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A Schläfli-Type Formula for Polytopes with Curved Faces and Its Application to the Kneser-Poulsen Conjecture

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Abstract. We prove a Schläfli-type formula for polytopes with curved faces lying in pseudo-Riemannian Einstein manifolds. This formula is applied to the Kneser-Poulsen conjecture claiming that the volume of the union of some balls cannot increase when the balls are rearranged in such a way that the distances between the centers decrease.

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1. Introduction

Let $P_1, \dots, P_k$ and $Q_1, \dots, Q_k$ be two k-tuples of points in the Euclidean space $\mathbb{R}^n$. Poulsen [17], Kneser [15] and Hadwiger [14] asked whether the inequalities $d(P_i, P_j) \geqslant d(Q_i, Q_j)$ for all $1 \leqslant i < j \leqslant k$ imply that

$$V_n\left(\bigcup_{i=1}^{k} B(P_i, r)\right) \geqslant V_n\left(\bigcup_{i=1}^{k} B(Q_i, r)\right),$$

where V_n is the n-dimensional volume, r is an arbitrary positive number, $B(P, r)$ is the ball of radius r around P.

Though this question is still open for $n \geqslant 3$, there have been many papers devoted to the verification of the conjecture and its generalizations in special cases. The first series of papers [6], [8], [3], [9], [13], [12], [10] considered the special case, when the balls are moved smoothly in such a way that during the motion all the center–center distances change in a monotonous way. In this special case monotonicity results were obtained not only for the volume of the union of congruent balls of the Euclidean space, but also for the volume of flowers, i.e. domains in the hyperbolic, Euclidean, or spherical space obtained from not necessarily congruent balls with the help of the operations $\cap$ and $\cup$ (see [10]).

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It is not true in general that if $P_1, \dots, P_k$ and $Q_1, \dots, Q_k$ are two configurations in $\mathbb{R}^n$, and $d(Q_i, Q_j) \leqslant d(P_i, P_j)$ for all $1 \leqslant i < j \leqslant k$, then there are continuous curves $\gamma_i : [0,1] \to \mathbb{R}^n$ connecting $\gamma_i(0) = P_i$ to $\gamma_i(1) = Q_i$ such that the distances $d(\gamma_i(t), \gamma_j(t))$ weakly decrease, however, one can always construct such curves in $\mathbb{R}^{2n}$. This simple observation, called the Leapfrog Lemma yields that if we want to show that a certain function on k-tuples of points in $\mathbb{R}^n$ does not increase when the points are contracted, it is enough to find an extension of that function to arbitrary k-tuples in $\mathbb{R}^{2n}$ which does not increase when the points are contracted continuously. This idea has been fruitfully applied in [1], [7], [4], [5]. For instance in the remarkable paper [4], Bezdek and Connelly could prove the Kneser-Poulsen conjecture in the plane using this method.

When the balls are moved smoothly, the change of the volume of the union of the balls can be controlled by a formula expressing the variation of the volume as a linear combination of the derivatives of the center–center distances with coefficients equal to the volumes of the walls of the Dirichlet-Voronoi decomposition of the union of the balls (see [9]). This formula and its extension for flowers obtained in [10] resemble the classical Schläfli formula expressing a multiple of the variation of the volume of a simplex in the hyperbolic or spherical space as a linear combination of the derivatives of the dihedral angles with coefficients equal to the volumes of the 2-codimensional faces. The author's primary motivation for the study of Schläfli-type formulae was to find a common root of these formulae. Roughly speaking, simplices and unions of balls belong to the class of polytopes with curved faces. The family of polytopes with curved faces contains also compact domains bounded by smooth hypersurfaces. For such domains lying in Einstein manifolds, Rivin and Schlenker proved a Schläfli-type formula in [18]. Analysing the sketch of the proof of the formula in [18] the author found a Schläfli-type formula for polytopes with curved faces in Riemannian Einstein manifolds which contained both the classical Schläfli formula and the formula of Rivin and Schlenker as a special case. This formula and some of its applications to the Kneser-Poulsen conjeture was presented at the Bolyai Bicentennial Conference in Budapest in 2002. Right after the conference the author received a preprint from R. Souam devoted to the proof of a Schläfli-type formula for piecewise smooth immersions of simplicial complexes into pseudo-Riemannian Einstein manifolds. The Schläfli-type formula we present in this paper is an extension of the formula presented at the Bolyai Conference to the pseudo-Riemannian case. Our formula is proved in a different setting and is slightly more general than that of R. Souam, which appeared recently in [19]. The main difference is that in Souam's formula the infinitesimal variation of the polytope is assumed to be continuous on the boundary, while in ours the faces of the polytope are deformed separately and the infinitesimal variations of the faces may not coincide on the walls between the faces. The formula in this paper contains exactly the same terms as Souam's formula together with an extra term which is related to the discontinuity of the infinitesimal variations of the faces along the walls between the faces. This extension of Souam's formula turns out to be quite useful when the formula is applied to the union of some smoothly moving balls. Since the balls move rigidly, we can choose Killing fields for the infinitesimal variations of the faces in our formula which simplifies the evaluation of the formula. However, in general it is impossible to choose a continuous infinitesimal variation compatible with the deformation of the boundary the restrictions of which onto the faces of the union are Killing fields.

The paper has the following structure. In Section 2 we give an exact definition of a polytope with curved faces in a manifold, and we introduce some basic notions related to polytopes (faces, walls between the faces, dihedral angles, variation of a polytope and compatible infinitesimal variations, etc.). Section 3 contains some variational formulae which culminate in the Schläfli-type formula in Theorem 2. In Section 3, our formula is specialized for polytopes made of some rigidly moving balls. The relation of the resulting formula to the Kneser-Poulsen conjecture is illuminated by an "Archimedean" formula for solids of revolution (Theorem 5). The main result of this section is Theorem 6 extending some of the results of [9], [10] and [4].

2. Polytopes with Curved Faces

Intuitively, a polytope with curved faces is a compact domain in a manifold, bounded by a finite number of smooth hypersurfaces. There are several mathematical models grasping this intuitive concept. We shall use the approach of Constructive Solid Geometry (CSG) and define polytopes with curved faces as solids that can be obtained by applying regularized Boolean set operations to regular domains of a manifold.

2.1. CSG representations of polytopes with curved faces. Let X be a topological space and denote by $\mathscr{C}_X$ the distributive lattice of closed subsets of X, by $\mathscr{I}_X$ the ideal in $\mathscr{C}_X$ formed by the nowhere dense closed subsets of X. The factor lattice $\mathscr{C}_X/\mathscr{I}_X$ is a Boolean algebra since a closed set intersects the closure of its complement in a nowhere dense set, thus we have complements in $\mathscr{C}_X/\mathscr{I}_X$.

A *regular closed subset* of X is a subset, which is equal to the closure of its interior. There is a *regularization operator* $\rho_X : \mathscr{P}_X \to \mathscr{R}_X$ from the power set $\mathscr{P}_X$ of X to the set $\mathscr{R}_X$ of regular closed subsets of X defined by $\rho_X(A) = \overline{\operatorname{int}(A)}$. Every (mod $\mathscr{I}_X$) congruence class in $\mathscr{C}_X$ contains a unique regular closed set, which can be obtained by the regularization of any of the closed sets belonging to the equivalence class, thus, there is a natural bijection between the Boolean algebra $\mathscr{C}_X/\mathscr{I}_X$ and the set $\mathscr{R}_X$. Transferring the Boolean algebra structure from $\mathscr{C}_X/\mathscr{I}_X$ to $\mathscr{R}_X$ we obtain a Boolean algebra structure on the set of regular closed subsets of X. The induced operations on $\mathscr{R}_X$ are the *regularized Boolean operations*

$$A \cup^* B = \rho_X(A \cup B), \qquad A \cap^* B = \rho_X(A \cap B), \qquad A \setminus^* B = \rho_X(A \setminus B).$$

A Boolean expression f is a formal expression (a finite sequence of symbols) such that f is either a symbol for a single variable or an expression of the form $(f_1 * f_2)$, where f_1 and f_2 are Boolean expressions, $*$ is one of the symbols $\cup$, $\cap$, $\setminus$. Each Boolean expression f corresponds to a regularized Boolean expression f^* obtained from f by replacing each operation symbol by the symbol of the corresponding regularized operation. Two Boolean expressions are considered to be the same if and only if they are identical as sequences of symbols. Thus, x, $(x \cup x)$,

$((x \cup y) \cap x)$ are different Boolean expressions. If $f(x_1, \ldots, x_k)$ is a Boolean expression in k variables, $A_1, \ldots, A_k \in \mathscr{P}_X$ are subsets of X, then we denote by $f(A_1, \ldots, A_k)$ and $f^*(A_1, \ldots, A_k)$ the evaluation of the expressions on the sets $A_1, \ldots, A_k$. If the sets $A_1, \ldots, A_k$ are contained in a subset Y of X, then we denote by $f_Y^*(A_1, \ldots, A_k)$ the evaluation of f^* in Y computed with operations regularized by ρ_Y instead of ρ_X. The following proposition is a straightforward corollary of the definitions.

