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On the Greatest Common Divisor of $u-1$ and $v-1$ with u and v Near $\mathscr{S}$-units

By

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Abstract. In [2], it was shown that if a and b are multiplicatively independent integers and $\varepsilon > 0$, then the inequality $\gcd(a^n - 1, b^n - 1) < \exp(\varepsilon n)$ holds for all but finitely many positive integers n. Here, we generalize the above result. In particular, we show that if $f(x), f_1(x), g(x), g_1(x)$ are non-zero polynomials with integer coefficients, then for every $\varepsilon > 0$, the inequality

$$\gcd\left(f(n)a^n + g(n),\ f_1(n)b^n + g_1(n)\right) < \exp(n\varepsilon)$$

holds for all but finitely many positive integers n.

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1. Introduction

In [2], it was shown that if a and b are multiplicatively independent positive integers, then for every fixed $\varepsilon > 0$ there exists a positive integer n_ε, depending on ε, a and b, such that the inequality $\gcd(a^n - 1, b^n - 1) < \exp(\varepsilon n)$ holds for $n > n_\varepsilon$. The method of [2] was extended to the instance when the integers a and b are replaced by $\mathscr{S}$-units. Namely, let $\mathscr{P} = \{p_1, \ldots, p_t\}$ be a finite set of prime numbers, and write $\mathscr{S} = \{\pm p_1^{\alpha_1} \cdots p_t^{\alpha_t} | \alpha_i \geqslant 0\}$ for the set of all integers whose prime factors belong to $\mathscr{P}$. The members of $\mathscr{S}$ are usually referred to as *$\mathscr{S}$-units*. It was then shown in Remark 1 in [4] and in [7], that for every fixed $\varepsilon > 0$ the inequality $\gcd(u-1, v-1) < (\max\{|u|, |v|\})^\varepsilon$ holds for all pairs of multiplicatively independent integers u and v from $\mathscr{S}$ with finitely many exceptions. This result was in turn sufficient to confirm a conjecture of Győry, Sárközy and Stewart [6], which asserted that the largest prime factor of expressions of the form $(ab+1)(ac+1)(bc+1)$, with distinct positive integers a, b and c, tends to infinity with $\max\{a, b, c\}$ (in fact, in [4], it was noted that this is true even for the expressions $(ab+1)(ac+1)$, with $a > b > c$). Some quantitative aspects of the above inequality appear in [3].

The results from [2] and [7] were extended to the setting of algebraic numbers in [5] and were used to give a non-trivial lower bound for the height of $h(u, v)$, where $h \in \mathbb{Q}[x, y]$ is non-constant, and u and v are $\mathscr{S}$-units.

In this paper, we extend the results from [2] but in a different direction. In particular, we prove that the same inequality as the one in [2] holds for the greatest common divisor of expressions of the type $f(n)a^n + g(n)$ and $f_1(n)b^n + g_1(n)$, where a and b are multiplicatively independent non-zero integers and f, f_1, g and g_1 are non-zero polynomials with integer coefficients. In order to prove such an inequality, we extend the results from [2] to study $\gcd(u-1, v-1)$, where u and v are *near $\mathscr{S}$-units*, in a sense which will be made more precise in the next section.

The rest of this paper is organized as follows. In Section 2, we present the main result of the paper, which is Theorem 2.1. In Section 3, we give a few applications of our main result. Section 4 is devoted to the proof of Theorem 2.1. We limit our analysis to the case of rational numbers.

2. The Main Result

Throughout this paper, u and v are non-zero rational numbers. We write $\mathscr{P} = \{p_1, \ldots, p_t\}$ for a fixed finite set of prime numbers, and we let $\mathscr{S} = \{\pm p_1^{\alpha_1} \cdots p_t^{\alpha_t} | \alpha_i \geqslant 0\}$ be the set of all $\mathscr{S}$-units; i.e., those integers whose prime factors are in $\mathscr{P}$. Every non-zero rational number u can be written uniquely in the form $u = u_{\mathscr{S}} \cdot u_{\overline{\mathscr{S}}}$, where $u_{\mathscr{S}}$ is a rational number in reduced form having both its numerator and denominator in $\mathscr{S}$, and $u_{\overline{\mathscr{S}}}$ is a rational number in reduced form having both its numerator and denominator free of primes from $\mathscr{S}$. For a non-zero rational number u we use $H(u)$ for its logarithmic height; i.e., if $u = x/y$ with x and y coprime integers, then $H(u) = \max\{1, \log|x|, \log|y|\}$. Here, log stands for the natural logarithm. We also use the notation $HR(u)$ for the number $\max\{\frac{H(x)}{H(y)}, \frac{H(y)}{H(x)}\}$. Notice that $HR(u) > 1$ unless $u = \pm 1$. Finally, we use $H_{\mathscr{S}}(u)$ and $H_{\overline{\mathscr{S}}}(u)$ for $H(u_{\mathscr{S}})$ and $H(u_{\overline{\mathscr{S}}})$, respectively. For two rational numbers u and v, we write $\gcd(u, v)$ for the greatest common divisor of the numerators of u and v. We use the Vinogradov symbols $\ll$, $\gg$ and $\asymp$, as well as the Landau symbols O and o with their usual meaning.

The main result of this paper is the following.

Theorem 2.1. *Let $\varepsilon \in (0, 1)$ be fixed. Let $\mathscr{L}_\varepsilon$ be the set of all pairs of rational numbers u and v such that*

$$\gcd(u-1, v-1) \geqslant \exp(\varepsilon \max\{H(u), H(v)\}).$$

There exist three positive constants K_1, K_2 and K_3 (which are ineffective) depending on $\mathscr{P}$ and ε, such that whenever $(u, v) \in \mathscr{L}_\varepsilon$, then with $H = \max\{H(u), H(v)\}$ one of the following three conditions holds:

1) $\max\{HR(u), HR(v)\} < K_1$;

2) *There exist two integers i and j, not both zero and with* $\max\{|i|, |j|\} < K_2$, *such that* $u^i = v^j$;

3) $\max\{H_{\overline{\mathscr{S}}}(u), H_{\overline{\mathscr{S}}}(v)\} > K_3 \frac{H}{\log H}$.

