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Real Hypersurfaces of Type B in Complex Two-Plane Grassmannians

By

Young Jin Suh

Kyungpook National University, Taegu, Korea

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Abstract. In this paper we give a characterization of real hypersurfaces of type B, that is, a tube over a totally real totally geodesic $\mathbb{H}P^n$ in complex two-plane Grassmannians $G_2(\mathbb{C}^{m+2})$, $m = 2n$ with the shape operator A satisfying $A\phi + \phi A = k\phi$, k is non-zero constant, for the structure tensor ϕ.

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0. Introduction

In the geometry of real hypersurfaces in complex space forms $M_m(c)$ or in quaternionic space forms there have been many characterizations of model hypersurfaces of type A_1, A_2, B, C, D and E in complex projective space $P_m(\mathbb{C})$, of type A_0, A_1, A_2 and B in complex hyperbolic space $H_m(\mathbb{C})$ or A_1, A_2, B in quaternionic projective space $\mathbb{H}P^m$, which are completely classified by Cecil and Ryan [6], Kimura [7], Berndt [2], Martinez and Pérez [8], respectively. Among them there were only a few characterizations of homogeneous real hypersurfaces of type B in complex projective space $P_m(\mathbb{C})$. For example, the condition that $A\phi + \phi A = k\phi$, k is non-zero constant, is a model characterization of this kind of type B, which is a tube over a real projective space $\mathbb{R}P^n$ in $P_m(\mathbb{C})$, $m = 2n$ (see Yano and Kon [13]).

Let M be a $(4m-1)$-dimensional Riemannian manifold with an almost contact structure (ϕ, ξ, η) and an associated Riemannian metric g. We put

$$\omega(X, Y) = g(\phi X, Y), \tag{0.1}$$

where ω defines a 2-form on M and rank $\omega =$ rank $\phi = 4m - 2$.

If there is a non-zero valued function ρ such that

$$\rho g(\phi X, Y) = \rho\omega(X, Y) = d\eta(X, Y), \tag{0.2}$$

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the rank of the matrix (ω) being $4m-2$, we have

$$\eta \wedge \overbrace{\omega \wedge \cdots \wedge \omega}^{2m-1 \text{ times}} = \eta \wedge \rho^{-(2m-1)} \overbrace{d\eta \wedge \cdots \wedge d\eta}^{2m-1 \text{ times}} \neq 0.$$

Let us denote by $G_2(\mathbb{C}^{m+2})$ the set of all complex 2-dimensional linear subspaces of $\mathbb{C}^{m+2}$. We call such a set $G_2(\mathbb{C}^{m+2})$ complex two-plane Grassmannians. This Riemannian symmetric space $G_2(\mathbb{C}^{m+2})$ has a remarkable geometrical structure. It is the unique compact irreducible Riemannian manifold being equipped with both a Kähler structure J and a quaternionic Kähler structure $\mathfrak{J} = \text{Span}\ \{J_1, J_2, J_3\}$ not containing J. In other words, $G_2(\mathbb{C}^{m+2})$ is the unique compact, irreducible, Kähler, quaternionic Kähler manifold which is not a hyperkähler manifold (see Berndt and Suh [4], [5]).

Now we consider a $(4m-1)$-dimensional real hypersurface M in complex two-plane Grassmannians $G_2(\mathbb{C}^{m+2})$. Then from the Kähler structure of $G_2(\mathbb{C}^{m+2})$ there exists an almost contact structure ϕ on M. If the non-zero function ρ satisfies (0.2), we call M a *contact* hypersurface of the Kähler manifold. Moreover, it can be easily verified that a real hypersurface M in $G_2(\mathbb{C}^{m+2})$ is *contact* if and only if there exists a non-zero constant function ρ defined on M such that

$$\phi A + A\phi = k\phi, \quad k = 2\rho. \qquad (^*)$$

The formula $(^*)$ means that

$$g((\phi A + A\phi)X, Y) = 2d\eta(X, Y),$$

where the exterior derivative $d\eta$ of the 1-form η is defined by

$$d\eta(X, Y) = (\nabla_X \eta)Y - (\nabla_Y \eta)X$$

for any vector fields X, Y on M in $G_2(\mathbb{C}^{m+2})$.

On the other hand, in $G_2(\mathbb{C}^{m+2})$ we are able to consider two kinds of natural geometric conditions for real hypersurfaces M that $[\xi] = \text{Span}\ \{\xi\}$ or $\mathfrak{D}^\perp = \text{Span}\ \{\xi_1, \xi_2, \xi_3\}$, $\xi_i = -J_i N$, $i = 1, 2, 3$, where N denotes a unit normal to M, is invariant under the shape operator A of M in $G_2(\mathbb{C}^{m+2})$. The first result in this direction is the classification of real hypersurfaces in $G_2(\mathbb{C}^{m+2})$ satisfying both two conditions. Namely, Berndt and the present author [4] have proved the following

Theorem A. *Let M be a connected real hypersurface in $G_2(\mathbb{C}^{m+2})$, $m \geqslant 3$. Then both $[\xi]$ and $\mathfrak{D}^\perp$ are invariant under the shape operator of M if and only if*

(*A*) *M is an open part of a tube around a totally geodesic $G_2(\mathbb{C}^{m+1})$ in $G_2(\mathbb{C}^{m+2})$, or*

(*B*) *m is even, say $m = 2n$, and M is an open part of a tube around a totally geodesic $\mathbb{H}P^n$ in $G_2(\mathbb{C}^{m+2})$.*

In Theorem A the vector ξ contained in the one-dimensional distribution $[\xi]$ is said to be a *Hopf* vector when it becomes a principal vector for the shape operator A of M in $G_2(\mathbb{C}^{m+2})$. Moreover in such a situation M is said to be a *Hopf* hypersurface. Besides this, a real hypersurface M in $G_2(\mathbb{C}^{m+2})$ also admits the 3-dimensional distribution $\mathfrak{D}^\perp$, which is spanned by *almost contact 3-structure* vector fields $\{\xi_1, \xi_2, \xi_3\}$, such that $T_xM = \mathfrak{D} \oplus \mathfrak{D}^\perp$. Also in the paper [5] due to Berndt and the present author we have given a characterization of real hypersurfaces of type A when the shape operator A of M in $G_2(\mathbb{C}^{m+2})$ commutes with the structure tensor ϕ, which is equivalent to the condition that the Reeb flow on M is isometric. Moreover, in the paper due to the present author [12] we have also given a characterization of type A by vanishing Lie derivative of the shape operator A in the direction of the structure vector field ξ.

Real hypersurfaces of type B in Theorem A is just the case that the one dimensional distribution $[\xi]$ is contained in $\mathfrak{D}^{\perp}$. It was shown in the paper [11] that the tube of type B satisfies the following formula on the orthogonal complement of the one-dimensional distribution $[\xi]$

$$A\phi_{\nu} - \phi_{\nu}A = 0, \quad \nu = 1, 2, 3.$$

From this view point, the present author [11] has given a characterization that the almost contact 3-structure tensors $\{\phi_1, \phi_2, \phi_3\}$ and the shape operator A of a real hypersurface M in $G_2(\mathbb{C}^{m+2})$ commute with each other as follows:

Theorem B. *Let M be a Hopf real hypersurface in $G_2(\mathbb{C}^{m+2})$ satisfying $(^*)$ on the orthogonal complement of the one-dimensional distribution $[\xi]$. Then M is locally congruent to an open part of a tube around a totally geodesic $\mathbb{H}P^m$ in $G_2(\mathbb{C}^{m+2})$, where $m = 2n$.*

Now in this paper as another characterization of real hypersurfaces of type B in complex two-plane Grassmannians $G_2(\mathbb{C}^{m+2})$ in terms of the *contact* hypersurface we want to assert the following remarkable fact:

Theorem. *Let M be a real hypersurface in $G_2(\mathbb{C}^{m+2})$ with constant mean curvature satisfying*

$$A\phi + \phi A = k\phi,$$

where the function k is non-zero and constant. Then M is congruent to an open part of a tube around a totally geodesic $\mathbb{H}P^n$ in $G_2(\mathbb{C}^{m+2})$, where $m = 2n$.