Proposition 1. *If $A_1, \ldots, A_k \in \mathscr{R}_X$, $B_1, \ldots, B_k \in \mathscr{P}_X$ are subsets such that the symmetric differences $A_i \triangle B_i$ are nowhere dense, $f(x_1, \ldots, x_k)$ is an arbitrary Boolean expression, $f^*(x_1, \ldots, x_k)$ its regularization, then we have*

$$f^*(A_1, \ldots, A_k) = \rho_X(f(A_1, \ldots, A_k)) = \rho_X(f(B_1, \ldots, B_k)).$$

Let M be an n-dimensional smooth manifold. A *regular domain* in M is a subset of the form $\{p \in M \mid f(p) \leqslant 0\}$, where $f : M \to \mathbb{R}$ is a smooth function on M, such that 0 is a regular value of f. (Another equivalent definition can be found in 4.8. [20].) M and $\emptyset$ are regular domains in M. Regular domains are n-dimensional manifolds with or without boundary embedded as regular closed sets in M. The boundary of a regular domain is either empty, or it is a smooth hypersurface in M.

A *CSG solid* in a manifold M is a compact set which can be obtained as the evaluation of a regularized Boolean expression on some regular domains of M. A *CSG representation* of a CSG solid P is a regularized Boolean expression $f^*(x_1, \ldots, x_k)$ together with a collection of regular domains $P_1, \ldots, P_k$ such that $P = f^*(P_1, \ldots, P_k)$.

In Constructive Solid Geometry, the regular domains $P_1, \ldots, P_k$ are called the *primitives* from which P is built, and CSG representations are depicted by a rooted

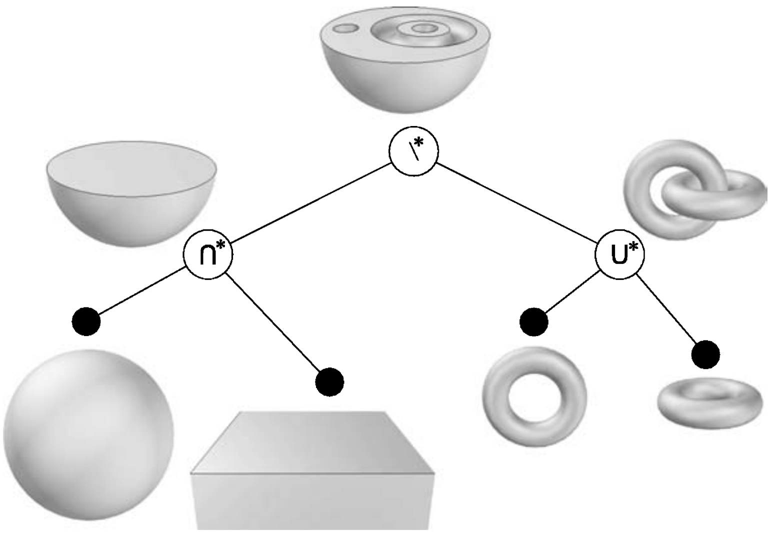

Figure 1. Example of a CSG tree

binary tree, called *CSG tree*, in which the leaves are marked with primitives, and the internal nodes are marked with regularized Boolean operations (see Fig. 1). The CSG representation of a CSG solid is not unique.

Definition 2.1. Let $s \geqslant 1$ be a natural number. A CSG solid P with a fixed CSG representation $f^*(P_1, \dots, P_k)$ will be called *s-transversal* if the following two properties hold:

(i) each of the variables $x_1, \dots, x_k$ occurs in $f^*(x_1, \dots, x_k)$ exactly once;
(ii) any $l \leqslant s$ of the hypersurfaces ∂P_i, $(1 \leqslant i \leqslant k)$ intersect transversally. (We say that the hypersurfaces $\Sigma_1, \dots, \Sigma_l$ intersect transversally if $\dim \bigcap_{i=1}^{l} T_p\Sigma_i = (n-l)$ for all $p \in \bigcap_{i=1}^{l} \Sigma_i$.)

Throughout this paper the term *polytope with curved faces* or simply *polytope* will be used for 3-transversal CSG solids.

2.2. Faces of a polytope and the walls between them

Definition 2.2. Let $P = f^*(P_1, \dots, P_k)$ be a 2-transversal CSG solid in M. The *ith face F_i* of P is the regularized contribution of the hypersurface $\Sigma_i = \partial P_i$ to the boundary of P, that is $F_i = \rho_{\Sigma_i}(\Sigma_i \cap \partial P)$.

Remark that the definition above is slightly different from the ones commonly used in CSG, where the faces are the $(n-1)$-dimensional cells of a CW structure on the boundary.

Proposition 2. *The faces of an s-transversal CSG solid with $s \geqslant 2$ have the following properties.*

(*i*) *F_i is an $(n-1)$-dimensional $(s-1)$-transversal CSG solid in Σ_i, that can be built up from the primitives $P_j \cap \Sigma_i$, $(1 \leqslant j \leqslant k)$, that is, there is a regularized Boolean expression $(\partial_i f^*)(x_1, \dots, x_k)$ in which each of its variables occurs exactly once and*

$$F_i = \partial_i f^*_{\Sigma_i}(P_1 \cap \Sigma_i, \dots, P_k \cap \Sigma_i).$$

$\partial_i f^*$ depends only on f^* and can be defined recursively by the following rules: $\partial_i x_i = x_i$ and if g^* and h^* are Boolean expressions with single use of disjoint sets of variables and x_i is a variable of g^*, then

$$\partial_i(g^* \cup^* h^*) = \partial_i(h^* \cup^* g^*) = \partial_i(g^* \setminus^* h^*) = (\partial_i g^*) \setminus^* h^*,$$
$$\partial_i(g \cap^* h) = (\partial_i g) \cap^* h, \qquad \partial_i(h \cap^* g) = \partial_i(h \setminus^* g) = h \cap^* (\partial_i g).$$

(*ii*) *The union of all the faces equals the boundary of P.*

Proof. (i) The proof of the first part of the proposition goes by induction on the length of f^*. Though it is a bit lengthy due to the several cases for the structure of f^*, it is quite straightforward, so we omit the details.

(ii) Set $N_i = \bigcup_{\substack{1 \leqslant j \leqslant k \\ j \neq i}} \Sigma_i \cap \Sigma_j$. Denote by F_i° the set of those boundary points of P which belong only to Σ_i, that is,

$$F_i^\circ = (\Sigma_i \cap \partial P) \setminus N_i.$$

F_i° is open in Σ_i and therefore $F_i^\circ \subset F_i$. We show that F_i° is dense in F_i. Indeed, if $U \subset \Sigma_i$ is an arbitrary open neighborhood of a point $p \in F_i$, then since F_i is regular closed in Σ_i, U contains a non-empty open (in Σ_i) subset $V \subset F_i \cap U$. Since V is $(n-1)$-dimensional, while the intersections $\Sigma_i \cap \Sigma_j$ are $(n-2)$-dimensional, V must have a point, which is not in N_i. Thus $F_i = \overline{F_i^\circ}$.

Finally we claim that $\bigcup_{1 \leqslant i \leqslant k} F_i^\circ$ is a dense subset of ∂P. To prove this, take a point $p \in \partial P$ and an arbitrary connected neighborhood $\widetilde{U}$ of it. Since P is regular closed, the interior of P is dense in P, therefore $\widetilde{U}$ contains an inner point q of P. $\widetilde{U}$ must contain also a point r in the exterior of P, since p is on the boundary. Connect q to r by a curve γ in $\widetilde{U}$ keeping away from the intersections $\Sigma_i \cap \Sigma_j$. This is possible since the intersections are $(n-2)$-dimensional. As γ connects an interior point of P to a point in its exterior, γ must cross the boundary of P at a point $s \in \partial P \cap \widetilde{U}$. Since ∂P is covered by $\bigcup_i \Sigma_i$, s is contained in Σ_i for an i. i is uniquely determined as γ misses the intersections $\Sigma_i \cap \Sigma_j$. Consequently, $s \in \widetilde{U}$ belongs to F_i°. We conclude that $\partial P = \overline{\bigcup_{i=1}^k F_i^\circ} = \bigcup_{i=1}^k \overline{F_i^\circ} = \bigcup_{i=1}^k F_i$. □

Using the above notation and the terminology of Chap. XVII of [16], the interior of a 2-transversal CSG solid P is a manifold with singular boundary in M. As $\bigcup_{i=1}^k F_i^\circ$ consists of regular frontier points, the set of singular frontier points is covered by the union of the $(n-2)$-dimensional intersections $\Sigma_i \cap \Sigma_j$. This means that Theorem 3.3 in Chap. XVII [16] is applicable to P and yields the following version of Stokes' theorem:

Proposition 3. *If M is oriented, and the regular frontier of the 2-transversal CSG solid P is equipped with the usual induced orientation, then*

$$\int_P d\omega = \sum_{i=1}^k \int_{F_i} \omega. \tag{1}$$

for any differential $(n-1)$-form ω on M.

