an n-dimensional linear subspace in the Euclidean space* R^{n+2} *which is parallel to a fixed* E^n.

Remark 2.1. When $H \equiv 0$, our Proposition 2.2 reduces to Theorem 4 of Otsuki in [10].

3. Proof of the Theorems

We first give the following lemmas

Lemma 3.1. *Equations* $(2.22)^{\pm}$ *are equivalent to their first order integral*

$$\left(\frac{dw}{ds}\right)^2 + (1 + H^2)w^2 + 2Hw^{2-n} + w^{2-2n} = C, \quad \text{for } \lambda - H > 0, \qquad (3.1)^+$$

or

$$\left(\frac{dw}{ds}\right)^2 + (1 + H^2)w^2 - 2Hw^{2-n} + w^{2-2n} = C, \quad \text{for } \lambda - H < 0, \qquad (3.1)^-$$

where C *is a constant. Moreover, the constant solution of* $(2.22)^{\pm}$ *corresponds to the Riemannian Product* $S^1(a) \times S^{n-1}(\sqrt{1-a^2})$.

Proof. From the assumption, and by making use of computations similar as in [10], we have $\nabla_{e_n} e_n = 0$. Hence, we know that any integral curve of the principal vector field corresponding to μ is a geodesic. Then we can get that $w(s)$ is a function defined in $(-\infty, +\infty)$ since M is complete and any integral curve of the principal vector field corresponding to μ is a geodesic.

The left hand side of equation $(2.22)^+$ multiplied by $2\frac{dw}{dv}$ is precisely the derivative of the left hand side of equation $(3.1)^+$. Similarly, the left hand side of equation $(2.22)^-$ multiplied by $2\frac{dw}{dv}$ is precisely the derivative of the left hand side of equation $(3.1)^-$. Combining $w(s) = w_0$ with $w = |\lambda - H|^{-1/n}$ and (2.16), we have that λ and μ are constant. Hence the constant solution of $(2.22)^{\pm}$ corresponds to the Riemannian product $S^1(a) \times S^{n-1}(\sqrt{1-a^2})$. We complete the proof of Lemma 3.1.

Lemma 3.2. *If M is an n-dimensional connected hypersurface $(n \geqslant 3)$ in $S^{n+1}(1)$ with constant mean curvature H and with two distinct principal curvatures λ and μ with multiplicities $(n-1)$ and 1, respectively. Then*

$$S \geqslant n + \frac{n^3H^2}{2(n-1)} + \frac{n(n-2)}{2(n-1)}\sqrt{n^2H^4+4(n-1)H^2} \tag{3.2}$$

holds if and only if

$$w^{-n} = |\lambda - H| \geqslant \frac{(n-2)|H| + \sqrt{n^2H^2+4(n-1)}}{2(n-1)}. \tag{3.3}$$

Similarly, we have

$$S \leqslant n + \frac{n^3H^2}{2(n-1)} - \frac{n(n-2)}{2(n-1)}\sqrt{n^2H^4+4(n-1)H^2} \tag{3.4}$$

holds if and only if

$$w^{-n} = |\lambda - H| \leqslant \frac{(2-n)|H| + \sqrt{n^2H^2+4(n-1)}}{2(n-1)}. \tag{3.5}$$

Proof. Using (2.16), we have the calculation that

$$\begin{aligned} S &= (n-1)\lambda^2 + \mu^2 \\ &= n[(n-1)\lambda^2 - 2(n-1)H\lambda + nH^2] \\ &= n[(n-1)w^{-2n} + H^2]. \end{aligned}$$

If

$$S \geqslant n + \frac{n^3H^2}{2(n-1)} + \frac{n(n-2)}{2(n-1)}\sqrt{n^2H^4+4(n-1)H^2},$$

then we obtain

$$w^{-2n} = |\lambda - H|^2 \geqslant \frac{1}{n-1} + \frac{n^2-2n+2}{2(n-1)^2}H^2 + \frac{(n-2)}{2(n-1)^2}\sqrt{n^2H^4+4(n-1)H^2}. \tag{3.6}$$

That is,

$$w^{-n} = |\lambda - H| \geqslant \frac{(n-2)|H| + \sqrt{n^2H^2+4(n-1)}}{2(n-1)}. \tag{3.7}$$

Similarly, we can get the other result. Lemma 3.2 is proved.