1. Riemannian Geometry of $G_2(\mathbb{C}^{m+2})$

In this section we summarize basic material about $G_2(\mathbb{C}^{m+2})$, for details we refer to [3], [4] and [5]. The special unitary group $G = SU(m+2)$ acts transitively on $G_2(\mathbb{C}^{m+2})$ with stabilizer isomorphic to $K = S(U(2) \times U(m)) \subset G$. Then $G_2(\mathbb{C}^{m+2})$ can be identified with the homogeneous space G/K, which we equip with the unique analytic structure for which the natural action of G on $G_2(\mathbb{C}^{m+2})$ becomes analytic. Denote by $\mathfrak{g}$ and $\mathfrak{k}$ the Lie algebra of G and K, respectively, and by $\mathfrak{m}$ the orthogonal complement of $\mathfrak{k}$ in $\mathfrak{g}$ with respect to the Cartan-Killing form B of $\mathfrak{g}$. Then $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$ is an $\mathrm{Ad}(K)$-invariant reductive decomposition of $\mathfrak{g}$. We put $o = eK$ and identify $T_oG_2(\mathbb{C}^{m+2})$ with $\mathfrak{m}$ in the usual manner. Since B is negative definite on $\mathfrak{g}$, its negative restricted to $\mathfrak{m} \times \mathfrak{m}$ yields a positive definite inner product on $\mathfrak{m}$. By $\mathrm{Ad}(K)$-invariance of B this inner product can be extended to a G-invariant Riemannian metric g on $G_2(\mathbb{C}^{m+2})$. In this way $G_2(\mathbb{C}^{m+2})$ becomes a Riemannian homogeneous space, even a Riemannian symmetric space.

For computational reasons we normalize g such that the maximal sectional curvature of $(G_2(\mathbb{C}^{m+2}), g)$ is eight. Since $G_2(\mathbb{C}^3)$ is isometric to the two-dimensional complex projective space $\mathbb{C}P^2$ with constant holomorphic sectional curvature eight we will assume $m \geqslant 2$ from now on. Note that the isomorphism $\mathrm{Spin}(6) \simeq \mathrm{SU}(4)$ yields an isometry between $G_2(\mathbb{C}^4)$ and the real Grassmann manifold $G_2^+(\mathbb{R}^6)$ of oriented two-dimensional linear subspaces of $\mathbb{R}^6$.

The Lie algebra $\mathfrak{k}$ has the direct sum decomposition $\mathfrak{k} = \mathfrak{su}(m) \oplus \mathfrak{su}(2) \oplus \mathfrak{R}$, where $\mathbb{R}$ is the center of $\mathfrak{k}$. Viewing $\mathfrak{k}$ as the holonomy algebra of $G_2(\mathbb{C}^{m+2})$, the center $\mathfrak{R}$ induces a Kähler structure J and the $\mathfrak{su}(2)$-part a quaternionic Kähler structure $\mathfrak{J}$ on $G_2(C^{m+2})$. If J_1 is any almost Hermitian structure in $\mathfrak{J}$, then $JJ_1 = J_1J$, and JJ_1 is a symmetric endomorphism with $(JJ_1)^2 = I$ and $\mathrm{Tr}(JJ_1) = 0$. This fact will be used frequently throughout this paper.

A canonical local basis J_1, J_2, J_3 of $\mathfrak{J}$ consists of three local almost Hermitian structures J_ν in $\mathfrak{J}$ such that $J_\nu J_{\nu+1} = J_{\nu+2} = -J_{\nu+1}J_\nu$, where the index is taken modulo 3. Since $\mathfrak{J}$ is parallel with respect to the Riemannian connection $\bar{\nabla}$ of $(G_2(\mathbb{C}^{m+2}), g)$, there exist for any canonical local basis J_1, J_2, J_3 of $\mathfrak{J}$ three local one-forms q_1, q_2, q_3 such that

$$\bar{\nabla}_X J_\nu = q_{\nu+2}(X)J_{\nu+1} - q_{\nu+1}(X)J_{\nu+2} \tag{1.1}$$

for all vector fields X on $G_2(\mathbb{C}^{m+2})$.

Let $p \in G_2(\mathbb{C}^{m+2})$ and W a subspace of $T_pG_2(\mathbb{C}^{m+2})$. We say that W is a quaternionic subspace of $T_pG_2(\mathbb{C}^{m+2})$ if $JW \subset W$ for all $J \in \mathfrak{J}_p$. And we say that W is a totally complex subspace of $T_pG_2(\mathbb{C}^{m+2})$ if there exists a one-dimensional subspace $\mathfrak{V}$ of $\mathfrak{J}_p$ such that $JW \subset W$ for all $J \in \mathfrak{V}$ and $JW \perp W$ for all $J \in \mathfrak{V}^\perp \subset \mathfrak{J}_p$. Here, the orthogonal complement of $\mathfrak{V}$ in $\mathfrak{J}_p$ is taken with respect to the bundle metric and orientation on $\mathfrak{J}$ for which any local oriented orthonormal frame field of $\mathfrak{J}$ is a canonical local basis of $\mathfrak{J}$. A quaternionic (resp. totally complex) submanifold of $G_2(\mathbb{C}^{m+2})$ is a submanifold all of whose tangent spaces are quaternionic (resp. totally complex) subspaces of the corresponding tangent spaces of $G_2(\mathbb{C}^{m+2})$.

The Riemannian curvature tensor $\bar{R}$ of $G_2(\mathbb{C}^{m+2})$ is locally given by

$$\begin{aligned}\bar{R}(X,Y)Z = {} & g(Y,Z)X - g(X,Z)Y + g(JY,Z)JX - g(JX,Z)JY - 2g(JX,Y)JZ \\ & + \sum_{\nu=1}^{3}\{g(J_\nu Y,Z)J_\nu X - g(J_\nu X,Z)J_\nu Y - 2g(J_\nu X,Y)J_\nu Z\} \\ & + \sum_{\nu=1}^{3}\{g(J_\nu JY,Z)J_\nu JX - g(J_\nu JX,Z)J_\nu JY\},\end{aligned} \tag{1.2}$$

where J_1, J_2, J_3 is any canonical local basis of $\mathfrak{J}$.

2. Some Fundamental Formulas for Real Hypersurfaces in $G_2(\mathbb{C}^{m+2})$

In this section we derive some fundamental formulas which will be used in the proof of our main theorem. Let M be a real hypersurface in $G_2(\mathbb{C}^{m+2})$, that is, a hypersurface in $G_2(\mathbb{C}^{m+2})$ with real codimension one. The induced Riemannian metric on M will also be denoted by g, and ∇ denotes the Riemannian connection of (M, g). Let N be a local unit normal field of M and A the shape operator of M with respect to N.

The Kähler structure J of $G_2(\mathbb{C}^{m+2})$ induces on M an almost contact metric structure (ϕ, ξ, η, g). Furthermore, let J_1, J_2, J_3 be a canonical local basis of $\mathfrak{J}$. Then each J_ν induces an almost contact metric structure $(\phi_\nu, \xi_\nu, \eta_\nu, g)$ on M. Using the above expression (1.2) for the curvature tensor $\bar{R}$, the Gauss and the Codazzi equations are respectively given by

$$\begin{aligned} R(X,Y)Z = {} & g(Y,Z)X - g(X,Z)Y + g(\phi Y,Z)\phi X - g(\phi X,Z)\phi Y - 2g(\phi X,Y)\phi Z \\ & + \sum_{\nu=1}^{3}\{g(\phi_\nu Y,Z)\phi_\nu X - g(\phi_\nu X,Z)\phi_\nu Y - 2g(\phi_\nu X,Y)\phi_\nu Z\} \\ & + \sum_{\nu=1}^{3}\{g(\phi_\nu\phi Y,Z)\phi_\nu\phi X - g(\phi_\nu\phi X,Z)\phi_\nu\phi Y\} \\ & - \sum_{\nu=1}^{3}\{\eta(Y)\eta_\nu(Z)\phi_\nu\phi X - \eta(X)\eta_\nu(Z)\phi_\nu\phi Y\} \\ & - \sum_{\nu=1}^{3}\{\eta(X)g(\phi_\nu\phi Y,Z) - \eta(Y)g(\phi_\nu\phi X,Z)\}\xi_\nu \\ & + g(AY,Z)AX - g(AX,Z)AY \end{aligned}$$

and

$$\begin{aligned} (\nabla_X A)Y - (\nabla_Y A)X = {} & \eta(X)\phi Y - \eta(Y)\phi X - 2g(\phi X,Y)\xi \\ & + \sum_{\nu=1}^{3}\{\eta_\nu(X)\phi_\nu Y - \eta_\nu(Y)\phi_\nu X - 2g(\phi_\nu X,Y)\xi_\nu\} \\ & + \sum_{\nu=1}^{3}\{\eta_\nu(\phi X)\phi_\nu\phi Y - \eta_\nu(\phi Y)\phi_\nu\phi X\}, \\ & + \sum_{\nu=1}^{3}\{\eta(X)\eta_\nu(\phi Y) - \eta(Y)\eta_\nu(\phi X)\}\xi_\nu, \end{aligned}$$

where R denotes the curvature tensor of a real hypersurface M in $G_2(\mathbb{C}^{m+2})$.