If P is a polytope, i.e. 3-transversal, then we can apply Proposition 2 to the faces F_i. The ith face of F_i is empty since $\partial_{\Sigma_i}(\Sigma_i \cap P_i) = \emptyset$, the jth face for $j \neq i$ is the regularized contribution of $\Sigma_i \cap \Sigma_j$ to the boundary of the face F_i, i.e. the set $W_{ij} = \rho_{\Sigma_{ij}}(\Sigma_i \cap \Sigma_j) \cap \partial_{\Sigma_i} F_i$. W_{ij} is an $(n-2)$-dimensional CSG solid in $\Sigma_i \cap \Sigma_j$. Comparision of the CSG representations of W_{ij} and W_{ji} derived in the proof of Proposition 2 reveals that $W_{ij} = W_{ji}$. The solid W_{ij} will be called *the wall between the faces F_i and F_j.*

2.3. Polytopes in pseudo-Riemannian manifolds. Assume that the manifold M is equipped with a pseudo-Riemannian metric $\{,\}$. For a tangent vector X of M, we define the *norm* $|X|$ and the *sign* $\epsilon(X)$ of X by $|X| = |\{X,X\}|^{1/2}$ and $\epsilon(X) = \operatorname{sgn}(\{X,X\})$ respectively.

Definition 2.3. A regular domain P of M will be called *pseudo-Riemannian* if the restriction of $\{,\}$ onto the boundary $\Sigma = \partial P$ is non-degenerate and has constant index, i.e., if Σ is a pseudo-Riemannian submanifold of M. A *pseudo-Riemannian s-transversal CSG solid* in a pseudo-Riemannian manifold M is an s-transversal CSG solid $P = f^*(P_1, \ldots, P_k)$ for which the intersection of any $l \leqslant s$

of the hypersurfaces ∂P_i is a pseudo-Riemannian submanifold of M. In particular, a *pseudo-Riemannian polytope* is a pseudo-Riemannian 3-transversal CSG solid.

A pseudo-Riemannian regular domain P has a unique *outer unit normal vector field* $\mathbf{N}_P \in \Gamma(TM|_\Sigma)$ along $\Sigma = \partial P$. If P is defined by the inequality $f \leqslant 0$, where f is a smooth function on M and 0 is a regular value of f, then $\mathbf{N}_P$ is given by the equation

$$\mathbf{N}_P = \epsilon(\operatorname{grad} f)\frac{\operatorname{grad} f}{|\operatorname{grad} f|}.$$

A pseudo-Riemannian 2-transversal CSG solid $P = f^*(P_1, \dots, P_k)$ has a well-defined smooth outer unit normal vector field along each face. If $\mathbf{N}_{P_i}$ is the outer unit normal vector field of P_i, then the outer unit normal vector field $\mathbf{N}_i$ of P along the face F_i is equal to $(-1)^{s_i}\mathbf{N}_{P_i}$, where s_i is the number of those subtractions in f where an expression containing the ith variable is subtracted from another expression.

Each face F_i of a pseudo-Riemannian polytope $P = f^*(P_1, \dots, P_k)$, is a pseudo-Riemannian 2-transversal CSG solid within the pseudo-Riemannian manifold Σ_i and the faces of F_i are the walls between F_i and the other faces. Applying the above reasoning to F_i we obtain that F_i has a well defined smooth outer unit normal vector field $\mathbf{n}_{ij} \in \Gamma(T\Sigma_i|_{W_{ij}})$ along the wall W_{ij}.

We are going to define the inner dihedral angle α_{ij} of the pseudo-Riemannian polytope P along the wall W_{ij}. α_{ij} is a smooth function on W_{ij} computed as follows. Set $\epsilon_{ij} = \epsilon(\mathbf{n}_{ij})$ and $\epsilon_i = \epsilon(\mathbf{N}_i)$. For any $p \in W_{ij}$, the vectors $\mathbf{n}_{ij}(p)$ and $\mathbf{N}_i(p)$ form an orthonormal basis of the 2-dimensional orthogonal complement of T_pW_{ij} in T_pM. Since $\mathbf{n}_{ji}(p)$ is also in this orthogonal plane, we can write the vector field $\mathbf{n}_{ji}$ as a linear combination of $\mathbf{n}_{ij}$ and $\mathbf{N}_i$:

$$\mathbf{n}_{ji} = \lambda\epsilon_{ij}\mathbf{n}_{ij} + \mu\mathbf{N}_i, \tag{2}$$

where λ and μ are smooth functions on the wall W_{ij}. As $|\mathbf{n}_{ji}| = |\mathbf{n}_{ij}| = |\mathbf{N}_i| = 1$ and $\{\mathbf{n}_{ij}, \mathbf{N}_i\} = 0$, Eq. (2) implies

$$\epsilon_{ji} = \lambda^2\epsilon_{ij} + \mu^2\epsilon_i. \tag{3}$$

We have three possibilities for Eq. (3) depending on the signs ϵ_{ji}, ϵ_{ij} and ϵ_i.

If $\lambda^2 + \mu^2 = 1$, then we define α_{ij} by the relations $0 < \alpha_{ij} < 2\pi$, $\epsilon_{ij}\lambda = \cos \circ \alpha_{ij}$ and $\epsilon_{ij}\mu = \sin \circ \alpha_{ij}$.

If $\lambda^2 - \mu^2 = 1$, then we define α_{ij} by the equations $|\lambda| = \cosh \circ \alpha_{ij}$, $\operatorname{sgn}(\lambda)\mu = \sinh \circ \alpha_{ij}$.

Finally, if $\mu^2 - \lambda^2 = 1$, then α_{ij} is the function determined by $\operatorname{sgn}(\mu)\lambda = \sinh \circ \alpha_{ij}$, $|\mu| = \cosh \circ \alpha_{ij}$.

The ordered basis $(\mathbf{n}_{ji}(p), \mathbf{N}_j(p))$ is orthonormal and has the same orientation as the ordered basis $(\mathbf{N}_i(p), \mathbf{n}_{ij}(p))$ for all $p \in W_{ij}$. These conditions allow us to compute that $\mathbf{N}_j$ is expressed as a linear combination of $\mathbf{n}_{ji}$ and $\mathbf{N}_j$ in the following way

$$\mathbf{N}_j = \epsilon_i\epsilon_{ji}\mu\mathbf{n}_{ij} - \epsilon_{ji}\lambda\mathbf{N}_i. \tag{4}$$

As a corollary of (2), (3) and (4) we obtain that

$$\mathbf{n}_{ij} = \lambda\epsilon_{ji}\mathbf{n}_{ji} + \mu\mathbf{N}_j, \tag{5}$$

which shows that $\alpha_{ij} = \alpha_{ji}$.

2.4. Variations of CSG solids. A CSG structure has a discrete component, the Boolean expression (or the CSG tree encoding it), which can not be deformed continuously. However, one can deform a CSG solid by deforming its primitives.

Throughout this section, I will denote an open interval around 0. When $H : X \times I \to Y$ is an arbitrary homotopy and $t \in I$, $H_t : X \to Y$ will denote the map defined by $H_t(p) = H(p,t)$.

Definition 2.4. A *variation of a regular domain* $P = P_0$ in M is a one-parameter family of regular domains P_t, $t \in I$, for which

(i) there is a proper isotopy $\Phi : \partial P \times I \to M$ such that Φ_0 is the inclusion of ∂P, and $\Phi_t(\partial P) = \partial P_t$ for all $t \in I$;

(ii) for any point $p \in M$ the sets $\{t \in I \mid p \in \operatorname{int} P_t\}$ and $\{t \in I \mid p \in \operatorname{ext} P_t\}$ are open in I.

Less formally, during a variation of a regular domain the boundary is transformed by a proper isotopy and the domain is not allowed to flip from one side of the boundary to the other.

The *trace of a variation* is the set $\mathbb{P} = \{(x,t) \in M \times I \mid x \in P_t\}$. The trace of a variation of a regular domain in M is a regular domain in $M \times I$.

The variation of the boundary of P determines the variation of P uniquely, so studying variations of regular domains we may focus on the variation of the boundary.

A vector field $X \in \Gamma(TM|_{\Sigma_0})$ along Σ_0 is said to be an *infinitesimal variation of* Σ_0, *compatible with the given variation* if one of the following equivalent conditions is fulfilled:

- The isotopy Φ in Definition 2.4 (i) can be chosen in such a way that $X_p = \left.\frac{\partial \Phi(p,t)}{\partial t}\right|_{t=0}$.
- The vector field $(X, \partial/\partial t) \in \Gamma(T(M \times I)|_{\Sigma_0 \times \{0\}})$ is tangential to the boundary of $\mathbb{P}$.

The difference of two infinitesimal variations compatible with the same variation can be an arbitrary tangential vector field along Σ_0.

In particular, if P_0 is a pseudo-Riemannian regular domain in a pseudo-Riemannian manifold M, $\mathbf{N}_P$ is the outer unit normal vector field of P, then every infinitesimal variation X compatible with the given variation has the same normal component $\nu\mathbf{N}_P$, where $\nu = \epsilon(\mathbf{N}_P)\{X, \mathbf{N}_P\}$.