The proof of Theorem 1.2. Since we see from Proposition 2.2 that

$$\frac{d^2w}{ds^2} + w[1 + H^2 + (2-n)Hw^{-n} + (1-n)w^{-2n}] = 0, \quad \text{for } \lambda - H > 0.$$

or

$$\frac{d^2w}{ds^2} + w[1 + H^2 + (n-2)Hw^{-n} + (1-n)w^{-2n}] = 0, \quad \text{for } \lambda - H < 0.$$

A direct calculation then gives

$$\frac{d^2w}{ds^2} \geqslant 0$$

if and only if

$$w^{-n} \geqslant \begin{cases} \frac{(2-n)H+\sqrt{n^2H^2+4(n-1)}}{2(n-1)}, & \text{for } \lambda - H > 0 \\ \frac{(n-2)H+\sqrt{n^2H^2+4(n-1)}}{2(n-1)}, & \text{for } \lambda - H < 0 \end{cases} \tag{3.8}$$

From Lemma 3.2, we have

$$w^{-n} \geqslant \frac{(n-2)|H| + \sqrt{n^2H^2 + 4(n-1)}}{2(n-1)}.$$

Summarizing the arguments above, we have

$$\frac{d^2w}{ds^2} \geqslant 0. \tag{3.9}$$

Thus $\frac{dw}{ds}$ is a monotonic function of $s \in (-\infty, +\infty)$. Therefore, $w(s)$ must be monotonic when s tends to infinity.

We see from $(3.1)^{\pm}$ that the positive function $w(s)$ is bounded. Since $w(s)$ is bounded and is monotonic when s tends to infinity, we find that both $\lim_{s\to-\infty} w(s)$ and $\lim_{s\to+\infty} w(s)$ exist and then we have

$$\lim_{s\to-\infty} \frac{dw(s)}{ds} = \lim_{s\to+\infty} \frac{dw(s)}{ds} = 0. \tag{3.10}$$

By the monotonicity of $\frac{dw}{ds}$, we see that $\frac{dw}{ds} \equiv 0$ and $w(s)$ is a constant. Then, according to Lemma 3.1, it is easily known that M is isometric to the Riemannian product $S^1(a) \times S^{n-1}(\sqrt{1-a^2})$. From Example, we have $S = n + \frac{n^3H^2}{2(n-1)} + \frac{n(n-2)}{2(n-1)}\sqrt{n^2H^4 + 4(n-1)H^2}$ and $a^2 = \frac{2+nH^2-\sqrt{n^2H^4+4(n-1)H^2}}{2n(1+H^2)}$. This proves Theorem 1.2.

The proof of Theorem 1.3. Since we see from Proposition 2.2 that

$$\frac{d^2w}{ds^2} + w[1 + H^2 + (2-n)Hw^{-n} + (1-n)w^{-2n}] = 0, \quad \text{for } \lambda - H > 0.$$

or

$$\frac{d^2w}{ds^2} + w[1 + H^2 + (n-2)Hw^{-n} + (1-n)w^{-2n}] = 0, \quad \text{for } \lambda - H < 0.$$

A direct calculation then gives

$$\frac{d^2w}{ds^2} \leqslant 0$$

if and only if

$$w^{-n} \leqslant \begin{cases} \frac{(2-n)H+\sqrt{n^2H^2+4(n-1)}}{2(n-1)}, & \text{for } \lambda - H > 0 \\ \frac{(n-2)H+\sqrt{n^2H^2+4(n-1)}}{2(n-1)}, & \text{for } \lambda - H < 0 \end{cases} \tag{3.11}$$

From Lemma 3.2, we have

$$0 < w^{-n} \leqslant \frac{(2-n)|H| + \sqrt{n^2H^2 + 4(n-1)}}{2(n-1)}.$$

Summarizing the arguments above, we have $\frac{d^2w}{ds^2} \leqslant 0$. We see from $(3.1)^{\pm}$ that the positive function $w(s)$ is bounded. Combining $\frac{d^2w}{ds^2} \leqslant 0$ with the boundedness of $w(s)$, we see that $w(s)$ is a constant. Then, according to Lemma 3.1, it is easily known that M is isometric to the Riemannian product $S^1(a) \times S^{n-1}(\sqrt{1-a^2})$. From Example, we have $S = n + \frac{n^3H^2}{2(n-1)} - \frac{n(n-2)}{2(n-1)}\sqrt{n^2H^4 + 4(n-1)H^2}$ and $a^2 = \frac{2+nH^2+\sqrt{n^2H^4+4(n-1)H^2}}{2n(1+H^2)}$. Theorem 1.3 is proved.

The proof of Theorem 1.4. From Proposition 2.1 and Theorem 1.3, we can easily get our result.

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