Remark 2.2. The above Theorem shows that the inequality $\gcd(u-1, v-1) < \max\{u, v\}^\varepsilon$ holds for all sufficiently large multiplicatively independent positive integers u and v such that $\max\{u_{\overline{S}}, v_{\overline{S}}\}$ is small with respect to $\max\{u, v\}$, which explains the title of this paper.

3. Some Applications

Before proving the above, somewhat technical statement of Theorem 2.1, we look at a few applications of it.

Corollary 3.1. *For every fixed* $\varepsilon > 0$ *there exists a positive constant* C_ε (*which is ineffective*) *depending only on* ε *and* $\mathscr{P}$, *such that if* u *and* v *are multiplicatively independent integers in* $\mathscr{S}$ *and* $\max\{|u|,|v|\} > C_\varepsilon$, *then*

$$\gcd(u-1, v-1) < (\max\{|u|,|v|\})^{\varepsilon}.$$

The above Corollary 3.1 follows immediately from Theorem 2.1.

Remark 3.2. The above Corollary 3.1 is the main result in [7], and appears also as Remark 1 in [4].

Corollary 3.3. *Let* a *and* b *be non-zero integers which are multiplicatively independent, and let* f, g, f_1 *and* g_1 *be non-zero polynomials with integer coefficients. For every positive integer* n *set*

$$u_n = f(n)a^n + g(n),$$

and

$$v_n = f_1(n)b^n + g_1(n).$$

Then, for every fixed $\varepsilon > 0$ *there exists a positive constant* $C_\varepsilon > 0$ *depending on* ε *and on the given data* a, b, f, f_1, g *and* g_1 (*which is ineffective*), *such that*

$$\gcd(u_n, v_m) < \exp(\varepsilon \max\{m,n\}) \tag{1}$$

holds for all pairs of positive integers (m,n) *with* $\max\{m,n\} > C_\varepsilon$.

Remark 3.4. Note first that by taking $f = g = 1$, $f_1 = g_1 = -1$, and $n = m$ in the above Corollary 3.3, we get a statement which is the main result in [2].

Proof. Assume that $\varepsilon > 0$ is fixed, and that m and n are positive integers such that inequality (1) does not hold. We set $\mathscr{P}$ to be the set of all prime divisors of ab. We obviously have that both inequalities

$$c_1 n < H(u_n) < c_2 n \quad \text{and} \quad c_3 m < H(v_m) < c_4 m$$

hold for all m and n sufficiently large, with c_1, c_2, c_3 and c_4 some computable constants depending only on the given data a, b, f, f_1, g and g_1, but not on ε. Since inequality (1) fails, we get that the inequality

$$c_5 < \frac{m}{n} < c_6$$

must hold when $\max\{m,n\}$ is sufficiently large, with c_5 and c_6 some computable constants depending on ε; in fact, one can take c_5 to be of the form $O(\varepsilon)$ and c_6 to be of the form $O(\varepsilon^{-1})$, where the constants implied by O above depend only on the data. The above inequality tells us that $\max\{m,n\} \asymp \min\{m,n\}$.

We now set $d_n = \gcd(f(n)a^n, g(n))$, $d_m = \gcd(f_1(m)b^m, g_1(m))$, $u = -\frac{f(n)a^n}{g(n)}$, and $v = -\frac{f_1(m)b^m}{g_1(m)}$, and notice that

$$\gcd(u_n, v_m) \leqslant d_n d_m \gcd(u-1, v-1).$$

Set also $H = \max\{H(u), H(v)\}$. Since obviously

$$\max\{H(d_n), H(d_m)\} \ll \max\{\log m, \log n\} \ll \log m,$$

while $H \gg \max\{m, n\} \gg m$, it follows that the failure of inequality (1) for large values of $\max\{m, n\}$ implies that the inequality

$$\gcd(u-1, v-1) > \exp(\varepsilon_1 H)$$

must hold, say with $\varepsilon_1 = \varepsilon/2$. So, the pair (u, v) belongs to $\mathscr{L}_{\varepsilon_1}$, and therefore, by Theorem 2.1, we conclude that one of the conditions 1, 2, or 3 must hold. However, notice that since both u and v are rational numbers of the form x_u/y_u and x_v/y_v, respectively, with $\min\{\log|x_u|, \log|x_v|\} \gg m$ and with $\max\{\log|y_u|, \log|y_v|\} \ll \log m$, it follows that

$$\min\{HR(u), HR(v)\} \gg \frac{m}{\log m},$$

and therefore condition 1 cannot hold for large values of m. Since $u_{\overline{\mathscr{S}}}$ and $v_{\overline{\mathscr{S}}}$ are rational numbers whose denominators and numerators divide $f(n)$ and $g(n)$, or $f_1(m)$ and $g_1(m)$, respectively, it follows that

$$\max\{H_{\overline{\mathscr{S}}}(u), H_{\overline{\mathscr{S}}}(v)\} \ll \log m = o\left(\frac{m}{\log m}\right) = o\left(\frac{H}{\log H}\right),$$

as m tends to infinity, therefore condition 3 cannot hold for large values of m either. It remains to check condition 2. The relation $u^i = v^j$, with fixed i and j such that at least one of them is non-zero and with large values of m and n, forces first of all that i and j have the same sign (in particular, none of them is zero), so we may assume that they are both positive. Secondly, the same relation above implies

$$\frac{a^{ni}}{b^{mj}} = h(m, n), \tag{2}$$

where $h(m,n) = \pm\frac{g(n)^i f_1(m)^j}{g_1(m)^j f(n)^i}$ is some rational function in m and n. Here, we assume that m and n are larger than any of the real roots of the polynomial $(ff_1gg_1)(x)$. For every prime number $p \in \mathscr{P}$ we set α_p and β_p to be the exponents at which p appears in the prime factorization of a and b, respectively. Relation (2), size arguments, and the fact that $h(m, n) \neq 0$, imply that the estimate

$$|\alpha_p in - \beta_p jm| \ll \log m$$

must hold for all $p \in \mathscr{P}$, therefore the estimate

$$\left|\alpha_p \frac{n}{m} - \frac{i}{j}\beta_p\right| \ll \frac{\log m}{m} \tag{3}$$

must hold for all $p \in \mathscr{P}$. Since $ij \neq 0$ and $n/m \gg 1$, we get that either $\alpha_p = \beta_p = 0$, or none of them is zero. Clearly, the relation $\alpha_p = \beta_p = 0$ is impossible (recall that