Definition 2.5. A *variation of a* (*pseudo-Riemannian*) *s-transversal CSG solid*

$$P = f^*(P_1, \ldots, P_k) \tag{6}$$

consists of variations $P_{1,t}, \ldots, P_{k,t}$, $t \in I$ of the primitives $P_1, \ldots, P_k$ respectively, such that

(i) $P_t = f^*(P_{1,t}, \ldots, P_{k,t})$ is a (pseudo-Riemannian) s-transversal CSG solid for all $t \in I$;

(ii) for any $l \leqslant s$ and $1 \leqslant i_1 < \cdots < i_l \leqslant k$, there is a proper isotopy Φ_t, $t \in I$ of $\bigcap_{j=1}^{l} \partial P_{i_j}$ in M such that Φ_0 is the inclusion map and $\Phi_t(\bigcap_{j=1}^{l} \partial P_{i_j}) = \bigcap_{j=1}^{l} \partial P_{i_j,t}$ for all t;

(iii) there is a compact set $K \subset M$ such that $P_t \subset K$ for all $t \in I$.

Proposition 4. *Let $\phi : M \times I \to \mathbb{R}$ be an arbitrary continuous function, $P_t = f^*(P_{1,t}, \ldots, P_{k,t})$ a variation of a CSG-solid in M. Then the integral*

$$\int_{P_t} \phi(p,t)dp \tag{7}$$

taken with respect to the measure induced by the pseudo-Riemannian metric of M is a continuous function of t.

Proof. Denote by χ_t the characteristic function of P_t. For any number $t_0 \in I$, the union of the boundaries of the primitives $P_{1,t_0}, \ldots, P_{k,t_0}$ has measure 0. However, for any point $p \in M \setminus \bigcup_{i=1}^{k} P_{i,t_0}$, the function $\chi_t(p)$ is constant in a small neighborhood of t_0 and therefore $\lim_{t \to t_0} \phi(p,t)\chi_t(p) = \phi(p,t_0)\chi_{t_0}(p)$ for almost all $p \in M$. Since the characteristic functions χ_t $(t \in I)$ have a common compact support, the statement follows from the Lebesgue lemma. □

Let $\mathbb{P}_i$ be the trace of the variation of the primitive P_i. The *trace of the variation of the solid* is defined to be the set $\mathbb{P} = f^*(\mathbb{P}_1, \ldots, \mathbb{P}_k)$. The trace of the variation of an s-transversal solid is not an s-transversal solid in M because of the lack of compactness. However, for any $[a,b] \subset I$, setting $\mathbb{H}_a = \{(p,t) \in M \times I \mid t \geqslant a\}$ and $\mathbb{H}^b = \{(p,t) \in M \times I \mid t \leqslant b\}$, we have:

Proposition 5. *The truncated trace $f^*(\mathbb{P}_1, \ldots, \mathbb{P}_k) \cap^* \mathbb{H}_a \cap^* \mathbb{H}^b$ of the variation of an s-transversal solid is an s-transversal solid in $M \times I$.* □

Consider now a variation of a pseudo-Riemannian polytope (6) in a pseudo-Riemannian manifold $(M, \{,\})$. Denote by Σ_i the boundary of the primitive P_i and let $X_i \in \Gamma(TM|_{\Sigma_i})$ be an infinitesimal variation of Σ_i compatible with the variation of P_i. A vector field $X \in \Gamma(TM|_{W_{ij}})$ along the wall W_{ij} is said to be an infinitesimal variation of the wall W_{ij} compatible with the variation of the polytope if the isotopy Φ_t of the intersection $\Sigma_i \cap \Sigma_j$ in part (ii) of Definition 2.5 can be chosen in such a way that $X_p = \left.\frac{d\Phi_t(p)}{dt}\right|_{t=0}$ for all $p \in W_{ij}$.

3. Variational Formulae

3.1. Variation of the volume. Consider a variation $P_t = f^*(P_{1,t}, \ldots, P_{k,t})$ of a pseudo-Riemannian polytope in a pseudo-Riemannian manifold $(M, \{,\})$ and denote by $V(t)$ the volume of P_t with respect to the measure induced by the pseudo-Riemannian metric.

Theorem 1. *If $F_1, \ldots, F_k$ are the faces of P_0, $\mathbf{N}_i$ is the outer unit normal vector field of P_0 along F_i and the vector fields $X_i \in \Gamma(TM|_{F_i})$ are compatible with the variation, then we have*

$$V'(0) = \sum_{i=1}^{k} \int_{F_i} \nu_i d\sigma_i = \sum_{i=1}^{k} \int_{F_i} \epsilon_i \{X, \mathbf{N}_i\} d\sigma_i. \tag{8}$$

Proof. We can use the standard idea to prove this theorem. If M is orientable, then we can take the volume form ω of M and apply Stokes' theorem for the pull-back $\pi^*\omega$ of ω by the canonical projection $\pi : M \times I \to M$ on the truncated trace $\mathbb{P}_0^t = \mathbb{P} \cap^* \mathbb{H}_0 \cap^* \mathbb{H}^t \subset M \times I$ of the variation. As $d(\pi^*\omega) = \pi^*(d\omega) = 0$, the integral of $\pi^*\omega$ on the boundary of $\mathbb{P}_0^t$ is 0. $\mathbb{P}_0^t$ has k lateral faces $\mathbb{F}_1, \ldots, \mathbb{F}_k$ and two additional faces $P_t \times \{t\}$ and $P_0 \times \{0\}$. Computing the integrals on the lateral faces of $\mathbb{P}_0^t$ by the Fubini theorem we obtain

$$V(t) - V(0) = \int_0^t \left(\sum_{i=1}^{k} \int_{F_{i,t}} \epsilon_i \{X, \mathbf{N}_i\} d\sigma_i \right) dt. \tag{9}$$

For every $0 \leqslant i \leqslant k$, $F_{i,t}$ is a 2-transversal CSG solid in $\partial P_{i,t}$ and there is an isotopy $\Phi : \partial P_i \times I \to M$ for which $\Phi_t(\partial P_i) = \partial P_{i,t}$. Pulling back the domain of the inner integral on the right hand side of (9) by Φ_t onto $\Phi_t^{-1}(F_{i,t})$, we obtain an integral of type (7). Therefore by Proposition 4, the integral in the bracket in (9) is a continuous function of t, thus, differentiating (9) gives (8).

If M is not orientable, then take its 2-fold orientable covering $\rho : \widetilde{M} \to M$ and apply the above arguments to the lift $\rho^{-1}(P_t)$ of the variation. □

3.2. Some useful formulae. Concerning the pseudo-Riemannian analogues of the fundamental equations of Riemannian submanifold theory, used below, we refer to [2].

Lemma 1. *Let M be an $(m+1)$-dimensional pseudo-Riemannian Einstein manifold, Σ be a smooth pseudo-Riemannian hypersurface in M with a fixed unit normal vector field* $\mathbf{N}$. *Denote by $H : \Sigma \to \mathbb{R}$ the trace of the Weingarten map and by II the second fundamental form of Σ. Then for any tangential vector field X along Σ, we have*

$$XH + \delta II(X) = 0, \tag{10}$$

where δII denotes the divergence of the second fundamental form.

Proof. To prove the identity at a given point of Σ choose a tangential orthonormal frame $E_1, \ldots, E_m$ in an open neighborhood U of that point and restrict attention to U. Since M is Einstein, the Ricci endomorphism $\overline{\mathrm{Ric}}$ and the curvature tensor R of M satisfies

$$\overline{\mathrm{Ric}}(X) = \sum_{i=1}^{m} \epsilon(E_i) R(X, E_i) E_i + \epsilon(\mathbf{N}) R(X, \mathbf{N})\mathbf{N} = \frac{s}{m+1} X,$$

where s is the scalar curvature of M. Taking the inner product of this equation with $\mathbf{N}$ we obtain

$$0 = \sum_{i=1}^{m} \epsilon(E_i) \{R(X, E_i)E_i, \mathbf{N}\}. \tag{11}$$

According to the Codazzi-Mainardi equation

$$\{R(X,E_i)E_i,\mathbf{N}\} = \epsilon(\mathbf{N})(\widehat{\nabla}_X II(E_i,E_i) - \widehat{\nabla}_{E_i} II(X,E_i)), \tag{12}$$

where $\widehat{\nabla}$ is the Levi-Civita connection of Σ. Plugging (12) into (11) we get

$$0 = \sum_{i=1}^{m} \epsilon(E_i)(\widehat{\nabla}_X II(E_i,E_i) - \widehat{\nabla}_{E_i} II(X,E_i)) = \{I, \widehat{\nabla}_X II\} + \delta II(X). \tag{13}$$

Since $H = \{I, II\}$ and $\widehat{\nabla}_X I = 0$, we also have

$$XH = \{\widehat{\nabla}_X I, II\} + \{I, \widehat{\nabla}_X II\} = \{I, \widehat{\nabla}_X II\}. \tag{14}$$

Equation (10) is an obvious consequence of (13) and (14). □

Lemma 2. *If Σ is a pseudo-Riemannian smooth hypersurface in an $(m+1)$-dimensional pseudo-Riemannian manifold $(M,\{,\})$, L is the Weingarten map with respect to a fixed unit normal field $\mathbf{N}$, I and II are the first and second fundamental forms, then for any tangent vector field X on Σ we have*

$$\delta II(X) = \frac{1}{2}\{\mathscr{L}_X I, II\} - \operatorname{div} L(X),$$

where $\mathscr{L}_X$ is the Lie derivative with respect to X.