The following identities can be proved in a straightforward method and will be used frequently in subsequent calculations:

$$\begin{aligned} \phi_{\nu+1}\xi_\nu = -\xi_{\nu+2}, \quad \phi_\nu\xi_{\nu+1} = \xi_{\nu+2}, \\ \phi\xi_\nu = \phi_\nu\xi, \quad \eta_\nu(\phi X) = \eta(\phi_\nu X), \\ \phi_\nu\phi_{\nu+1}X = \phi_{\nu+2}X + \eta_{\nu+1}(X)\xi_\nu, \\ \phi_{\nu+1}\phi_\nu X = -\phi_{\nu+2}X + \eta_\nu(X)\xi_{\nu+1}. \end{aligned} \tag{2.1}$$

Then in this section let us give some basic formulas which will be used later.

Now let us put

$$JX = \phi X + \eta(X)N, \qquad J_\nu X = \phi_\nu X + \eta_\nu(X)N$$

for any tangent vector X of a real hypersurface M in $G_2(\mathbb{C}^{m+2})$, where N denotes a normal vector of M in $G_2(\mathbb{C}^{m+2})$. Then from this and the formulas in Section 1 we have that

$$(\nabla_X\phi)Y = \eta(Y)AX - g(AX, Y)\xi, \qquad \nabla_X\xi = \phi AX, \tag{2.2}$$

$$\nabla_X\xi_\nu = q_{\nu+2}(X)\xi_{\nu+1} - q_{\nu+1}(X)\xi_{\nu+2} + \phi_\nu AX, \tag{2.3}$$

$$(\nabla_X\phi_\nu)Y = -q_{\nu+1}(X)\phi_{\nu+2}Y + q_{\nu+2}(X)\phi_{\nu+1}Y + \eta_\nu(Y)AX - g(AX, Y)\xi_\nu. \tag{2.4}$$

Summing up these formulas, we know the following

$$\begin{aligned}\nabla_X(\phi_\nu\xi) &= (\nabla_X\phi_\nu)\xi + \phi_\nu(\nabla_X\xi)\\ &= -q_{\nu+1}(X)\phi_{\nu+2}\xi + q_{\nu+2}(X)\phi_{\nu+1}\xi + \eta_\nu(\xi)AX - g(AX, \xi)\xi_\nu + \phi_\nu\phi AX.\end{aligned} \tag{2.5}$$

Moreover, from $JJ_\nu = J_\nu J$, $\nu = 1, 2, 3$, it follows that

$$\phi\phi_\nu X = \phi_\nu\phi X + \eta_\nu(X)\xi - \eta(X)\xi_\nu. \tag{2.6}$$

3. Some Key Propositions

Before going to give the proof of our main Theorem in the introduction let us check that "What kind of model hypersurfaces given in Theorem A satisfy the formula $(^*)$." In other words, it will be an interesting problem to know whether there exist any real hypersurfaces in $G_2(\mathbb{C}^{m+2})$ satisfying the condition $(^*)$.

In this section we will show that only real hypersurfaces of type B in $G_2(\mathbb{C}^{m+2})$, that is, a tube over a quaternionic projective space $\mathbb{H}P^n$ in $G_2(\mathbb{C}^{m+2})$ satisfies the formula $A\phi + \phi A = k\phi$, $m = 2n$, where the function k is non-zero and constant.

Now in order to solve such a problem let us recall some Propositions given by Berndt and the present author [4] as follows:

For a tube of type A in Theorem A we have the following

Proposition A. *Let M be a connected real hypersurface of $G_2(\mathbb{C}^{m+2})$. Suppose that $A\mathfrak{D} \subset \mathfrak{D}$, $A\xi = \alpha\xi$, and ξ is tangent to $\mathfrak{D}^\perp$. Let $J_1 \in \mathfrak{J}$ be the almost Hermitian structure such that $JN = J_1N$. Then M has three (if $r = \pi/2\sqrt{8}$) or four (otherwise) distinct constant principal curvatures*

$$\alpha = \sqrt{8}\cot(\sqrt{8}r), \qquad \beta = \sqrt{2}\cot(\sqrt{2}r), \qquad \lambda = -\sqrt{2}\tan(\sqrt{2}r), \qquad \mu = 0$$

with some $r \in (0, \pi/4)$. The corresponding multiplicities are

$$m(\alpha) = 1, \qquad m(\beta) = 2, \qquad m(\lambda) = 2m - 2 = m(\mu),$$

and the corresponding eigenspaces we have

$$T_\alpha = \mathbb{R}\xi = \mathbb{R}JN = \mathbb{R}\xi_1,$$
$$T_\beta = \mathbb{C}^\perp\xi = \mathbb{C}^\perp N = \mathbb{R}\xi_2 \oplus \mathbb{R}\xi_3,$$
$$T_\lambda = \{X \,|\, X \perp \mathbb{H}\xi, JX = J_1X\},$$
$$T_\mu = \{X \,|\, X \perp \mathbb{H}\xi, JX = -J_1X\},$$

where $\mathbb{R}\xi$, $\mathbb{C}\xi$ *and* $\mathbb{Q}\xi$ *repectively denotes real, complex and quaternionic span of the structure vector* ξ *and* $\mathbb{C}^\perp\xi$ *denotes the orthogonal complement of* $\mathbb{C}\xi$ *in* $\mathbb{H}\xi$.

For such kind of real hypersurfaces of type A mentioned above let us check whether this type satisfies the formula $(^*)$ or not.

Now let us assume that real hypersurfaces of type A satisfies the formula $(^*)$. In Proposition A let us put $X = \xi_2 \in T_\beta$, $\beta = \beta_2 = \beta_3$, and $\xi = \xi_1$. Then by the formula (2.1) we have

$$\begin{aligned} A\phi\xi_2 + \phi A\xi_2 &= A\phi_2\xi_1 + \phi A\xi_2 \\ &= -A\xi_3 + \beta_2\phi\xi_2 \\ &= -\beta_3\xi_3 - \beta_2\xi_3 \\ &= -2\sqrt{2}\cot\sqrt{2}r\xi_3. \end{aligned}$$

From this, together with the formula $(^*)$ we know

$$-k\xi_3 = k\phi\xi_2 = -2\sqrt{2}\cot\sqrt{2}r\ \xi_3$$

which means $k = 2\sqrt{2}\cot\sqrt{2}r$.

On the other hand, by the paper [5] of Berndt and the present author we know that the distributions T_λ and T_μ in Proposition A are ϕ-invariant, that is $\phi T_\lambda \subset T_\lambda$ and $\phi T_\mu \subset T_\mu$ respectively. By virtue of this fact we know that for any $X \in T_\lambda$, $\lambda = -\sqrt{2}\tan\sqrt{2}r$

$$A\phi X + \phi AX = -2\sqrt{2}\tan\sqrt{2}r\phi X.$$

Then in this time $k = -2\sqrt{2}\tan\sqrt{2}r$. From this, together with the above formula we get $\cot^2\sqrt{2}r = -1$, which makes a contradiction. So real hypersurfaces of type A can not satisfy the formula $(^*)$.

Moreover, for a tube of type B in Theorem A we introduce the following

Proposition B. *Let* M *be a connected real hypersurface of* $G_2(\mathbb{C}^{m+2})$. *Suppose that* $A\mathfrak{D} \subset \mathfrak{D}$, $A\xi = \alpha\xi$, *and* ξ *is tangent to* $\mathfrak{D}$. *Then the quaternionic dimension* m *of* $G_2(\mathbb{C}^{m+2})$ *is even, say* $m = 2n$, *and* M *has five distinct constant principal curvatures*

$$\alpha = -2\tan(2r), \quad \beta = 2\cot(2r), \quad \gamma = 0, \quad \lambda = \cot(r), \quad \mu = -\tan(r)$$

with some $r \in (0, \pi/4)$. *The corresponding multiplicities are*

$$m(\alpha) = 1, \qquad m(\beta) = 3 = m(\gamma), \qquad m(\lambda) = 4n-4 = m(\mu)$$

and the corresponding eigenspaces are

$$T_\alpha = \mathbb{R}\xi, \qquad T_\beta = \mathfrak{J}J\xi, \qquad T_\gamma = \mathfrak{J}\xi, T_\lambda, T_\mu,$$

where

$$T_\lambda \oplus T_\mu = (\mathbb{HC}\xi)^\perp, \qquad \mathfrak{J}T_\lambda = T_\lambda, \qquad \mathfrak{J}T_\mu = T_\mu, \qquad JT_\lambda = T_\mu.$$

Of course we have proved that all of the principal curvatures and its eigenspaces of the tube of type A (resp. the tube of type B) in Theorem A satisfies all of the properties in Proposition A (resp. Proposition B).