$\mathscr{P}$ was precisely the set of *all* prime divisors of ab), therefore none of the numbers α_p and β_p is zero. The above relation (3) now implies that the estimate

$$\left| \frac{n}{m} - \frac{i\alpha_p}{j\beta_p} \right| \ll \frac{\log m}{m} \tag{4}$$

holds for all $p \in \mathscr{P}$. In particular, since i, j, α_p and β_p are fixed and the right hand side of (4) tends to zero when m tends to infinity, we get that if m is large then the above relation (4) cannot hold for all primes $p \in \mathscr{P}$ unless the rational number β_p/α_p is independent of the prime number p. Writing $\beta_p/\alpha_p = \kappa/\ell$ for all $p \in \mathscr{P}$, we get $a^\kappa = \pm b^\ell$, therefore $a^{2\kappa} = b^{2\ell}$, which contradicts the fact that a and b are multiplicatively independent and completes the proof of Corollary 3.3. □

Finally, we look again at the problem of Győry, Sárközy and Stewart concerning the largest prime factor of an expression of the form $(ab+1)(ac+1)(bc+1)$, with a, b and c distinct positive integers. For any non-zero integer k let $P(k)$ be the largest prime factor of k with the convention that $P(\pm 1) = 1$. As we mentioned in the Introduction, it was conjectured in [6] that $P((ab+1)(ac+1)(bc+1))$ tends to infinity when $a > b > c > 0$ and a tends to infinity. The fact that this is indeed so has been confirmed in both [4] and [7]. In fact, in [4], it was shown that even $P((ab+1)(ac+1))$ tends to infinity when $a > b > c > 0$ and a tends to infinity. In particular, for every fixed finite set of prime numbers $\mathscr{P}$ there exists an ineffective constant $C_{\mathscr{P}}$ depending on $\mathscr{P}$, such that $(ab+1)(ac+1)$ is not in $\mathscr{S}$ whenever $a > b > c > 0$ are integers and $a > C_{\mathscr{P}}$.

Here, we give a somewhat more effective version of this statement. Let $\mathscr{P}$ be again any fixed finite set of prime numbers. The result is the following:

Corollary 3.5. *For every fixed finite set of prime numbers $\mathscr{P}$ there exist two ineffective constants $C_{\mathscr{P}}$ and $C'_{\mathscr{P}}$ such that the inequality*

$$((ab+1)(ac+1))_{\overline{\mathscr{S}}} > \exp\left(C_{\mathscr{P}} \frac{\log a}{\log\log a} \right)$$

holds for all positive integers $a > b > c > 0$ with $a > C'_{\mathscr{P}}$.

Remark 3.6. The above Corollary 3.5 is still ineffective but it is a stronger statement than the fact that $P((ab+1)(ac+1))$ tends to infinity when $a > b > c > 0$ and a tends to infinity. Indeed, the conjecture from [6] is merely the statement that for every fixed finite set of prime numbers $\mathscr{P}$ the inequality $((ab+1)(ac+1)(bc+1))_{\overline{\mathscr{S}}} > 1$ holds whenever $a > b > c > 0$ and a is large enough, while Corollary 3.5 above provides a specific lower bound for the largest divisor of $(ab+1)(ac+1)$ which is free of prime factors from $\mathscr{P}$, and which holds uniformly for all triples of distinct positive integers $a > b > c$ with a sufficiently large.

Proof. Let $a > b > c > 0$ be arbitrary and set $u = ab+1$, $v = ac+1$. Notice that $a | \gcd(u-1, v-1)$, and that $a > u^{1/2} = (\max\{u, v\})^{1/2}$. With Theorem 2.1 for $\varepsilon = 1/2$, we get that there must exist three ineffective constants K_1, K_2 and K_3 depending only on $\mathscr{P}$ such that one of the conditions 1, 2, or 3 holds. Notice that condition 1 simply tells us that $a < K_4 = \exp(K_1)$. We assume that $K_1 > e$, therefore $K_4 > e^e$. To see that condition 2 is impossible, assume that $u^i = v^j$ holds with some positive integers i and j. Clearly, $j > i$, and, in particular, $j \geqslant 2$. We may assume that i and j are coprime, and we deduce the existence of a positive integer ρ such that $u = \rho^j$, $v = \rho^i$. But in this case,

$$\gcd(u-1, v-1) = \gcd(\rho^i - 1, \rho^j - 1) = \rho - 1 < \rho \leqslant \rho^{j/2} = u^{1/2}.$$

Thus, $a < u^{1/2} = (ab+1)^{1/2}$, which is impossible because $a > b$. And so, we conclude that if $a > K_4$, then condition 3 must hold and, in particular, we must have

$$\begin{aligned}\log\left((ab+1)(ac+1)\right)_{\overline{\mathscr{S}}} &= H_{\overline{\mathscr{S}}}(u) + H_{\overline{\mathscr{S}}}(v)\\ &> \max\{H_{\overline{\mathscr{S}}}(u), H_{\overline{\mathscr{S}}}(v)\} > K_3 \frac{H}{\log H}\\ &= K_3 \frac{\log(ab+1)}{\log\log(ab+1)} > K_3 \frac{\log a}{\log\log a}.\end{aligned}$$

The last inequality follows because the function $x \longmapsto \log x / \log\log x$ is increasing for $x > e^e$. We conclude that Corollary 3.5 holds with the choices $C_{\mathscr{P}} = K_3$ and $C'_{\mathscr{P}} = K_4$. □

4. The Proof of Theorem 2.1

We may as well assume that $0 < \varepsilon < 1$ and that ε is below any bound of our choice. We put $K_1 = A\varepsilon^{-3}$, with $A \geqslant 4$, and $K_3 = B\varepsilon^5$, with $B \leqslant 1/4$. The constants A and B will be specified more precisely later; they depend only on ε and $\mathscr{P}$.

Throughout this proof, we shall use the usual symbols $\ll$, $\gg$, O and o with the meaning that they are either absolute or depend only on $\mathscr{P}$ but not on ε, and $\ll_\varepsilon$, $\gg_\varepsilon$, O_ε and o_ε with the meaning that the inequalities implied by them depend on both ε and $\mathscr{P}$. We use $c_1, c_2, \ldots$ for positive constants which depend on both ε and $\mathscr{P}$.