Proof. Choose a local orthonormal frame $E_1,\dots,E_m$ on Σ. Then

$$\delta II(X) = -\sum_{i=1}^{m} \epsilon(E_i)\widehat{\nabla}_{E_i} II(X,E_i),$$

and

$$\begin{aligned}
\mathscr{L}_X I(E_i,E_j) &= -\{\mathscr{L}_X E_i, E_j\} - \{E_i, \mathscr{L}_X E_j\} = -\{[X,E_i],E_j\} - \{E_i,[XE_j]\} \\
&= \{-\widehat{\nabla}_X E_i + \widehat{\nabla}_{E_i} X, E_j\} + \{E_i, -\widehat{\nabla}_X E_j + \widehat{\nabla}_{E_j} X\} \\
&= -X\{E_i,E_j\} + \{\widehat{\nabla}_{E_i} X, E_j\} + \{E_i, \widehat{\nabla}_{E_j} X\} \\
&= \{\widehat{\nabla}_{E_i} X, E_j\} + \{E_i, \widehat{\nabla}_{E_j} X\},
\end{aligned}$$

from which

$$\begin{aligned}
\frac{1}{2}\{\mathscr{L}_X I, II\} &= \sum_{i,j=1}^{m} \epsilon(E_i)\epsilon(E_j)\{\widehat{\nabla}_{E_i} X, E_j\} II(E_i,E_j) \\
&= \sum_{i=1}^{m} \epsilon(E_i) II(E_i, \widehat{\nabla}_{E_i} X).
\end{aligned}$$

Finally

$$\begin{aligned}
\operatorname{div} L(X) &= \sum_{i=1}^{m} \epsilon(E_i)\{\widehat{\nabla}_{E_i}(L(X)), E_i\} \\
&= \sum_{i=1}^{m} \epsilon(E_i)(E_i(II(X,E_i)) - II(X, \widehat{\nabla}_{E_i} E_i)) \\
&= \sum_{i=1}^{m} \epsilon(E_i)(\widehat{\nabla}_{E_i} II(X,E_i)) + II(\widehat{\nabla}_{E_i} X, E_i)).
\end{aligned}$$

Combining these formulae we obtain the statement. □

Lemmas 1 and 2 yield the following.

Corollary 1. *For pseudo-Riemannian hypersurfaces lying in pseudo-Riemannian Einstein manifolds we have*

$$0 = XH + \frac{1}{2}\{\mathscr{L}_X I, II\} - \operatorname{div} L(X) \quad \textit{for any } X \in \Gamma(T\Sigma).$$

Let Σ be a smooth pseudo-Riemannian hypersurface in a pseudo-Riemannian manifold M, $\mathbf{N} \in \Gamma(TM|_\Sigma)$ be a unit normal vector field along Σ. Consider an isotopy $\Phi_t : \Sigma \to M$, $t \in (-\varepsilon, \varepsilon)$ and the induced vector field $X \in \Gamma(TM|_\Sigma)$, $X_p = \frac{\mathrm{d}}{\mathrm{d}t}\Phi_t(p)|_{t=0}$. Denote by I_t, II_t and $H_t = \{I_t, II_t\}$ the pull-backs of the first and second fundamental forms and the Minkowski curvature of $\Phi_t(\Sigma)$ onto Σ by Φ_t. Denote by $I = I_0$, $II = II_0$ and III the first, second and third fundamental forms of Σ, and let the symbols I', II' and H' stand for $\frac{\mathrm{d}I_t}{\mathrm{d}t}(0)$, $\frac{\mathrm{d}II_t}{\mathrm{d}t}(0)$ and $\frac{\mathrm{d}H_t}{\mathrm{d}t}(0)$ respectively. Let $X = X^T + \nu\mathbf{N}$ be the decomposition of X into components tangential and normal to Σ.

Choose a point $p \in \Sigma$ and let $\gamma : t \mapsto \Phi_t(p)$ be the curve drawn by p during the isotopy. Extend $\mathbf{N}_p$ to a smooth vector field $\mathbf{N}_\gamma$ along γ for which $\mathbf{N}_\gamma(t)$ is a unit normal of $\Phi_t(\Sigma)$ at $\gamma(t)$.

Lemma 3. *Using the notation introduced above we have*

$$I' = -2\nu II + \mathscr{L}_{X^T} I, \tag{15}$$

$$\nabla_t \mathbf{N}_\gamma(0) = -\epsilon(\mathbf{N}_p)\operatorname{grad}\nu - L_p(X_p^T), \tag{16}$$

$$II' = \epsilon(\mathbf{N})\nabla^2\nu + \nu(R(\mathbf{N}, \dots, \mathbf{N}) - III) + \mathscr{L}_{X^T} II, \tag{17}$$

$$H' = \nu\{II, II\} - \epsilon(\mathbf{N})\Delta\nu + \epsilon(\mathbf{N})\nu r(\mathbf{N}) + X^T H, \tag{18}$$

where L_p is the Weingarten map of Σ at p, the gradient of ν is taken in Σ, $\nabla^2\nu$ is the Hessian of ν with respect to the Levi-Civita connection, $\Delta\nu = -\operatorname{div}(\operatorname{grad}\nu) = -\{I, \nabla^2\nu\}$ is the Laplacian of ν, $r(\mathbf{N})$ is the Ricci curvature of M in the direction $\mathbf{N}$.

Proof. Observe first that (15) and (17) claim the equality of symmetric (0,2) tensor fields, for which it is enough to show the equality of the corresponding quadratic forms. Choose an arbitrary tangent vector $Y \in T_p\Sigma$ and a curve $\eta : (-\delta, \delta) \to \Sigma$ such that $\eta(0) = p$ and $\eta'(0) = Y$. Consider the parameterized surface $\mathbf{r} : (-\epsilon, \epsilon) \times (-\delta, \delta) \to M$, $\mathbf{r}(t, u) = \Phi_t(\eta(u))$ Differentiating the equation $I_t(Y, Y) = \{\partial_u \mathbf{r}(t, 0), \partial_u \mathbf{r}(t, 0)\}$ at $t = 0$ we get

$$I'(Y, Y) = 2\{\nabla_t \partial_u \mathbf{r}(0, 0), Y\} = 2\{\nabla_u \partial_t \mathbf{r}(0, 0), Y\} = 2\{\nabla_Y X, Y\}.$$

Plugging in the decomposition of X, we obtain

$$I'(Y, Y) = 2\nu\{\nabla_Y \mathbf{N}, Y\} + 2\{\nabla_Y X^T, Y\} = -2\nu II(Y, Y) + \mathscr{L}_{X^T} I(Y, Y).$$

This proves (15).

Extend $\mathbf{N}_\gamma$ to a smooth vector field $\mathbf{N}_\mathbf{r}$ along $\mathbf{r}$ in such a way that $\mathbf{N}_\mathbf{r}(t, u)$ is a unit normal of the hypersurface $\Phi_t(\Sigma)$ at $\Phi_t(\gamma(u))$. Since $\{\mathbf{N}_\mathbf{r}, \mathbf{N}_\mathbf{r}\}$ is constant ± 1, $\nabla_t \mathbf{N}_\mathbf{r}(0,0) = \nabla_t \mathbf{N}_\gamma(0)$ is tangent to Σ at p. On the other hand, differentiating $\{\mathbf{N}_\mathbf{r}, \partial_u \mathbf{r}\} \equiv 0$ with respect to t we obtain

$$\{\nabla_t \mathbf{N}_\gamma(0), Y\} = -\{\mathbf{N}_p, \nabla_t \partial_u \mathbf{r}(0,0)\} = -\{\mathbf{N}_p, \nabla_u \partial_t \mathbf{r}(0,0)\} = -\{\mathbf{N}_p, \nabla_Y X\}.$$