Now by using this Proposition B we show that a tube of type B in Theorem A, that is, a tube over a totally geodesic $\mathbb{H}P^n$ in $G_2(\mathbb{C}^{m+2})$, $m = 2n$ satisfies the formula $(^*)$ for a constant $k = 2\cot\ 2r$ as follows:

For any $\xi \in T_\alpha$, $\alpha = -2\tan 2r$, we have

$$A\phi\xi + \phi A\xi = 0 = k\phi\xi.$$

For any $\xi_\nu \in T_\beta$, $\beta = 2\cot\ 2r$, the eigen space $T_\gamma = \mathfrak{J}\xi$ gives $\phi\xi_\nu \in T_\gamma$. This implies $A\phi\xi_\nu = 0$ for any $\nu = 1, 2, 3$. From this we have the following for $k = 2\cot\ 2r$

$$A\phi\xi_\nu + \phi A\xi_\nu = 2\cot 2r\, \phi\xi_\nu.$$

For any $X \in T_\lambda$, $\lambda = \cot\ r$ we know that $JT_\lambda = T_\mu$ gives

$$A\phi X + \phi AX = -\tan r\phi X + \cot r\phi X = 2\cot 2r\phi X.$$

This means that the formula $(^*)$ holds for $k = 2\cot 2r$.

Finally, for the case $\phi\xi_\nu \in T_\gamma$, $\nu = 1, 2, 3$, the formula $(^*)$ also holds for $k = 2\cot 2r$.

4. Some Key Lemmas and Theorems

In order to give a characterization of type B among the classes of real hypersurfaces M in complex two-plane Grassmannians $G_2(\mathbb{C}^{m+2})$ we will prepare some lemmas and a proposition as follows:

Lemma 1. *Let M be a real hypersurface in $G_2(\mathbb{C}^{m+2})$ satisfying*

$$A\phi + \phi A = k\phi,$$

where the function k is non-zero and constant. Then $Tr\ A = \alpha + nk$, where $n = 2m - 1$

Proof. Now suppose that $\phi A + A\phi = k\phi$. By applying ϕ to the left, we have

$$\phi^2 A + \phi A\phi = k\phi^2.$$

Then it follows that

$$AX - \phi A\phi X - kX + (k - \alpha)\eta(X)\xi = 0.$$

Now let us take an orthonormal basis $\{e_i \mid i = 1, \ldots, 4m-1\}$ for M in above formula. Then we have

$$\text{Tr}\ A - \text{Tr}\ \phi A\phi - (4m-1)k + (k - \alpha) = 0. \tag{4.1}$$

On the other hand, we know

$$\text{Tr}\ \phi A\phi = \text{Tr}\ A\phi^2 = -\text{Tr}\ A + \alpha.$$

Because we have

$$A\phi^2 X = -AX + \eta(X)A\xi = -AX + \alpha\eta(X)\xi.$$

From this, together with (4.1), we have

$$\mathrm{Tr}\, A = (2m-1)k + \alpha,$$

which completes the proof of Lemma 1. □

Now let us assume that the structure vector ξ is principal and denote by $\mathfrak{H}$ the orthogonal complement of the real span $[\xi]$ of the structure vector ξ in TM. Then taking an inner product of the Codazzi equation in section 2 with ξ and using $A\xi = \alpha\xi$ imply

$$\begin{aligned} &-2g(\phi X, Y) + 2\sum_{\nu=1}^{3}\{\eta_\nu(X)\eta_\nu(\phi Y) - \eta_\nu(Y)\eta_\nu(\phi X) - g(\phi_\nu X, Y)\eta_\nu(\xi)\} \\ &\quad = g((\nabla_X A)Y - (\nabla_Y A)X, \xi) \\ &\quad = g((\nabla_X A)\xi, Y) - g((\nabla_Y A)\xi, X) \\ &\quad = (X\alpha)\eta(Y) - (Y\alpha)\eta(X) + \alpha g((A\phi + \phi A)X, Y) - 2g(A\phi AX, Y). \end{aligned} \tag{4.2}$$

Putting $X = \xi$, we have

$$Y\alpha = (\xi\alpha)\eta(Y) - 4\sum_{\nu=1}^{3}\eta_\nu(\xi)\eta_\nu(\phi Y) \tag{4.3}$$

for any tangent vector field Y on M. Substituting this formula into (4.2), then we have

$$\begin{aligned} &-2g(\phi X, Y) + 2\sum_{\nu=1}^{3}\{\eta_\nu(X)\eta_\nu(\phi Y) - \eta_\nu(Y)\eta_\nu(\phi X) - g(\phi_\nu X, Y)\eta_\nu(\xi)\} \\ &\quad = 4\sum_{\nu=1}^{3}\{\eta(X)\eta_\nu(\phi Y) - \eta(Y)\eta_\nu(\phi X)\}\eta_\nu(\xi) + \alpha g((A\phi + \phi A)X, Y) \\ &\qquad - 2g(A\phi AX, Y). \end{aligned}$$

From this formula we are able to assert

Lemma 2. *If $A\xi = \alpha\xi$ and $X \in \mathfrak{H}$ with $AX = \lambda X$, then*

$$\begin{aligned} 0 = (2\lambda - \alpha)A\phi X - (2 + \lambda\alpha)\phi X + 2\sum_{\nu=1}^{3}\{&2\eta_\nu(\xi)\eta_\nu(\phi X)\xi - \eta_\nu(X)\phi_\nu\xi \\ &- \eta_\nu(\phi X)\xi_\nu - \eta_\nu(\xi)\phi_\nu X\}. \end{aligned}$$

Lemma 3. *Let M be a real hypersurface in $G_2(\mathbb{C}^{m+2})$ satisfying $A\phi + \phi A = k\phi$, k is non-zero and constant. Then ξ is principal. Moreover, the principal curvature function α is constant provided that $\xi \in \mathfrak{D}^\perp$ or $\xi \in \mathfrak{D}$.*

Proof. Then by applying the structure vector ξ to the above assumption in the right side, we know $\phi A\xi = 0$. This means $A\xi = \alpha\xi$, that is, the structure vector

ξ is principal. Then we are able to use (4.2) and (4.3). The formula (4.3) means that

$$\text{grad}\ \alpha = (\xi\alpha)\xi + 4\sum_{\nu=1}^{3}\eta_\nu(\xi)\phi\xi_\nu. \tag{4.4}$$

For the case where $\xi \in \mathfrak{D}^\perp$. We may put $\xi = \xi_1$. Then (4.4) implies

$$\text{grad}\ \alpha = (\xi\alpha)\xi. \tag{4.5}$$

For the case where $\xi \in \mathfrak{D}$. Then naturally the formula (4.4) gives (4.5). Now differentiating (4.5), we have

$$\nabla_X(\text{grad}\ \alpha) = X(\xi\alpha)\xi + (\xi\alpha)\phi AX.$$

Then this implies

$$\begin{aligned} 0 &= g(\nabla_X(\text{grad}\ \alpha), Y) - g(\nabla_Y(\text{grad}\ \alpha), X) \\ &= X(\xi\alpha)\eta(Y) - Y(\xi\alpha)\eta(X) + (\xi\alpha)g((\phi A + A\phi)X, Y). \end{aligned}$$

This gives

$$k(\xi\alpha)g(\phi X, Y) = Y(\xi\alpha)\eta(X) - X(\xi\alpha)\eta(Y).$$

From this, putting $X = \xi$, we have $Y(\xi\alpha) = \xi(\xi\alpha)\eta(Y)$. Then it follows that

$$k(\xi\alpha)g(\phi X, Y) = 0.$$

By virtue of $k \neq 0$, we have $\xi\alpha = 0$. From this, together with (4.5) we have

$$\text{grad}\ \alpha = 0,$$

which means that the principal curvature α is constant. □

Then by using Lemmas 1 and 3 we have the following Proposition.