For simplicity, we assume that both u and v are positive (if not, we may replace u, v, ε by $u^2, v^2, \varepsilon/2$, respectively, in light of the fact that $\gcd(u-1, v-1) | \gcd(u^2-1, v^2-1)$). Replacing one or both of u, v by u^{-1} and v^{-1}, if needed, we may assume that $u > 1$ and $v > 1$.

We assume that a pair (u, v) in $\mathscr{L}_\varepsilon$ has H large enough (depending on ε), and that it does not fulfill either condition 1 or condition 3 from the statement of Theorem 2.1. Up to interchanging u with v, we may assume that $H = H(u)$. We write

$$u = \frac{x_u}{y_u}, \quad x_u = p_u r_u, \quad y_u = q_u s_u, \quad \gcd(x_u, y_u) = 1, \quad u_{\mathscr{S}} = \frac{r_u}{s_u}, \quad u_{\overline{\mathscr{S}}} = \frac{p_u}{q_u},$$

and

$$v = \frac{x_v}{y_v}, \quad x_v = p_v r_v, \quad y_v = q_v s_v, \quad \gcd(x_v, y_v) = 1, \quad v_{\mathscr{S}} = \frac{r_v}{s_v}, \quad v_{\overline{\mathscr{S}}} = \frac{p_v}{q_v}.$$

We clearly have $H(u)=\log x_u$, $H(v)=\log x_v$. But we also know that $\log y_u \leqslant (\log x_u)/4$ and $\log y_v \leqslant (\log x_v)/4$, and therefore both estimates

$$\log(x_u-y_u)=\log x_u+\log\left(1-\frac{y_u}{x_u}\right)=\left(1+O\left(\frac{1}{x_u^{3/4}\log x_u}\right)\right)\log x_u,$$

and

$$\log(x_v-y_v)=\log x_v+\log\left(1-\frac{y_v}{x_v}\right)=\left(1+O\left(\frac{1}{x_v^{3/4}\log x_v}\right)\right)\log x_v$$

hold. We may assume that $H(v)$ is large, since if $H(v)$ remains bounded, then v remains bounded, while u becomes large as H gets large. In particular, with the fact that condition 1 fails, the numerator of $u-1$ increases while the numerator of $v-1$ stays bounded, and since $(u,v)\in\mathscr{L}_\varepsilon$, it follows that u can take only finitely many values. Hence, for large values of H we have that $H(u-1)=H(x_u-y_u)=(1-o_\varepsilon(1))H$ and $H(v-1)=H(x_v-y_v)=(1-o_\varepsilon(1))H(v)$, where the dependence in o_ε above is only on H for fixed ε. Since $(u,v)\in\mathscr{L}_\varepsilon$, we get

$$H(v)\geqslant c_1H \tag{5}$$

holds for large H, where one can take $c_1=\varepsilon/2$. Clearly,

$$\max\{\log y_u,\ \log y_v\}\leqslant K_1^{-1}H\leqslant\frac{\varepsilon^3H}{4}, \tag{6}$$

which shows that q_u, q_v, s_u, s_v are small with respect to x_u. We now use the fact that condition 3 does not hold with $K_3=B\varepsilon^5<\varepsilon^5$ to conclude that

$$\max\{\log(q_up_u),\ \log(q_vp_v)\}<\varepsilon^5\frac{H}{\log H}. \tag{7}$$

Since $H=\log(p_ur_u)$, it follows, by (5) and (7), that

$$\frac{\log r_u}{\log r_v}\in[c_2,c_3] \tag{8}$$

holds for sufficiently large H, where we can take $c_2=1/2$ and $c_3=4\varepsilon^{-1}$. It is also plain that

$$\min\{\log r_u,\ \log r_v\}\gg\varepsilon H, \tag{9}$$

and the constant implied by $\gg$ above can be taken to be $1/4$ if H is sufficiently large, where the above estimate follows easily from (5) and (7).

We now have everything we need to apply the machinery from [2].

We fix an integer h and start with the approximation

$$\frac{1}{v-1}=\frac{1}{v(1-v^{-1})}=\frac{1}{v}\sum_{i\geqslant 0}\frac{1}{v^i}=\sum_{i=1}^{h}\frac{1}{v^i}+O\left(\frac{1}{v^{h+1}}\right), \tag{10}$$

where the constant in O above can be taken to be absolute once H is large enough (since $K_1>2$). We shall take it to be 2, and then (10) holds whenever $v>2$.

Multiplying the above approximation by $u^j - 1$, we get

$$\left| \frac{u^j - 1}{v - 1} - \sum_{i=1}^{h} \frac{u^j}{v^i} + \sum_{i=1}^{h} \frac{1}{v^i} \right| = O\left(\frac{u^j}{v^{h+1}} \right), \tag{11}$$

where the constant implied in O above can again be taken to be absolute, say equal to 4, for example, once $K_1 > 2$, and H is large enough.

We now set

$$z_j(u,v) = \frac{u^j - 1}{v - 1} = \frac{x_u^j - y_u^j}{x_v - y_v} \cdot \frac{y_v}{y_u^j} = \frac{c_j(u,v)}{d} \cdot \frac{y_v}{y_u^j} \quad (j = 1, \ldots, k). \tag{12}$$

Here, we write $(x_u - y_u)/(x_v - y_v) = e/d$, where e and d are integers with $\gcd(e,d) = 1$ and $c_j(u,v) = e(x_u^j - y_u^j)/(x_v - y_v)$. Thus, $c_j(u,v)$ is an integer, and d is independent of j. We recall that we are assuming that $x_u - y_u$ and $x_v - y_v$ have the greatest common divisor at least as large as $x_u^{\varepsilon} \geqslant x_v^{\varepsilon}$, and therefore we will assume that $d \leqslant x_v^{1-\varepsilon}$.

We let j run from 1 to some positive integer k to be fixed later, and we shall apply W. Schmidt's Subspace Theorem, viewing the left sides of (17) as *small* linear forms in the variables $z_j(u,v)$, u^j/v^i, $1/v^i$. We recall the following particular case of the Subspace Theorem (see [8], [9]).