Decomposing X gives

$$-\{\mathbf{N}_p, \nabla_Y X\} = -Y(\nu)\epsilon(\mathbf{N}_p) - II(X_p^T, Y) = \{-\epsilon(\mathbf{N}_p) \operatorname{grad} \nu - L_p(X_p^T), Y\},$$

which yields (16). Differentiating the equation

$$II_t(Y,Y) = -\{\nabla_u \mathbf{N}_\mathbf{r}(t,0), \partial_u \mathbf{r}(t,0)\}$$

with respect to t at $t=0$ we obtain

$$\begin{aligned}
II'(Y,Y) &= -\{\nabla_t \nabla_u \mathbf{N}_\mathbf{r}(0,0), Y\} - \{\nabla_Y \mathbf{N}, \nabla_t \partial_u \mathbf{r}(0,0)\} \\
&= -\{\nabla_u \nabla_t \mathbf{N}_\mathbf{r}(0,0), Y\} - R(X_p, Y, \mathbf{N}_p, Y) - \{\nabla_Y \mathbf{N}, \nabla_u \partial_t \mathbf{r}(0,0)\} \\
&= \epsilon(\mathbf{N}_p)\{\nabla_Y \operatorname{grad} \nu, Y\} + \{\nabla_Y (L(X^T)), Y\} + \nu(p) R(\mathbf{N}_p, Y, Y, \mathbf{N}_p) \\
&\quad - \{\nabla_Y \mathbf{N}, \nabla_Y X\} - R(X_p^T, Y, \mathbf{N}_p, Y) \\
&= \epsilon(\mathbf{N}_p)\{\nabla_Y \operatorname{grad} \nu, Y\} + \nu(p)(R(\mathbf{N}_p, Y, Y, \mathbf{N}_p) - \{\nabla_Y \mathbf{N}, \nabla_Y \mathbf{N}\}) \\
&\quad + \{\nabla_Y (L(X^T)), Y\} - \{\nabla_Y \mathbf{N}, \nabla_Y X^T\} - R(X_p^T, Y, \mathbf{N}_p, Y).
\end{aligned}$$

In the special case when X is tangent to Σ and Φ_t is the flow generated by X on Σ, the above formula reduces to

$$\mathscr{L}_{X^T} II(Y,Y) = \{\nabla_Y (L(X^T)), Y\} - \{\nabla_Y \mathbf{N}, \nabla_Y X^T\} - R(X_p^T, Y, \mathbf{N}_p, Y),$$

from which

$$\begin{aligned}
II'(Y,Y) &= \epsilon(\mathbf{N}_p)\{\nabla_Y \operatorname{grad} \nu, Y\} + \nu(p)(R(\mathbf{N}_p, Y, Y, \mathbf{N}_p) - \{\nabla_Y \mathbf{N}, \nabla_Y \mathbf{N}\}) \\
&\quad + \mathscr{L}_{X^T} II(Y,Y),
\end{aligned}$$

proving (17).

Let $\mathscr{G}_t$ and $\mathscr{B}_t$ be the matrices of I_t and II_t with respect to a local orthonormal frame over an open subset U of Σ. Then $H_t = \operatorname{tr} \mathscr{G}_t^{-1} \mathscr{B}_t$. Differentiating this equation we obtain

$$\begin{aligned}
H' &= \operatorname{tr}(\mathscr{G}_0^{-1} \mathscr{B}_0' - \mathscr{G}_0^{-1} \mathscr{G}_0' \mathscr{G}_0^{-1} \mathscr{B}_0) = \{I, II'\} - \{I', II\} \\
&= 2\nu\{II, II\} - \epsilon(\mathbf{N})\Delta\nu + \epsilon(\mathbf{N})\nu r(\mathbf{N}) - \nu\{I, III\} + X^T H \\
&= \nu\{II, II\} - \epsilon(\mathbf{N})\Delta\nu + \epsilon(\mathbf{N})\nu r(\mathbf{N}) + X^T H.
\end{aligned}$$

□

Corollary 2. *If M is an $(m+1)$-dimensional pseudo-Riemannian Einstein manifold with sectional curvature s, Σ is a smooth pseudo-Riemannian hypersurface, X is an infinitesimal variation of Σ as above, then*

$$\frac{s}{m+1}\{X, \mathbf{N}\} = H' + \frac{1}{2}\{I', II\} - \operatorname{div}(\epsilon(\mathbf{N}) \operatorname{grad} \nu + L(X^T)).$$

Proof. Using (18), (15) and Corollary 1

$$\begin{aligned}\frac{s}{m+1}\{X,\mathbf{N}\} &= \epsilon(\mathbf{N})\nu r(\mathbf{N}) = H' - \nu\{II,II\} + \epsilon(\mathbf{N})\Delta\nu - X^T H\\ &= H' + \frac{1}{2}\{I',II\} + \epsilon(\mathbf{N})\Delta\nu - \frac{1}{2}\{\mathscr{L}_{X^T}I,II\} - X^T H\\ &= H' + \frac{1}{2}\{I',II\} - \operatorname{div}(\epsilon(\mathbf{N})\operatorname{grad}\nu + L(X^T)).\end{aligned}$$

□

3.3. Variation of the dihedral angle. Consider a variation $P_t = f^*(P_{1,t},\dots,P_{k,t})$ of a pseudo-Riemannian polytope in a pseudo-Riemannian manifold $(M,\{,\})$, let W_{ij} be the wall of $P = P_0$ between the i-th and j-th faces. Choose an isotopy $\Phi : (\partial P_i \cap \partial P_j)\times I \to M$ such that $\Phi_t(\partial P_i \cap \partial P_j) = (\partial P_{i,t} \cap \partial P_{j,t})$ and Φ_0 is the inclusion, and let $X_{ij} \in \Gamma(TM|_{W_{ij}})$ be the initial speed vector field $X_{ij}(p) = \frac{\mathrm{d}\Phi}{\mathrm{d}t}\big|_{t=0}$. Define the signs ϵ_i, ϵ_j, ϵ_{ij}, ϵ_{ji}, the functions α_{ij}, λ, μ and the vector fields $\mathbf{N}_i$, $\mathbf{N}_j$, $\mathbf{n}_{ij}$, $\mathbf{n}_{ji}$ along W_{ij} as in subsection 2.3. These functions and vector fields can be extended to functions and vector fields along the isotopy Φ. For example we extend α_{ij} to a smooth function $\alpha_{ij}^{\Phi} : (\partial P_i \cap \partial P_j)\times I \to \mathbb{R}$ in such a way that $\alpha_{ij}^{\Phi}(p,0) \equiv \alpha_{ij}$ and $\alpha_{ij}^{\Phi}(p,t)$ is one of the angles between the $\partial P_{i,t}$ and $\partial P_{j,t}$ at the point $\Phi(p,t)$. Derivatives of the extended functions α_{ij}^{Φ}, λ^{Φ}, μ^{Φ} and covariant derivative of the extended vector fields $\mathbf{N}_i^{\Phi}$, $\mathbf{N}_j^{\Phi}$, $\mathbf{n}_{ij}^{\Phi}$, $\mathbf{n}_{ji}^{\Phi}$ with respect to t at $t=0$ will be denoted by $X_{ij}\alpha_{ij}^{\Phi}$, $X_{ij}\lambda^{\Phi}$, $X_{ij}\mu^{\Phi}$ $\nabla_{X_{ij}}\mathbf{N}_i^{\Phi}$, $\nabla_{X_{ij}}\mathbf{N}_j^{\Phi}$, $\nabla_{X_{ij}}\mathbf{n}_{ij}^{\Phi}$ and $\nabla_{X_{ij}}\mathbf{n}_{ji}^{\Phi}$ respectively. It will be seen that these derivatives depend only on X_{ij}, so to simplify notation we drop the Φ's from the upper indices.

Our aim is to find an expression for the derivative $X_{ij}\alpha_{ij}$ of the dihedral angle. Differentiating the relation

$$\{\mathbf{N}_i,\mathbf{N}_j\} = -\epsilon_i\epsilon_{ji}\lambda$$

and making use of

$$X_{ij}\lambda = -\epsilon_i\epsilon_{ij}(X_{ij}\alpha_{ij})\mu$$

we obtain

$$\epsilon_{ij}\epsilon_{ji}(X_{ij}\alpha_{ij})\mu = \{\nabla_{X_{ij}}\mathbf{N}_i,\mathbf{N}_j\} + \{\mathbf{N}_i,\nabla_{X_{ij}}\mathbf{N}_j\}. \tag{19}$$

A computation similar to the one used in the proof of (16) shows that

$$\nabla_{X_{ij}}\mathbf{N}_i = -\epsilon_i\operatorname{grad}\nu_i - L_i(X_{ij} - \nu_i\mathbf{N}_i),$$

where $\nu_i = \epsilon_i\{X_{ij},\mathbf{N}_i\}$, L_i is the Weingarten map of the hypersurface ∂P_i. Taking the inner product with (4) and using $\{\nabla_{X_{ij}}\mathbf{N}_i,\mathbf{N}_i\} = 0$ this yields

$$\{\nabla_{X_{ij}}\mathbf{N}_i,\mathbf{N}_j\} = -\epsilon_i\epsilon_{ji}\mu\{\epsilon_i\operatorname{grad}\nu_i + L_i(X_{ij} - \nu_i\mathbf{N}_i),\mathbf{n}_{ij}\} \tag{20}$$

Plugging (20) into (19) and dividing by μ ($\neq 0$ by 2-transversality) we get

$$\begin{aligned}X_{ij}\alpha_{ij} = &-\epsilon_i\{\epsilon_i\operatorname{grad}\nu_i + L_i(X_{ij} - \nu_i\mathbf{N}_i),\epsilon_{ij}\mathbf{n}_{ij}\}\\ &-\epsilon_j\{\epsilon_j\operatorname{grad}\nu_j + L_j(X_{ij} - \nu_j\mathbf{N}_j),\epsilon_{ji}\mathbf{n}_{ji}\}.\end{aligned} \tag{21}$$