Proposition 4. *Let M be a real hypersurface in $G_2(\mathbb{C}^{m+2})$ satisfying $A\phi + \phi A = k\phi$, k is non-zero and constant. Then we have*

$$\begin{aligned} &2A^2X - 2kAX + (\alpha k + 2)X \\ &\quad - \left[\eta(X)(2\alpha^2 - \alpha k + 2) + 4\sum_\nu \eta_\nu(\xi)\eta_\nu(X) - 4\sum_\nu \eta_\nu^2(\xi)\eta(X)\right]\xi \\ &\quad - 2\sum_\nu \left\{\eta_\nu(\phi X)\phi\xi_\nu - \eta_\nu(X)\xi_\nu + \eta(X)\eta_\nu(\xi)\xi_\nu + \eta_\nu(\xi)\phi_\nu\phi X\right\} = 0, \end{aligned}$$

where $\sum_\nu$ denotes the sum from $\nu = 1$ to $\nu = 3$.

Proof. Now substituting (4.3) into (4.2), we have

$$\begin{aligned} &-2g(\phi X, Y) + 2\sum_\nu\{\eta_\nu(X)\eta_\nu(\phi Y) - \eta_\nu(Y)\eta_\nu(\phi X) - g(\phi_\nu X, Y)\eta_\nu(\xi)\} \\ &\quad = -4\sum_\nu \eta_\nu(\xi)\eta_\nu(\phi X)\eta(Y) + 4\sum_\nu \eta_\nu(\xi)\eta_\nu(\phi Y)\eta(X) \\ &\qquad + \alpha g((A\phi + \phi A)X, Y) - 2g(A\phi AX, Y). \end{aligned} \tag{4.6}$$

On the other hand, from the assumption we have

$$g(A\phi AX, Y) = kg(A\phi X, Y) - g(A^2\phi X, Y).$$

Then from this together with formula (4.6) we have

$$\begin{aligned} &2A^2\phi X - 2kA\phi X + (\alpha k + 2)\phi X - 4\sum_\nu \eta_\nu(\xi)\eta_\nu(\phi X)\xi - 4\sum_\nu \eta_\nu(\xi)\eta(X)\phi\xi_\nu \\ &\quad = 2\sum_\nu \{-\eta_\nu(X)\phi\xi_\nu - \eta_\nu(\phi X)\xi_\nu - \eta_\nu(\xi)\phi_\nu X\}. \end{aligned}$$

Replacing X by ϕX, we have

$$\begin{aligned} 2A^2X &= 2\eta(X)A^2\xi + 2kAX - 2k\eta(X)A\xi - (\alpha k + 2)X \\ &\quad + \eta(X)(\alpha k + 2)\xi + 4\sum_\nu \eta_\nu(\xi)\eta_\nu(X)\xi - 4\sum_\nu \eta_\nu^2(\xi)\eta(X)\xi \\ &\quad + 2\sum_\nu\{\eta_\nu(\phi X)\phi\xi_\nu - \eta_\nu(X)\xi_\nu + \eta(X)\eta_\nu(\xi)\xi_\nu + \eta_\nu(\xi)\phi_\nu\phi X\} \\ &= \left[\eta(X)(2\alpha^2 - \alpha k + 2) + 4\sum_\nu \eta_\nu(\xi)\eta_\nu(X) - 4\sum_\nu \eta_\nu^2(\xi)\eta(X)\right]\xi \\ &\quad - (\alpha k + 2)X + 2kAX + 2\sum_\nu\{\eta_\nu(\phi X)\phi\xi_\nu \\ &\quad - \eta_\nu(X)\xi_\nu + \eta(X)\eta_\nu(\xi)\xi_\nu + \eta_\nu(\xi)\phi_\nu\phi X\}. \end{aligned}$$

□

Now we are going to prove a key Lemma which will be useful in the proof of our Main Theorem.

Lemma 5. *Under the same assumption as in Proposition 4 we have*

$$X(k - Tr\ A) = \eta(X)\xi(k - Tr\ A) + 4\sum_\nu \eta_\nu(\phi X)\eta_\nu(\xi).$$

Proof. Differentiating $(\phi A + A\phi)X = k\phi X$ covariantly, we have

$$(\nabla_Y\phi)AX + \phi(\nabla_Y A)X + (\nabla_Y A)\phi X + A(\nabla_Y\phi)X = (Yk)\phi X + k(\nabla_Y\phi)X.$$

Then substituting the formula (2.2) into the above equation, we have

$$\begin{aligned} &\eta(X)\{A^2Y + \alpha AY - kAY\} - g(A^2X + \alpha AX - kAX, Y)\xi + \phi(\nabla_Y A)X \\ &\quad + (\nabla_Y A)\phi X = (Yk)\phi X. \end{aligned}$$

From this, using Proposition 4, we have

$$\begin{aligned} \eta(X)\Bigg[\alpha AY - \frac{\alpha k + 2}{2}Y + \Bigg\{\eta(Y)\left(\alpha^2 - \frac{\alpha}{2}k + 1\right) \\ + 2\sum_\nu \eta_\nu(\xi)\eta_\nu(Y) - 2\sum_\nu \eta_\nu^2(\xi)\eta(Y)\Bigg\}\xi \end{aligned}$$

$$
\begin{aligned}
&+\sum_{\nu}\{\eta_\nu(\phi Y)\phi\xi_\nu-\eta_\nu(Y)\xi_\nu+\eta(Y)\eta_\nu(\xi)\xi_\nu+\eta_\nu(Y)\phi_\nu\phi Y\}\Big]\\
&-g\left(\alpha AX-\frac{\alpha k+2}{2}X,Y\right)\xi\\
&+g\left(\left\{\eta(X)\left(\alpha^2-\frac{\alpha}{2}k+1\right)+2\sum_{\nu}\eta_\nu(\xi)\eta_\nu(X)-2\sum_{\nu}\eta_\nu^2(\xi)\eta(X)\right\}\xi,Y\right)\xi\\
&+\sum_{\nu}g(\{\eta_\nu(\phi X)\phi\xi_\nu-\eta_\nu(X)\xi_\nu+\eta(X)\eta_\nu(\xi)\xi_\nu+\eta_\nu(\xi)\phi_\nu\phi X\},Y)\xi\\
&+\phi(\nabla_Y A)X+(\nabla_Y A)\phi X\\
&=(Yk)\phi X.
\end{aligned}
$$

From this, contracting, we have

$$
\begin{aligned}
\sum_i(E_ik)\phi E_i=\alpha A\xi-\frac{\alpha k+2}{2}\xi-\alpha\sum_i g(AE_i,E_i)\xi&\\
+\frac{\alpha k+2}{2}\sum_i g(E_i,E_i)\xi-\sum_{i,\nu}\eta_\nu(\phi E_i)g(\phi\xi_\nu,E_i)\xi&\\
+\sum_{i,\nu}\eta_\nu(E_i)\eta_\nu(E_i)\xi-\sum_{i,\nu}\eta(E_i)\eta_\nu(\xi)\eta_\nu(E_i)\xi&\\
-\sum_{i,\nu}\eta_\nu(\xi)g(\phi_\nu\phi E_i,E_i)\xi+\phi(\nabla_{E_i}A)E_i+(\nabla_{E_i}A)\phi E_i,&
\end{aligned}
$$

where $\sum_i$ (resp. $\sum_\nu$) denotes the sum from $i=1$ to $i=4m-1$ (resp. from $\nu=1$ to $\nu=3$). Then by virtue of formulas defined in (2.1) we have

$$
\begin{aligned}
\sum_i(E_ik)\phi E_i=\alpha A\xi-\frac{\alpha k+2}{2}\xi-\alpha(\mathrm{Tr}\,A)\xi+\frac{\alpha k+2}{2}(4m-1)\xi+6\xi-2\sum_\nu\eta_\nu^2(\xi)\xi&\\
-\sum_\nu\eta_\nu(\xi)(\mathrm{Tr}\,\phi_\nu\phi)\xi+\sum_i\phi(\nabla_{E_i}A)E_i+\sum_i(\nabla_{E_i}A)\phi E_i.&\qquad(4.7)
\end{aligned}
$$

On the other hand, the first term in the fourth line of (4.7) becomes

$$
\sum_i g(\phi(\nabla_{E_i}A)E_i,X)=-\sum_i g((\nabla_{E_i}A)E_i,\phi X)=-\sum_i g(E_i,(\nabla_{E_i}A)\phi X).
$$