Theorem 4.1. (The Subspace Theorem) Let $\mathscr{P}' = \mathscr{P} \cup \{\infty\}$ *be a finite set of absolute values of* $\mathbb{Q}$ *containing the infinite one* (*and normalized so that* $|p|_p = p^{-1}$ *holds for all* $p \in \mathscr{P}$), *and let* $N \in \mathbb{N}$. *For each* $p \in \mathscr{P}'$, *let* $L_{1,p}, \ldots, L_{N,p}$ *be* N *linearly independent linear forms in* N *variables with rational coefficients. Given* $\delta > 0$, *there are only finitely many proper subspaces* $T_1, \ldots, T_w$ *of* $\mathbb{Q}^N$, *such that every non-zero integer point* $\mathbf{x} = (x_1, \ldots, x_N) \in \mathbb{Z}^N \backslash \{\mathbf{0}\}$ *satisfying*

$$\prod_{p \in \mathscr{P}'} \prod_{i=1}^{N} |L_{i,p}(\mathbf{x})|_p < (\max\{|x_i| \mid i = 1, \ldots, N\})^{-\delta} \tag{13}$$

belongs to one of the subspaces T_j *with* $j \in \{1, \ldots, w\}$.

We shall apply the above Subspace Theorem with $N = hk + h + k$. For convenience, we shall denote an integer point $\mathbf{x} \in \mathbb{Z}^N$ by writing

$$\mathbf{x} = (x_1, \ldots, x_N) = (z_1, \ldots, z_k, y_{01}, \ldots, y_{0h}, \ldots, y_{k1}, \ldots, y_{kh}). \tag{14}$$

With this notation, we choose the linear forms with rational coefficients as follows. For $i = 1, \ldots, k$ we let

$$L_{i,\infty}(\mathbf{x}) = z_i + y_{01} + \cdots + y_{0h} - y_{i1} - \cdots - y_{ih},$$

while for $(i,p) \notin \{(1,\infty), \ldots, (k,\infty)\}$ we set

$$L_{i,p}(\mathbf{x}) = x_i.$$

It is easy to see that for each $p \in \mathscr{P}'$ the linear forms $L_{1,p}, \ldots, L_{N,p}$ are indeed linearly independent. We now set

$$\mathbf{x} = d y_u^k x_v^h (z_1(u,v), \ldots, z_k(u,v), v^{-1}, \ldots, v^{-h}, \ldots, u^k v^{-1}, \ldots, u^k v^{-h}).$$

We observe that $\mathbf{x} \in \mathbb{Z}^N$. Indeed, the fact that for $j \leqslant k$ the number $dy_u^k x_v^h z_j(u,v)$ is an integer follows immediately from (12), while the fact that the other components of $\mathbf{x}$ are integral follows from the fact that for $i \leqslant h$ and $j \leqslant k$ we have

$$\frac{u^j}{v^i} = \frac{x_u^j y_v^i}{y_u^j x_v^i}, \tag{15}$$

and the denominator of (15) obviously divides $y_u^k x_v^h$.

We shall now estimate the double product appearing in the left hand side of inequality (13) for our linear forms $L_{i,p}$ and our non-zero integer point $\mathbf{x}$.

For $i > k$, we have that $L_{i,p}(\mathbf{x}) = x_i$, which is of the form

$$dy_u^{k-j_1} x_v^{h-i_1} x_u^{j_1} y_v^{i_1} = dq_u^{k-j_1} p_u^{j_1} q_v^{i_1} p_v^{h-i_1} (s_u^{k-j_1} r_u^{j_1} s_v^{i_1} r_v^{h-i_1})$$

for some indices $i_1 \leqslant h$ and $j_1 \leqslant k$, and the number $w_{i_1,j_1} = s_u^{k-j_1} r_u^{j_1} s_v^{i_1} r_v^{h-i_1}$ is a member of $\mathscr{S}$. From the product formula $\prod_{p \in \mathscr{P}'} |w_{i_1,j_1}|_p = 1$, we conclude that

$$\prod_{p \in \mathscr{P}'} |L_{i,p}(\mathbf{x})|_p \leqslant d(q_u p_u)^k (q_v p_v)^h. \tag{16}$$

On the other hand, for $j \leqslant k$, we have that $x_j = z_j(u,v) = c_j(u,v) y_u^{k-j} y_v x_v^h$, whence again by the product formula

$$\prod_{p \in \mathscr{P}} |x_i|_p \leqslant \frac{1}{r_v^h}, \tag{17}$$

and from (11) we have that

$$|L_{i,\infty}(\mathbf{x})|_\infty \ll dx_v^h y_u^k \cdot \frac{u^i}{v^{h+1}} \ll d \cdot \frac{y_u^{k-i} x_u^i y_v^{h+1}}{x_v}. \tag{18}$$

Thus, from inequalities (16)–(18), we find that

$$\begin{aligned}
\prod_{p \in \mathscr{P}'} \prod_{i=1}^{N} |L_{i,p}(\mathbf{x})|_p &\leqslant d^{N-k} (q_u p_u)^{kN} (q_v p_v)^{hN} \cdot \prod_{p \in \mathscr{P}'} \prod_{i=1}^{k} |L_{i,p}(\mathbf{x})|_p \\
&= d^{N-k} (q_u p_u)^{kN} (q_v p_v)^{hN} \prod_{i=1}^{k} |L_{i,\infty}(\mathbf{x})|_\infty \prod_{p \in \mathscr{P}} \prod_{i=1}^{k} |x_i|_p \\
&\ll d^{N-k} \cdot d^k \cdot (q_u p_u)^{kN} (q_v p_v)^{hN} |r_v|^{-kh} \prod_{i=1}^{k} \frac{y_u^{k-i} x_u^i y_v^{h+1}}{x_v} \\
&\ll d^N (q_u p_u)^{kN} (q_v p_v)^{hN} (y_u p_u)^{k^2} y_v^N r_u^{k^2} r_v^{-hk}.
\end{aligned} \tag{19}$$

From what we have said before, the constant implied by the last $\ll$ from (19) can be taken as 4^k. We now use the fact that $d \leqslant (x_v - y_v)^{1-\varepsilon} \ll x_v^{1-\varepsilon} = r_v p_v^{1-\varepsilon}$, and introducing this in (19), we get

$$\begin{aligned}
&\prod_{p \in \mathscr{P}'} \prod_{i=1}^{N} |L_{i,p}(\mathbf{x})|_p \\
&\qquad \ll (q_u p_u)^{kN} (q_v p_v)^{hN+(1-\varepsilon)N} (y_u p_u)^{k^2} y_v^N r_u^{k^2} r_v^{(1-\varepsilon)N-hK},
\end{aligned} \tag{20}$$

and we can take the constant implied by $\ll$ in (20) above as 4^{k+1}. Notice now that

$$\begin{aligned}\max\{|x_i|\,|i=1,\ldots,N\} &\leqslant dy_u^k x_v^h \max\left\{u^k v^{-1}, \frac{u^k-1}{v-1}\right\}\\ &\ll (x_v-y_v)y_u^k x_v^h u^k v^{-1}\\ &\ll x_v^h x_u^k y_v\\ &\ll x_v^{h+1}x_u^k\\ &= r_v^{h+1}r_u^k p_v^{h+1}p_u^k. \end{aligned}\tag{21}$$

The constant implied in (21) above can be taken to be 2 because $u^k-1<u^k$, $x_v-y_v<x_v$ and $v-1>v/2$.