3.4. Schläfli formula for polytopes

Theorem 2. *We have the following formula for the variation of the volume of a polytope in an $(m+1)$-dimensional pseudo-Riemannian Einstein manifold with scalar curvature s*

$$\frac{s}{m+1}V'(0) = \sum_{i=1}^{k} \epsilon_i \int_{F_i} \left(H_i' + \frac{1}{2}\{I_i', II_i\} \right) d\sigma_i + \sum_{1 \leqslant i<j \leqslant k} \int_{W_{ij}} (X_{ij}\alpha_{ij}) d\sigma_{ij}$$
$$+ \sum_{i=1}^{k} \sum_{\substack{j=1 \\ j \neq i}}^{k} \int_{W_{ij}} \epsilon_i \epsilon_{ij} \{L_i(X_{ij} - X_i), \mathbf{n}_{ij}\} d\sigma_{ij}. \tag{22}$$

Proof. By Theorem 1 and Corollary 2, we have

$$\frac{s}{m+1}V'(0) = \sum_{i=1}^{k} \int_{F_i} \epsilon_i \frac{s}{m+1} \{X, \mathbf{N}_i\} d\sigma_i = \sum_{i=1}^{k} \epsilon_i \int_{F_i} \left(H_i' + \frac{1}{2}\{I_i', II_i\} \right) d\sigma_i$$
$$- \sum_{i=1}^{k} \epsilon_i \int_{F_i} \operatorname{div}(\epsilon_i \operatorname{grad} \nu_i + L_i(X_i^T)) d\sigma_i.$$

The last family of integrals can be computed by the divergence theorem

$$\int_{F_i} \operatorname{div}(\epsilon_i \operatorname{grad} \nu_i + L_i(X_i^T)) d\sigma_i = \sum_{\substack{1 \leqslant j \leqslant k \\ j \neq i}} \int_{W_{ij}} \{\epsilon_i \operatorname{grad} \nu_i + L_i(X_i^T), \epsilon_{ij}\mathbf{n}_{ij}\} d\sigma_{ij}$$

and thus the theorem follows from (21) and the equation $X_i^T = X_i - \nu_i \mathbf{N}_i$. □

4. Application to the Kneser-Poulsen Conjecture

Let M be the $(m+1)$-dimensional hyperbolic, Euclidean or spherical space of constant sectional curvature $K = s/(m^2+m)$. Complete connected umbilical hypersurfaces in M will be called $*$-spheres. The term $*$-ball will be used for any ball if $K > 0$ and for any convex regular domain bounded by a $*$-sphere if $K \leqslant 0$. In this section we study the consequences of the Schläfli-type formula (22) for polytopes made of $*$-balls in M.

Let $P = f^*(B_1, \ldots, B_k)$ be a polytope made of $*$-balls in M, such that the boundary of B_i is a $*$-sphere with constant normal curvature κ_i. Consider a variation P_t of P, obtained by moving the primitives B_i rigidly. In this case we can choose Killing fields for the infinitesimal variations X_i. Doing so, the integrals $\int_{F_i} \left(H_i' + \frac{1}{2}\{I_i', II_i\} \right) d\sigma_i$ in (22) will all vanish. The intersection angle of $*$-spheres is constant along the intersection, so the funtions α_{ij} depend only on t and the derivative $\alpha_{ij}'(0) = X_{ij}\alpha_{ij}$ does not depend on the actual choice of X_{ij}. By umbilicity, the Weingarten map L_i of ∂B_i is simply a multiplication by κ_i, thus, formula (22) reduces to the following form

$$\frac{s}{m+1}V'(0) = \sum_{1 \leqslant i<j \leqslant k} \int_{W_{ij}} \alpha_{ij}'(0)\sigma_{ij}(W_{ij}) + \sum_{i=1}^{k} \sum_{\substack{j=1 \\ j \neq i}}^{k} \int_{W_{ij}} \kappa_i \{(X_{ij} - X_i), \mathbf{n}_{ij}\} d\sigma_{ij}.$$

Since X_i is a Killing field, and X_{ij} is a compatible infinitesimal variation of the wall W_{ij}, $\sum_{\substack{j=1\\ j\neq i}}^{k}\int_{W_{ij}}\{(X_{ij}-X_i),\mathbf{n}_{ij}\}d\sigma_{ij}$ is the derivative of the m-dimensional volume of the i-th face at $t=0$ (cf. Theorem 1). Thus we obtain the following special case of Theorem 2.

Theorem 3. *For a polytope built of $*$-balls in M, we have the following variational formula*

$$\frac{d}{dt}\left(\frac{s}{m+1}V-\sum_{i=1}^{k}\kappa_i\sigma_i(F_i)\right)\Bigg|_{t=0}=\sum_{1\leqslant i<j\leqslant k}\alpha'_{ij}(0)\sigma_{ij}(W_{ij}). \tag{23}$$

Corollary 3. *If a polytope $P=f^*(B_1,\dots,B_k)$ is varied within the class of polytopes by moving the $*$-balls B_i smoothly and rigidly in such a way that the inner dihedral angles α_{ij} of the polytope weakly decrease, then the quantity $(sV/(m+1)-\sum_i\kappa_i\sigma_i(F_i))$ does not increase.* □

For simplicity call CSG objects made of ordinary balls *flowers*. We are going to extend Corollary 3 for piecewise analytic variations of flowers.

The boundary of a flower is typically not a smooth submanifold of M, but the singular points always form a negligible set (see [16]). At smooth points of the boundary, the boundary is umbilical and its principal curvature κ with respect to the outer unit normal is well defined. The function which assigns to a flower P the number $sV(P)/(m+1)-\int_{\partial P}\kappa d\sigma$ is a natural extension of the function "$sV/(m+1)-\sum_i\kappa_i\sigma_i(F_i)$", which was defined only for polytopes. It is not difficult to see that this function varies continuously under a variation of the flower.

For ordinary intersecting balls $B_i=B(P_i,r_i)$ and $B_j=B(P_j,r_j)$, the inner dihedral angle $\alpha(B_i,B_j)$ of the union $B_i\cup B_j$ is an increasing function of the distance $d(P_i,P_j)$ between the centers. Taking a regularized Boolean expression f^* in which each variable is used exactly once, one can define a sign $\epsilon_{ij}^{f^*}$ depending only on f^* such that for any 3-transversal family of balls the inner dihedral angle α_{ij} of the polytope $f^*(B_1,\dots,B_k)$ coincides with $\epsilon_{ij}^{f^*}\alpha(B_i,B_j)$ plus a constant multiple of π. Therefore, the inner dihedral angle α_{ij} of a 3-transversal flower built of balls with fixed radii is an increasing function of the signed distance $\epsilon_{ij}^{f^*}d(P_i,P_j)$ of the centers.

Theorem 4. *If a flower $P=f^*(B_1,\dots,B_k)$ is varied by moving the balls $B_i=B(P_i,r_i)$ by piecewise analytic rigid motions in such a way that the signed distances $\epsilon_{ij}^{f^*}d_{ij}(t)=\epsilon_{ij}^{f^*}d(P_{i,t},P_{j,t})$ between the moving centers weakly decrease, then the quantity $sV(P_t)/(m+1)-\int_{\partial P_t}\kappa\mathrm{d}\sigma$ does not increase.*

Proof. Since the quantity $sV(P_t)/(m+1)-\int_{\partial P_t}\kappa d\sigma$ varies continuously during the variation, it is enough to show monotonicity on intervals where the variation is analytic. Configurations of balls not satisfying the 3-transversality condition can be characterized by the vanishing of some analytic functions so by the unicity theorem, if P_t is a polytope for at least one moment of time, then it will break the 3-transversality condition only at a finite number of moments. These singular moments cut the time interval into a finite number of intervals. On each subinterval, $sV(P_t)/(m+1) - \int_{\partial P_t} \kappa d\sigma$ does not increase, so we are done by continuity.

If the configuration of the balls is singular during the whole variation, then we can perturb the radii. The set $\mathscr{S}$ of vectors $\delta = (\delta_1, \dots, \delta_k)$ for which $r_i + \delta_i > 0$ and the balls $B^{\delta}_{i,t} = B(P_i, r_i + \delta_i)$ do not satisfy the 3-transversality condition is nowhere dense in $\mathbb{R}^k$. For $\delta \in \mathbb{R}^k_+ \setminus \mathscr{S}$, we know the monotonicity of "$sV(P^{\delta}_t)/(m+1) - \int_{\partial P^{\delta}_t} \kappa d\sigma$" for the variation $P^{\delta}_t = f^*(B^{\delta}_{1,t}, \dots, B^{\delta}_{k,t})$. Taking the limit as $\delta \in \mathbb{R}^k_+ \setminus \mathscr{S}$ tends to $\mathbf{0}$ we get the required monotonicity in the general case. □

Theorem 4 has the same flavour as the Kneser-Poulsen conjecture. It will turn out below that the connection is more organic than just a superficial similarity.