Also by virtue of the Codazzi equation in section 2 the last term of the above equation can be changed into

$$
\begin{aligned}
&\sum_i g((\nabla_{E_i}A)\phi X-(\nabla_{\phi X}A)E_i,E_i)\\
&\quad=\sum_\nu\eta(X)\eta_\nu^2(\xi)-\sum_\nu\eta_\nu(X)\eta_\nu(\xi)+\sum_\nu\eta_\nu(X)\mathrm{Tr}\,\phi_\nu\phi\\
&\qquad-\eta(X)\sum_\nu\eta_\nu(\xi)\mathrm{Tr}\,\phi_\nu\phi-\sum_\nu\eta_\nu(X)\eta_\nu(\xi)+\eta(X)\sum_\nu\eta_\nu^2(\xi).\qquad(4.8)
\end{aligned}
$$

Also let us use the Codazzi equation in the final term of the fourth line of (4.7). Then it follows that

$$\begin{aligned}\sum_i g(\phi E_i, (\nabla_{E_i}A)X) &= \sum_i g(\phi E_i, (\nabla_X A)E_i) - \eta(X)\sum_i g(\phi E_i, \phi E_i) \\ &+ \sum_{\nu,i} \eta_\nu(\phi E_i) g(\phi_\nu \phi X, \phi E_i) + \sum_{\nu,i}\{\eta_\nu(E_i) g(\phi_\nu X, \phi E_i) \\ &- \eta_\nu(X) g(\phi_\nu E_i, \phi E_i) - 2g(\phi_\nu E_i, X)\eta_\nu(\phi E_i)\} \\ &+ \sum_{\nu,i}\{\eta(E_i)\eta_\nu(\phi X) - \eta(X)\eta_\nu(\phi E_i)\}\eta_\nu(\phi E_i). \qquad (4.9)\end{aligned}$$

Now from the third term in the right side of (4.9) let us calculate term by term as follows:

$$\sum_\nu \eta_\nu(\phi E_i) g(\phi_\nu \phi X, \phi E_i) = -\sum_\nu \eta_\nu(\phi^2 \phi_\nu \phi X) = \sum_\nu \eta_\nu(X)\eta_\nu(\xi) - \sum_\nu \eta(X)\eta_\nu^2(\xi),$$

$$\sum_{i,\nu} \eta_\nu(E_i) g(\phi_\nu X, \phi E_i) = -\sum_\nu \eta_\nu(\phi\phi_\nu X) = 3\eta(X) - \sum_\nu \eta_\nu(X)\eta(\xi_\nu),$$

$$-\sum_{i,\nu} \eta_\nu(X) g(\phi_\nu E_i, \phi E_i) = \sum_\nu \eta_\nu(X) \mathrm{Tr}\phi\phi_\nu,$$

$$-2\sum_{i,\nu} g(\phi_\nu E_i, X)\eta_\nu(\phi E_i) = -6\eta(X) + 2\sum_\nu \eta_\nu(\xi)\eta_\nu(X),$$

and

$$-\sum_\nu \eta(X)\eta_\nu(\phi E_i)\eta_\nu(\phi E_i) = -3\eta(X) + \sum_\nu \eta(X)\eta(\xi_\nu)^2.$$

Substituting all of these formulas into (4.9), we have the following

$$\begin{aligned}\sum_i g((\nabla_{E_i}A)\phi E_i, X) &= \sum_i g(\phi E_i, (\nabla_X A)E_i) - (4m-1)\eta(X) \\ &+ 3\eta(X) - \sum_\nu \eta_\nu(X)\eta(\xi_\nu) + \sum_\nu \eta_\nu(X)\mathrm{Tr}\ \phi\phi_\nu - 6\eta(X) \\ &+ 2\sum_\nu \eta_\nu(\xi)\eta_\nu(X) - 3\eta(X) + \sum_\nu \eta(X)\eta(\xi_\nu)^2 \\ &+ \sum_\nu \eta_\nu(X)\eta_\nu(\xi) - \sum_\nu \eta(X)\eta_\nu(\xi)^2. \qquad (4.10)\end{aligned}$$

Now substituting (4.8) and (4.10) into (4.7), then by Lemma 1 and Lemma 3 we have

$$\begin{aligned}\sum_i (E_i k) g(\phi E_i, X) &= (4m+4)\eta(X) - 2\sum_\nu \eta_\nu^2(\xi)\eta(X) - \sum_\nu \eta_\nu(\xi)\mathrm{Tr}(\phi_\nu\phi)\eta(X) \\ &- \sum_i g(E_i, (\nabla_{\phi X}A)E_i) - \sum_\nu \eta(X)\eta_\nu^2(\xi) + \sum_\nu \eta_\nu(X)\eta_\nu(\xi)\end{aligned}$$

$$
\begin{aligned}
&-\sum_\nu \eta_\nu(X)\mathrm{Tr}\,\phi_\nu\phi+\eta(X)\sum_\nu \eta_\nu(\xi)\mathrm{Tr}\,\phi_\nu\phi+\sum_\nu \eta_\nu(X)\eta_\nu(\xi)\\
&-\eta(X)\sum_\nu \eta_\nu(\xi)^2+\sum_i g(\phi E_i,(\nabla_X A)E_i)\\
&-(4m-1)\eta(X)+3\eta(X)-\sum_\nu \eta_\nu(X)\eta(\xi_\nu)\\
&+\sum_\nu \eta_\nu(X)\mathrm{Tr}\,\phi\phi_\nu-6\eta(X)+2\sum_\nu \eta_\nu(\xi)\eta_\nu(X)\\
&-3\eta(X)+\sum_\nu \eta(X)\eta(\xi_\nu)^2+\sum_\nu \eta_\nu(X)\eta_\nu(\xi)-\sum_\nu \eta(X)\eta_\nu(\xi)^2\\
=&-\eta(X)-4\sum_\nu \eta_\nu^2(\xi)\eta(X)+4\sum_\nu \eta_\nu(X)\eta_\nu(\xi)-\mathrm{Tr}\,(\nabla_{\phi X}A),
\end{aligned}
$$

where we have used that the structure vector ξ is principal and $\mathrm{Tr}\,(\nabla_X A)\phi=0$. Then it can be written as follows:

$$
\phi X(k)=\phi X(\mathrm{Tr}\,A)+\eta(X)+4\sum_\nu \eta_\nu^2(\xi)\eta(X)-4\sum_\nu \eta_\nu(X)\eta_\nu(\xi).
$$

From this, replacing X by ϕX, we have

$$
\phi^2 X(k-\mathrm{Tr}\,A)=-4\sum_\nu \eta_\nu(\phi X)\eta_\nu(\xi).
$$

Finally, we have arrived at the following formula

$$
X(k-\mathrm{Tr}\,A)=\eta(X)\xi(k-\mathrm{Tr}\,A)+4\sum_\nu \eta_\nu(\phi X)\eta_\nu(\xi).
$$

From this we complete the proof of Lemma 5. □

By Lemma 1 we know that the mean curvature is constant if and only if the function α is constant. By the result in Lemma 5 we know that if the function $k-\mathrm{Tr}\,A$ is constant, then

$$
\sum_\nu \eta_\nu(\phi X)\eta_\nu(\xi)=0 \tag{4.11}
$$

for any $X\in T_xM$. Then the formula (4.11) is equal to

$$
\sum_\nu \eta_\nu(\xi)\phi\xi_\nu=0. \tag{4.12}
$$

On the other hand, the formula $\sum_\nu \eta_\nu(\xi)\phi^2\xi_\nu=0$ is equivalent to

$$
\sum_\nu \eta_\nu(\xi)\phi\xi_\nu=0,
$$

because $\sum_\nu \eta_\nu(\xi)\phi\xi_\nu$ is orthogonal to the structure vector field ξ. From this, (4.12) is equivalent to

$$
\eta(Y)\sum_\nu \eta_\nu^2(\xi)=\sum_\nu \eta_\nu(\xi)\eta_\nu(Y)=0
$$

for any $Y \in \mathfrak{D}$. By virtue of this formula (4.11) is also equivalent to

$$\xi \in \mathfrak{D} \quad \text{or} \quad \xi \in \mathfrak{D}^{\perp}. \tag{4.13}$$

Accordingly, by Lemma 5 we know that the constancy of the function α implies the formula (4.13). Moreover, conversely, by Lemma 3 we are able to see that (4.13) implies that the function α is constant. Now we summarize this content as follows:

Theorem 4.1. *Let M be a real hypersurface in $G_2(\mathbb{C}^{m+2})$ satisfying the formula*

$$A\phi + \phi A = k\phi,$$

where the function k is non-zero and constant. Then the following are equivalent to each other

(1) *the mean curvature is constant,*
(2) *the function α is constant,*
(3) $\xi \in \mathfrak{D}$ *or* $\xi \in \mathfrak{D}^{\perp}$.