We now show that if H is sufficiently large, we can then choose k and h depending on ε such that inequality (13) holds for our double product appearing in the left hand side of (20) with $\delta=1/2$.

To see this, it suffices to show, via (20) and (21), that we can find k and h such that

$$\begin{aligned}&4^{k+1}(q_up_u)^{kN}(q_vp_v)^{hN+(1-\varepsilon)N}(y_up_u)^{k^2}y_v^N r_u^{k^2} r_v^{(1-\varepsilon)N-hk}\\ &\quad\leqslant \frac{1}{4}\left(r_v^{h+1}r_u^k p_v^{h+1}p_u^k\right)^{-1/2}.\end{aligned}\tag{22}$$

We now separate in (22) the contribution of r_v in the right hand side, put everything else on the left, and take logarithms, to infer that (22) is implied by

$$\begin{aligned}&\frac{1}{\log r_v}\Bigg((k+2)\log 4 + kN\log(q_up_u)+2hN\log(q_vp_v)+k^2\log(y_uq_u)+N\log y_v\\ &\quad+\left(k^2+\frac{k}{2}\right)\log r_u+\frac{k}{2}\log p_u+\frac{h+1}{2}\log p_v\Bigg)<\left(hk-(1-\varepsilon)N-\frac{h+1}{2}\right).\end{aligned}\tag{23}$$

The right hand side of (23) is

$$hk-hk+\varepsilon hk-(1-\varepsilon)(h+k)-\frac{h+1}{2}>\frac{h}{2}(2\varepsilon k-3)-k-1,$$

and we choose k such that $\lfloor 2\varepsilon k\rfloor=5$. Here, we assume that $\varepsilon<1/2$. With this choice of k, the right hand side of (23) is larger than $h-k-1$, and in order to show that there exists h such that inequality (23) holds for large H, it suffices to show first that for some absolute constant $c_4<1$ depending on ε, and for large H, the left hand side of inequality (23) is bounded above by c_4h when k is fixed, and then to choose h such that $h-k-1>c_4h$.

But obviously,

$$\frac{(k+2)\log 4}{\log r_v}\ll\frac{k\varepsilon}{H}\ll\frac{1}{H}\tag{24}$$

(by inequality (9) and the fact that $k\ll\varepsilon^{-1}$),

$$\frac{kN\log(q_up_u)}{\log r_v}\ll\frac{k^2h\log(q_up_u)}{\log r_v}\ll\frac{h\varepsilon^2}{\log H}\tag{25}$$

(by inequalities (7), (9) and the facts that $k \ll \varepsilon^{-1}$ and $N \ll hk$),

$$\frac{2hN\log(q_v p_v)}{\log r_v} \ll \frac{kh^2\log(q_v p_v)}{\log r_v} \ll \frac{h^2\varepsilon^3}{\log H} \tag{26}$$

(by inequalities (7), (9) and the facts that $k \ll \varepsilon^{-1}$ and $N \ll hk$),

$$\frac{k^2\log(y_u q_u)}{\log r_v} \leqslant \frac{k^2(\log y_u + \log q_u)}{\log r_v} \ll A^{-1} + \frac{\varepsilon^2}{\log H} \tag{27}$$

(by inequalities (6), (7), (9), the definitions of K_1 and K_2, and the fact that $k \ll \varepsilon^{-1}$),

$$\frac{N\log y_v}{\log r_v} \ll A^{-1}\varepsilon^2 N \ll A^{-1}\varepsilon^2 hk \ll A^{-1}\varepsilon h, \tag{28}$$

(by the choice of K_1, inequalities (6), (9) and the facts that $k \ll \varepsilon^{-1}$ and $N \ll hk$),

$$\left(k^2 + \frac{k}{2}\right)\frac{\log r_u}{\log r_v} \leqslant \left((3\varepsilon^{-1})^2 + \frac{3\varepsilon^{-1}}{2}\right)c_3 < (9+2)\cdot 4\varepsilon^{-3} = 44\varepsilon^{-3} \tag{29}$$

(by the fact that $2\varepsilon k < 6$ together with inequality (8)),

$$\frac{k}{2}\cdot\frac{\log p_u}{\log r_v} \ll \frac{k\varepsilon^4}{\log H} \ll \frac{\varepsilon^3}{\log H}, \tag{30}$$

(by inequalities (7), (9), and the fact that $k \ll \varepsilon^{-1}$), and finally

$$\frac{h+1}{2}\cdot\frac{\log p_v}{\log r_v} \ll \frac{\varepsilon^4 h}{\log H}. \tag{31}$$

(by inequalities (7) and (9)). All the constants implied by $\ll$ in (24) through (31) are absolute. We shall see that we may choose $h = O(\varepsilon^{-3})$ if H is sufficiently large, where the constant implied by O above is absolute. Thus, we see that if H is sufficiently large with respect to ε, then the left hand side of (24) is bounded above by 1, the left hand side of (25) by $h/5$, the left hand side of (26) by $h/5$, the left hand sides of (27) and (28) by 1 and $h/5$, respectively, if $A > c_5$, where $c_5 > 0$ is some absolute constant, the left hand side of (30) by 1, and the left hand side of (31) by $h/5$. We thus get that for large H and small A, the left hand side of inequality (23) is bounded above by

$$1 + \frac{h}{5} + \frac{h}{5} + 1 + \frac{h}{5} + \frac{44}{\varepsilon^3} + 1 + \frac{h}{5} = 3 + \frac{4h}{5} + \frac{44}{\varepsilon^3}.$$

Thus, if

$$3 + \frac{4h}{5} + \frac{44}{\varepsilon^3} < h - k - 1,$$

then inequality (23) holds. The above inequality simply tells us to choose h such that

$$h = \left\lfloor 5\left(k + 4 + \frac{44}{\varepsilon^3}\right)\right\rfloor + 1. \tag{32}$$

With this choice, we notice that the condition $h = O(\varepsilon^{-3})$ is satisfied. The conclusion is then that if A is sufficiently large, and if we first choose h as shown at (32), and we then let H be sufficiently large, then the inequality (22) does indeed hold.