Let N be an $(m-1)$-dimensional complete totally geodesic subspace of M. In the spherical case, let N^* be the polar circle of N and set $N^* = \emptyset$ for $K \leqslant 0$. The group G of orientation preserving isometries of M leaving each point of N fixed consists of rotations of M about N, hence G is isomorphic to the circle group S^1.

Theorem 5. *Suppose that a the set of singular points of the boundary of a G-invariant CSG object P in M is negligible and that the boundary hypersurfaces of the primitives of P intersect N transversally. If a non-singular boundary point $p \in \partial P$ is not in N, then denote by $k_G(p)$ the normal curvature of ∂P in the direction tangent to the circle Gp, relative to the exterior unit normal vector field of P. Then the following identity holds*

$$\frac{s}{m+1} V_{m+1}(P) - \int_{\partial P} k_G d\sigma = 2\pi V_{m-1}(P \cap N), \tag{24}$$

where V_{m+1}, V_{m-1} and σ are the volume measures induced by the Riemannian metric on M, N and the smooth part of ∂P respectively.

Proof. First we recall some formulas concerning tubular hypersurfaces around N. We refer to [11] for details. Tubular hypersurfaces around N are obtained as the level sets of the function $\rho : M \to \mathbb{R}$, which assigns to a point in M its geodesic distance from N.

By the symmetry properties of the tubular hypersurfaces around N, the tubular hypersurface of radius a ($a < \pi/(2\sqrt{K})$ in the spherical case) has two principal curvatures $\kappa_1(a)$ and $\kappa_2(a)$ with multiplicities 1 and $(m-1)$ respectively. These principal curvature functions satisfy the Riccati differential equation $\kappa'_* = \kappa_*^2 + K$ with initial condition $\kappa_1(0) = -\infty$ and $\kappa_2(0) = 0$, from which $\kappa_1(a) = -\sqrt{K}\cot(\sqrt{K}a)$ and $\kappa_2(a) = \sqrt{K}\tan(\sqrt{K}a)$. In particular, the trace of the shape operator of the tubular hypersurface of radius a around N is $\sqrt{K}((m-1)\tan(\sqrt{K}a) - \cot(\sqrt{K}a))$. Observe that $\kappa_1(a)$ is the principal curvature of a sphere of radius a with respect to the outer unit normal vector field.

The normal bundle of N is trivial and carries a natural Riemannian product metric, so it can be identified with $N \times \mathbb{R}^2$. The exponential map of the normal bundle gives a diffeomorphism between $N \times \mathbb{R}^2$ in the non-positively curved cases and between the tube of radius $\pi/(2\sqrt{K})$ around the zero section of the bundle and $M \setminus N^*$ in the spherical case. The pull-back of the volume measure of M onto

$N \times \mathbb{R}^2$ by the exponential map is the volume measure of $N \times \mathbb{R}^2$ multiplied by a smooth function of the form $\vartheta \circ \tilde{\rho}$, where $\tilde{\rho} : N \times \mathbb{R}^2 \to \mathbb{R}$ is the pull-back of the distance function ρ, i.e. $\tilde{\rho}(p, \mathbf{x}) = |\mathbf{x}|$, ϑ is a smooth function whose logarithmic derivative is

$$\frac{\vartheta'}{\vartheta}(a) = -(a^{-1} + \sqrt{K}((m-1)\tan(\sqrt{K}a) - \cot(\sqrt{K}a))). \tag{25}$$

Integrating (25) taking care of the initial condition $\vartheta(0) = 1$ we obtain

$$\vartheta(a) = \frac{\sin(\sqrt{K}a)\cos^{m-1}(\sqrt{K}a)}{\sqrt{K}a}. \tag{26}$$

Let X be the unit vector field on $M \setminus (N \cup N^*)$ whose restriction onto a tubular hypersurface around N is the unit normal vector field of the hypersurface pointing away from N.

The vector field $Y = (\kappa_1 \circ \rho)X$ has a singularity along N, but extends smoothly to N^* in the spherical case. By symmetry, the divergence of Y depends only on the distance of a point from N, thus $\operatorname{div} Y = g \circ \rho$ for a suitable function g. To determine g explicitly, take the solid tubes A_a and A_b of radii $a > b > 0$ around an arbitrary compact regular domain A in N and apply the divergence theorem for the vector field Y over $A_a \setminus^* A_b$. Then dividing by the volume of A we get

$$2\pi \int_b^a g(t) \frac{\sin(\sqrt{K}t)\cos^{m-1}(\sqrt{K}t)}{\sqrt{K}} \mathrm{d}t = 2\pi[\kappa_1(t)t\vartheta(t)]_b^a,$$

from which $\operatorname{div} Y = g \circ \rho$ turns out to be constant $Km = s/(m+1)$.

Let ε be a positive number and consider the solid tube N_ε around N. If ε is sufficiently small, then the boundary of N_ε intersects the smooth part of the boundary of P almost orthogonally and therefore transversally and the intersection $\partial N_\varepsilon \cap (\partial P)$ is covered by the 2ε-neighborhood of $N \cap \partial P$. In that case we can apply the divergence theorem for the domain $P \setminus^* N_\varepsilon$:

$$\int_{P \setminus^* N_\varepsilon} \frac{s}{m+1} \mathrm{d}V_{m+1} = \int_{\partial P \setminus N_\varepsilon} \kappa_1 \circ \rho\{X, \mathbf{N}\} \mathrm{d}\sigma - \kappa_1(\varepsilon)\sigma(\partial N_\varepsilon \cap P), \tag{27}$$

where $\mathbf{N}$ is the outer unit normal vector field on the smooth part of the boundary of P, $\{,\}$ is the Riemannian metric of M, σ is the m dimensional volume measure induced by the Riemannian metric on smooth hypersurfaces of M.

Study the limit of the components of the Eq. (27) as ε tends to 0. The limit of the left hand side is just $sV_{m+1}(P)/(m+1)$. If $p \in \partial P$ is a smooth boundary point of P, then ∂P contains the circle Gp. The curvature of this circle is $\kappa_1(\rho(p))$ provided that the orientations are chosen in such a way that X is the second Frenet vector field of the circle. According to Meusnier's theorem, the curvature of the circle is related to the normal curvature of the hypersurface in the direction tangent to this circle by the equation $\kappa_1 \circ \rho\{X, \mathbf{N}\} = k_G$. This implies that the integral on the right hand side of (27) converges to $\int_{\partial P} k_G d\sigma$.

Consider the intersection $P \cap N$ and the 2ε-neighborhood $U_{2\varepsilon}$ of its relative boundary in N. The orthogonal projection of $\partial N_\varepsilon \cap P$ is contained in

$(P \cap N) \cup U_{2\varepsilon}$ and it contains $(P \cap N) \setminus U_{2\varepsilon}$, therefore $\kappa_1(\varepsilon)\sigma(\partial N_\varepsilon \cap P)$ is between $2\pi\kappa_1(\varepsilon)\varepsilon\vartheta(\varepsilon)V_{m-1}((P \cap N) \cup U_{2\varepsilon})$ and $2\pi\kappa_1(\varepsilon)\varepsilon\vartheta(\varepsilon)V_{m-1}((P \cap N)\setminus U_{2\varepsilon})$. Since

$$\lim_{\varepsilon \to 0} V_{m-1}((P \cap N) \cup U_{2\varepsilon}) = \lim_{\varepsilon \to 0} V_{m-1}((P \cap N) \setminus U_{2\varepsilon}) = V_{m-1}(P \cap N)$$

and $\lim_{\varepsilon \to 0} -\kappa_1(\varepsilon)\varepsilon\vartheta(\varepsilon) = 1$, the limit of (27) as ε tends to 0 yields (24). □

Corollary 4. *Suppose that $P = f^*(B_1, \ldots, B_k)$ is a CSG object in M made of G-invariant $*$-balls. Denote by κ the principal curvature function on the smooth part of the boundary ∂P and by σ the m-dimensional volume measure on ∂P. Then we have*

$$\frac{s}{m+1} V_{m+1}(P) - \int_{\partial P} \kappa \mathrm{d}\sigma = 2\pi V_{m-1}(f^*(B_1 \cap N, \ldots, B_k \cap N)).$$

□

Combining Theorem 4 and Corollary 4 we obtain

Theorem 6. *Let f^* be a Boolean expression as usual, $P_1, \ldots, P_k$ and $Q_1, \ldots, Q_k$ be points in the $(m-1)$-dimensional subspace N. If there exist piecewise analytic curves $\gamma_i : [0,1] \to M$ connecting the points $\gamma_i(0) = P_i$ to the points $\gamma_i(1) = Q_i$ in such a way that the signed distances $\epsilon_{ij}^{f^*} \mathrm{d}(\gamma_i(t), \gamma_j(t))$ weakly decrease as t increase, then for any choice of the radii, we have the following inequality between the volumes of the $(m-1)$-dimensional flowers built from the balls $B_i^N = B(P_i, r_i) \cap N$ and $\widetilde{B}_i^N = B(Q_i, r_i) \cap N$:*

$$V_{m-1}(f^*(B_1^N, \ldots, B_k^N)) \leqslant V_{m-1}(f^*(\widetilde{B}_1^N, \ldots, \widetilde{B}_k^N)).$$

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