By virtue of this theorem we also assert the following

Theorem 4.2. *Let M be a real hypersurface in $G_2(\mathbb{C}^{m+2})$ with constant mean curvature satisfying the formula*

$$A\phi + \phi A = k\phi,$$

where the function k is non-zero and constant. Then we have the following

(1) *The structure vector field ξ is principal,*
(2) *The function α is constant,*
(3) $\xi \in \mathfrak{D}$ *or* $\xi \in \mathfrak{D}^{\perp}$.

5. Proof of the Main Theorem

Let M be a real hypersurface in a two-plane complex Grassmannians $G_2(\mathbb{C}^{m+2})$ with *constant mean curvature*. Now let us denote by $\mathfrak{H}$ the orthogonal component of the structure vector ξ in the tangent space of M in $G_2(\mathbb{C}^{m+2})$. Then by Theorem 4.2 let us consider the following two cases:

Now we consider the first case $\xi \in \mathfrak{D}^{\perp}$. In this case we may put $\xi = \xi_1$. Then by Proposition 4 we have for any $X \in \mathfrak{H} = [\xi]^{\perp}$

$$2A^2X - 2kAX + (\alpha k + 2)X - 2\sum_{\nu}\{\eta_\nu(\phi X)\phi\xi_\nu - \eta_\nu(X)\xi_\nu + \eta_\nu(\xi)\phi_\nu\phi X\} = 0. \tag{5.1}$$

From this formula we are able to assert the following

Proposition 5.1. *Let M be a real hypersurface in $G_2(\mathbb{C}^{m+2})$ satisfying the formula $(^*)$ with constant mean curvature. Then the principal curvature α is constant and for all $X \in \mathfrak{H}$ with $AX = \lambda X$ one of the following two statements holds*:

(1) $2\lambda^2 - 2k\lambda + \alpha k = 0$ *and* $\phi\mathfrak{D}X = -\phi_1\mathfrak{D}X$,
(2) $2\lambda^2 - 2k\lambda + (\alpha k + 4) = 0$ *and* $\phi_1\mathfrak{D}X = \phi\mathfrak{D}X$.

Proof. In order to prove this Proposition we use the formulas in (2.1) to the formula (5.1). Then for any principal vector $X \in \mathfrak{H}$ such that $AX = \lambda X$ the equation (5.1) can be given by

$$\{2\lambda^2 - 2k\lambda + (\alpha k + 2)\}X + 4\{\eta_2(X)\xi_2 + \eta_3(X)\xi_3\} - 2\phi_1\phi X = 0. \quad (5.2)$$

Now we decompose the vector $X \in \mathfrak{H}$ as follows:

$$X = \mathfrak{D}X + \eta_2(X)\xi_2 + \eta_3(X)\xi_3,$$

where $\mathfrak{D}X$ denotes the $\mathfrak{D}$ component of the vector $X \in \mathfrak{H}$. Then by the formula (2.1) again we have

$$\phi_1\phi X = \phi_1\phi\mathfrak{D}X + \eta_2(X)\xi_2 + \eta_3(X)\xi_3.$$

From this, together with (5.2) it follows that

$$\begin{aligned}\{2\lambda^2 - 2k\lambda + \alpha k + 2\}\mathfrak{D}X + \{2\lambda^2 - 2k\lambda + \alpha k + 4\}\eta_2(X)\xi_2 \\ + \{2\lambda^2 - 2k\lambda + \alpha k + 4\}\eta_3(X)\xi_2 - 2\phi_1\phi\mathfrak{D}X = 0.\end{aligned}$$

From this, together with the fact that $\phi_1\phi\mathfrak{D} \subset \mathfrak{D}$ we have the following

$$\begin{aligned}\{2\lambda^2 - 2k\lambda + (\alpha k + 2)\}\mathfrak{D}X - 2\phi_1\phi\mathfrak{D}X = 0, \\ \{2\lambda^2 - 2k\lambda + (\alpha k + 4)\}\eta_2(X)\xi_2 = 0, \\ \{2\lambda^2 - 2k\lambda + (\alpha k + 4)\}\eta_3(X)\xi_3 = 0.\end{aligned}$$

If $2\lambda^2 - 2k\lambda + (\alpha k + 4) = 0$, from the first equation we know $\mathfrak{D}X = -\phi_1\phi\mathfrak{D}X$, that is, $\phi_1\mathfrak{D}X = \phi\mathfrak{D}X$. Thus we assert the formula (2) in our Proposition.

Now when we consider $2\lambda^2 - 2k\lambda + (\alpha k + 4) \neq 0$, then $\eta_2(X) = \eta_3(X) = 0$. This means $X \in \mathfrak{D}$. Then by the first equation we know that $\phi_1\phi\mathfrak{D}X$ and $\mathfrak{D}X$ are proportional. From this we have

$$\phi_1\phi\mathfrak{D}X = \pm\mathfrak{D}X.$$

If $\phi_1\phi\mathfrak{D}X = -\mathfrak{D}X$, then $(2\lambda^2 - 2k\lambda + \alpha k + 4)\mathfrak{D}X = 0$, which makes a contradiction. So $\phi_1\phi\mathfrak{D}X = \mathfrak{D}X$, that is $\phi\mathfrak{D}X = -\phi_1\mathfrak{D}X$. Then we have our assertion (1). From this we complete the proof of our Proposition. □

In this section we have assumed that the mean curvature of M in $G_2(\mathbb{C}^{m+2})$ is constant. Then by Theorem 4.2 we know that the function α is constant. Accordingly, all principal curvatures satisfying the formulas (1) and (2) in Proposition 5.1 are constant. Also by virtue of these two formulas the number of principal curvatures in the subspace $\mathfrak{H}$ is at most four. Since the function k is given by 2ρ as in the introduction, the formulas in Proposition 5.1 can be written by

$$\lambda^2 - k\lambda + \rho\alpha = 0 \quad (5.3)$$

and

$$\lambda^2 - k\lambda + \rho\alpha + 2 = 0. \quad (5.4)$$

In (5.3) the function $k = 2\rho$ is given by the sum of two roots of the quadratic equation. Then it follows that two roots are equal to each other, that is $\rho = \alpha$. By virtue of this fact we also know that there cannot exist any roots satisfying the formula (5.4). So we are able to assert that $\mathfrak{H} = T_\alpha$, where $T_\alpha = \{X \in \mathfrak{H} \,|\, AX = \alpha X\}$.

Since we know that the structure vector ξ is principal with principal curvature α, we assert that M is locally congruent to a totally umbilic hypersurface in $G_2(\mathbb{C}^{m+2})$. But in a paper [10] due to the present author it is proved that there does not exist such a real hypersurface in $G_2(\mathbb{C}^{m+2})$. So we conclude here that the first case $\xi \in \mathfrak{D}^\perp$ cannot appear.

Now let us consider the second case $\xi \in \mathfrak{D}$. Then by Lemma 2 for any $X \in \mathfrak{H}$ and $A\xi = \alpha\xi$, we have

$$0 = (2\lambda - \alpha)A\phi X - (2 + \lambda\alpha)\phi X - 2\sum_\nu \{\eta_\nu(X)\phi_\nu\xi + \eta_\nu(\phi X)\xi_\nu\}, \tag{5.5}$$

where $\mathfrak{H} = [\xi_1, \xi_2, \xi_3, \phi\xi_1, \phi\xi_2, \phi\xi_3] \oplus \mathscr{G}$ and $\mathscr{G}$ is the orthogonal complement of the subspace $[\xi_1, \ldots, \ldots, \phi\xi_3]$ in $\mathfrak{H}$. Then any vector $X \in \mathfrak{H}$ can be expressed by

$$X = \mathscr{G}X + \sum_\nu \eta_\nu(X)\xi_\nu - \sum_\nu \eta_\nu(\phi X)\phi\xi_\nu,$$

where $\mathscr{G}X$ denotes the $\mathscr{G}$-component of the vector $X \in \mathfrak{H}$. If $AX = \lambda X$, then by the assumption $(A\phi + \phi A)X = k\phi X$ we know that $A\phi X = (k - \lambda)\phi X$. From this, together with (5.5) we have

$$0 = \{(2\lambda - \alpha)(k - \lambda) - (2 + \lambda\alpha)\}\phi X - 2\sum_\nu \{\eta_\nu(X)\phi_\nu\xi + \eta_\nu(\phi X)\xi_\nu\}. \tag{5.6}$$