We now simply apply the Subspace Theorem to conclude that there exist some rational coefficients C_i, not all zero, such that if there are infinitely many pairs (u, v) failing both conditions 1 and conditions 3, then infinitely many of those pairs must satisfy

$$\sum_{i=1}^{N} C_i x_i = 0,$$

with $\mathbf{x} = (x_1, \ldots, x_N)$ given by (14). Thus, we get a non-trivial relation of the form

$$\sum_{j=1}^{k} \alpha_j \frac{u-1}{v-1} + \sum_{\substack{1 \leqslant i \leqslant h \\ 0 \leqslant j \leqslant k}} \beta_{i,j} \frac{u^j}{v^i} = 0,$$

in which not all coefficients $\alpha_i, \beta_{i,j}$ are zero. This gives us a non-trivial relation of the form

$$\sum_{\substack{0 \leqslant i \leqslant h \\ 0 \leqslant j \leqslant k}} \gamma_{i,j} v^i u^j = 0, \tag{33}$$

in which not all coefficients $\gamma_{i,j}$ are zero. Indeed, the coefficients $\gamma_{i,j}$ come from formally looking at the rational function

$$\sum_{j=1}^{k} \alpha_j \frac{X^j - 1}{Y - 1} + \sum_{\substack{1 \leqslant i \leqslant h \\ 0 \leqslant j \leqslant k}} \beta_{i,j} \frac{X^j}{Y^i} \tag{34}$$

in $\mathbb{Q}[X, Y]$, multiplying it by $(Y-1)Y^h$ and expressing the resulting polynomial (notice that the result is really a polynomial) as a sum of monomials. If all the coefficients $\gamma_{i,j}$ arising in this way were zero, then the rational function shown at (34) must have been zero, but it is easy to see that this can be so only when all the coefficients α_j and $\beta_{i,j}$ were zero, which is not the case (for this argument, see [2]). Let D be the subset of all pairs (i,j) with $0 \leqslant i \leqslant h$ and $0 \leqslant j \leqslant k$ for which $\gamma_{i,j} \neq 0$. Since (33) has infinitely many solutions (u, v), there must exist a non-empty subset D' of D for which the relation

$$\sum_{(i,j) \in D'} \gamma_{i,j} v^i u^j = 0 \tag{35}$$

holds for infinitely many pairs (u, v) satisfying (33), such that relation (35) is non-degenerate for each one of these pairs (u, v), in the sense that there is no proper subset D'' of D' for which the relation

$$\sum_{(i,j) \in D''} \gamma_{i,j} v^i u^j = 0$$

holds.

We shall now assume that the pairs (u,v) fail also condition 2 with $K_2=\max\{h,k\}$ in the sense that $u^i\neq v^j$ for any pair $(i,j)\neq(0,0)$ with $\max\{|i|,|j|\}\leqslant K_2$.

We shall then show that if A is sufficiently large and B is sufficiently small (in computable ways which depend on ε, $\mathscr{P}$ and the coefficients $\gamma_{i,j}$ for $(i,j)\in D'$), and when H is sufficiently large, then (u,v) cannot belong to $\mathscr{L}_\varepsilon$, which will give us the final contradiction.

To proceed, we may assume that D' has cardinality $t\geqslant 2$. We shall show that:

Claim. *If equation* (35) *admits infinitely many solutions* (u,v) *in* $\mathscr{L}_\varepsilon$ *failing all three conditions from Theorem* 2.1 *with some appropriate constants A and B, then the Newton polygon of the polynomial*

$$\sum_{(i,j)\,\in D'}\gamma_{i,j}X^iY^j$$

is a line.

Proof of the Claim. We use induction on the parameter $\ell\geqslant 2$ with $\ell=2,\ldots,t$, to show that (i_ℓ,j_ℓ) must be collinear with (i_w,j_w) for all $w\leqslant\ell-1$.

Of course the Newton polygon is a line when $t=2$.

From now on, we assume that $t\geqslant 3$. We fix one of the $t!$ total orderings of D', and assume that we have labeled the pairs of D' as $D'=\{(i_\ell,j_\ell)\mid\ell=1,\ldots,t\}$, such that $v^{i_1}u^{j_1}>u^{i_2}v^{j_2}>\cdots>u^{i_t}v^{j_t}$. Clearly, any solution of (35) leads to a total ordering of D' (because $u^i\neq v^j$ for any non-negative integers i,j not both zero with $\max\{i,j\}\leqslant K_2$), and since we have infinitely many such solutions, we may assume that infinitely many of them lead to the same fixed total ordering.

For $\ell=1,\ldots,t$, we set

$$\Lambda_\ell=i_\ell\log v+j_\ell\log u,$$

and notice that we have $\Lambda_1>\Lambda_2>\cdots>\Lambda_t\geqslant 0$, where the last inequality here holds because $u>1$, $v>1$, and $\min\{i_t,j_t\}\geqslant 0$. We label $\gamma_\ell=\gamma_{i_\ell,j_\ell}$. We write (35) as

$$\sum_{\ell=1}^{t}\gamma_\ell\exp\Lambda_\ell=0. \tag{36}$$

In particular,

$$|\gamma_1|\exp(\Lambda_1-\Lambda_2)\leqslant\sum_{\ell=2}^{t}|\gamma_\ell|\exp(\Lambda_\ell-\Lambda_2)=O_\varepsilon(1),$$

therefore

$$0<\Lambda_1-\Lambda_2\ll_\varepsilon 1;$$

hence,

$$0<(i_1-i_2)\log v+(j_1-j_2)\log u\ll_\varepsilon 1,$$

or, equivalently,

$$\left|\frac{i_1-i_2}{j_1-j_2}+\frac{\log u}{\log v}\right| = O_\varepsilon\left(\frac{1}{H}\right). \tag{37}$$

Here, we assume that $j_2 \neq j_1$, for if not, we may interchange u with v and the is with the js and get a relation of the same type (notice that it is not possible that $i_1 = i_2$ and $j_1 = j_2$ at the same time because the pairs (i_1, j_1) and (i_2, j_2) must be different). A refined argument of the same type is now to rewrite (36) as

$$\begin{aligned} 0 &< \left|\frac{-\gamma_1}{\gamma_2}\cdot\exp(\Lambda_1-\Lambda_2)-1\right| \\ &\leqslant \frac{1}{|\gamma_2|}\sum_{\ell=3}^{t}|\gamma_\ell|\exp(\Lambda_\ell-\Lambda_2) = O_\varepsilon\left(\frac{1}{\exp(\Lambda_2-\Lambda_3)}\right). \end{aligned} \tag{38}$$

The inequality on the right is clear, and the one on the left follows from the fact that (35) is non-degenerate.