From this, multiplying ϕ and using $\phi_\nu\xi = \phi\xi_\nu$, $\nu = 1, 2, 3$, we have

$$\begin{aligned}\{2\lambda^2 - 2k\lambda + (\alpha k + 2)\}\mathscr{G}X + \{2\lambda^2 - 2k\lambda + (\alpha k + 2) + 2\}\sum_\nu \eta_\nu(X)\xi_\nu \\ - \{2\lambda^2 - 2k\lambda + (\alpha k + 2) + 2\}\sum_\nu \eta_\nu(\phi X)\phi\xi_\nu = 0,\end{aligned}$$

where we have used the above decomposition for the expression of $X \in \mathfrak{H}$. Accordingly, we are able to assert the following:

$$\begin{aligned}\{2\lambda^2 - 2k\lambda + (\alpha k + 2)\}\mathscr{G}X = 0, \\ \{2\lambda^2 - 2k\lambda + (\alpha k + 4)\}\eta_\nu(X)\xi_\nu = 0, \quad \nu = 1, 2, 3 \\ \{2\lambda^2 - 2k\lambda + (\alpha k + 4)\}\eta_\nu(\phi X)\phi\xi_\nu = 0, \quad \nu = 1, 2, 3.\end{aligned} \tag{5.7}$$

From these equations we know that if $2\lambda^2 - 2k\lambda + (\alpha k + 4) \neq 0$, then the vector X is orthogonal to ξ_ν and $\phi\xi_\nu$ for any $\nu = 1, 2, 3$. Then naturally $X = \mathscr{G}X$. From this fact we know that all of principal curvatures corresponding to eigenspaces in the space $\mathfrak{H}$ satisfy one of the following equations:

$$2\lambda^2 - 2k\lambda + (\alpha k + 2) = 0 \quad \text{or} \quad 2\lambda^2 - 2k\lambda + (\alpha k + 4) = 0.$$

On the other hand, by Theorem 4.2 the functions α and k are known to be constant. From this together with the above equation all of the principal curvatures are constant.

Now without loss of generality we may put $\alpha = -2\tan 2r$ and $\lambda = \cot r$ for a real number r with $0 < r < \frac{\pi}{4}$. Then by (5.6) (also by Lemma 2), we know for any $X \in \mathscr{G}$ with$AX = \lambda X$ that

$$A\phi X = \mu\phi X, \quad \mu = \frac{\alpha\lambda + 2}{2\lambda - \alpha}.$$

Then the function $\mu = -\tan r$. So it follows that

$$k = \lambda + \mu = \cot r - \tan r = 2\cot 2r,$$

which implies $\alpha k = -4$. Then its principal curvatures in $\mathfrak{H}$ satisfy

$$\lambda^2 - k\lambda - 1 = 0 \quad \text{or} \quad \lambda^2 - k\lambda = 0.$$

Then, including principal curvature α the real hypersurface has at most five distinct constant principal curvatures. Then by the above formulas and the quadratic equations the other possible principal curvatures are

$$\beta = 2\cot 2r, \qquad \gamma = 0, \qquad \lambda = \cot r, \qquad \mu = -\tan r.$$

Note that the principal curvature λ and μ are two different roots of the equation

$$2x^2 - 2kx + (\alpha k + 2) = 0,$$

where $k = 2\cot 2r$.

A basic role in the geometry of Riemannian symmetric space is played by the so-called maximal flats. In the case of $G_2(\mathbb{C}^{m+2})$, a maximal flat is a two-dimensional totally geodesic submanifold isometric to some flat two-dimensional torus. A non-zero tangent vector X of $G_2(\mathbb{C}^{m+2})$ is said to be *singular* if X is tangent to more than one maximal flat of $G_2(\mathbb{C}^{m+2})$. In $G_2(\mathbb{C}^{m+2})$ there are two types of singular tangent vectors X which are characterized by the properties $JX \perp \mathfrak{J}X$ and $JX \in \mathfrak{J}X$. We will have to compute explicitly Jacobi vector fields along geodesics whose tangent vectors are all singular. For this we need the eigenvalues and eigenspaces of the Jacobi operator $\bar{R}_X := \bar{R}(.,X)X$, where $\bar{R}$ denotes the curvature tensor of $G_2(\mathbb{C}^{m+2})$ mentioned in Section 1. If $JN \perp \mathfrak{J}N$ then the eigenvalues and eigenspaces of $\bar{R}_N$ are given by (see Berndt and Suh [4])

$$\begin{array}{ll} 0 & \mathbb{R}N \oplus \mathfrak{J}JN = N \oplus [\phi_1\xi, \phi_2\xi, \phi_3\xi], \\ 1 & (\mathbb{H}\mathbb{C}N)^{\perp} = [N, \xi, \xi_1, \xi_2, \xi_3, \phi_1\xi, \phi_2\xi, \phi_3\xi]^{\perp}, \\ 4 & \mathbb{R}JN \oplus \mathfrak{J}N = \mathbb{R}\xi \oplus [\xi_1, \xi_2, \xi_3], \end{array} \tag{5.8}$$

where $\mathbb{H}\mathbb{C}N = \mathbb{R}N \oplus \mathbb{R}JN \oplus \mathfrak{J}N \oplus \mathfrak{J}JN$ and $[\ldots,\ldots,\ldots]$ denotes the linear real span of the given vectors.

For $p \in M$ denote by c_p the geodesic in $G_2(\mathbb{C}^{m+2})$ with $c_p(0) = p$ and $\dot{c}_p(0) = N_p$, and by F the smooth map

$$F : M \to G_2(\mathbb{C}^{m+2}), \qquad p \mapsto c_p(r).$$

Geometrically, F is the displacement of M at distance r in direction of the normal field N. For each $p \in M$ the differential d_pF of F at p can be computed by means of Jacobi vector fields by

$$d_pF(X) = Z_X(r).$$

Here, Z is the Jacobi vector field along c_p with initial values $Z_X(0) = X$ and $Z_X'(0) = -AX$. Using the explicit descriptions (5.8) of the Jacobi operator $\bar{R}_N$ for the case $JN \perp \mathfrak{J}N$ mentioned above and of the shape operator A of M we get

$$Z_X(r) = \begin{cases} \{\cos(2r) - \frac{\rho}{2}\sin(2r)\}E_X(r) & \text{if } \rho \in \{\alpha, \beta\} \\ \{\cos(r) - \rho\sin(r)\}E_X(r) & \text{if } \rho \in \{\lambda, \mu\} \\ \{1 - \rho\}E_X(r) & \text{if } \rho \in \{\gamma\} \end{cases}$$

where E_X denotes the parallel vector field along c_p with $E_X(0) = X$. This shows that the kernel of dF is $T_\beta \oplus T_\lambda = \mathfrak{J}N \oplus T_\lambda$ and that F is of constant rank $\dim(T_\alpha \oplus T_\gamma \oplus T_\mu) = 4n$. So, locally, F is a submersion onto a $4n$-dimensional submanifold B of $G_2(\mathbb{C}^{m+2})$. Moreover, the tangent space of B at $F(p)$ is obtained by parallel translation of $(T_\alpha \oplus T_\gamma \oplus T_\mu)(p) = (\mathbb{H}\xi \oplus T_\mu)(p)$, which is a quaternionic and real subspace of $T_pG_2(\mathbb{C}^{m+2})$.

Since both J and $\mathfrak{J}$ are parallel along c_p, also $T_{F(p)}B$ is a quaternionic and real subspace of $T_{F(p)}G_2(\mathbb{C}^{m+2})$. Thus B is a quaternionic and real submanifold of $G_2(\mathbb{C}^{m+2})$. Since B is quaternionic, it is totally geodesic in $G_2(\mathbb{C}^{m+2})$ (see Alekseevski [1]). The only quaternionic totally geodesic submanifolds of $G_2(\mathbb{C}^{m+2})$, $m = 2n \geqslant 4$, of half dimension are $G_2(\mathbb{C}^{n+2})$ and $\mathbb{H}P^n$ (see Berndt [3]). But only $\mathbb{H}P^n$ is embedded in $G_2(\mathbb{C}^{m+2})$ as a real submanifold. So we conclude that B is an open part of a totally geodesic $\mathbb{H}P^n$ in $G_2(\mathbb{C}^{m+2})$. Rigidity of totally geodesic submanifolds finally implies that M is an open part of the tube with radius r around a totally geodesic $\mathbb{H}P^n$ in $G_2(\mathbb{C}^{m+2})$. Thus we have proved our main Theorem. □

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Author's address: Y. J. Suh, Department of Mathematics, Kyungpook National University, Taegu 702-701, Korea, e-mail: yjsuh@mail.knu.ac.kr