The only instance in which the left hand side of (38) might have any chance of being small is when $-\gamma_1/\gamma_2 = |\gamma_1/\gamma_2|$ and $|\gamma_1/\gamma_2|\exp(\Lambda_1-\Lambda_2)$ is close to 1. If this is so, then the left hand side of (38) is

$$\begin{aligned} \left|\frac{-\gamma_1}{\gamma_2}\cdot\exp(\Lambda_1-\Lambda_2)-1\right| &= |\exp(\log|\gamma_1/\gamma_2|+\Lambda_1-\Lambda_2)-1| \\ &\gg |\log|\gamma_1/\gamma_2|+\Lambda_1-\Lambda_2|. \end{aligned} \tag{39}$$

To get a lower bound on the right hand side of (39), we write

$$\begin{aligned} |\log|\gamma_1/\gamma_2|+\Lambda_1-\Lambda_2| = \Bigg|&(i_1-i_2)\log(r_v/s_v)+(j_1-j_2)\log(r_u/s_u) \\ &+\log\left|\left(\frac{p_v}{q_v}\right)^{i_1-i_2}\left(\frac{p_u}{q_u}\right)^{j_1-j_2}\left(\frac{\gamma_1}{\gamma_2}\right)\right|\Bigg|, \end{aligned} \tag{40}$$

and use a lower bound for linear forms in logarithms (see [1]) to find a lower bound on (40). Clearly, (40) is non-zero. The key observation is that r_u/s_u and r_v/s_v are numbers composed solely from primes from $\mathscr{P}$ and the maximal exponent at which the prime numbers from $\mathscr{P}$ can appear in either r_u, s_u, r_v or s_v is certainly $\ll H$. Finally, the height of the last rational number appearing inside the logarithm in (40) is, by the failure of condition 3 from the statement of Theorem 2.1,

$$\ll_\varepsilon \max\{h,k\}B\varepsilon^5\frac{H}{\log H} \ll_\varepsilon B\frac{H}{\log H}$$

by the fact that k and h depend only on ε. With the classical lower bound for the linear form in logarithms of algebraic numbers from [1], we get therefore that (40) is bounded from below by

$$\exp\left(-c_6 B\log(\max\{h,k\}H)\frac{H}{\log H}\right) \geqslant \exp(-c_7 BH),$$

where c_7 depends only on ε, $\mathscr{P}$, and the coefficients γ_1 and γ_2. With (38), we get that the estimate

$$0<\Lambda_2-\Lambda_3<c_7BH+O_\varepsilon(1)$$

holds.

We thus get

$$0<(i_2-i_3)\log v+(j_2-j_3)\log u<c_7BH+O_\varepsilon(1). \tag{41}$$

If $j_2-j_3=0$, we get that $i_2-i_3>0$ and

$$\log v \leqslant \frac{c_7BH}{(i_2-i_3)}. \tag{42}$$

Since $\log v = \log x_v - \log y_v \geqslant \varepsilon H/3$ for H sufficiently large (see inequalities (5) and (6)), we get that (42) implies that

$$\frac{\varepsilon}{3}<c_7B+O_\varepsilon\left(\frac{1}{H}\right), \tag{43}$$

but we can prevent (43) from happening by first choosing H to be large enough so that the contribution from the term $O_\varepsilon(1/H)$ is smaller than $\varepsilon/6$, and then by choosing $B<\varepsilon/(6c_7)$. So, when B is small enough, then $j_2-j_3\neq 0$, and now (41) implies that the inequality

$$\left|\frac{i_2-i_3}{j_2-j_3}+\frac{\log u}{\log v}\right|<\frac{c_7BH}{\log v}+O_\varepsilon\left(\frac{1}{\log v}\right)<\frac{3c_7B}{\varepsilon}+O_\varepsilon\left(\frac{1}{H}\right), \tag{44}$$

holds for sufficiently large values of H. We now choose B such that $3c_7B/\varepsilon$ is smaller than $(2|(j_1-j_2)(j_2-j_3)|)^{-1}$. But then, if H is large enough, the right hand side of (37) is smaller than the number $(4|(j_1-j_2)(j_2-j_3)|)^{-1}$, while the contribution of the $O_\varepsilon(1/H)$ from the right hand side of (44) is also smaller than $(4|(j_1-j_2)(j_2-j_3)|)^{-1}$, and thus, with (37), (44), and the absolute value inequality, we get

$$\left|\frac{i_1-i_2}{j_1-j_2}-\frac{i_2-i_3}{j_2-j_3}\right|<\frac{1}{|(j_1-j_2)(j_2-j_3)|}\left(\frac{1}{2}+\frac{1}{4}+\frac{1}{4}\right)=\frac{1}{|(j_1-j_2)(j_2-j_3)|},$$

which forces

$$\frac{i_1-i_2}{j_1-j_2}=\frac{i_2-i_3}{j_2-j_3};$$

i.e., $(i_1,j_1),(i_2,j_2)$ and (i_3,j_3) are collinear.

The remainder of the proof of the claim is now obvious. Assume, by induction, that for some $3\leqslant \ell<t$ all points (i_w,j_w) are collinear for $1\leqslant w\leqslant \ell$. In this case, the points (i_w-i_ℓ,j_w-j_ℓ) are collinear, and if we set r_1/s_1 to be the slope of the line passing through these points (if the line is vertical, we simply interchange again u with v and the is with the js), then there exist integers α_w such that $i_w-i_\ell=\alpha_w r_1$ and $j_w-j_\ell=\alpha_w s_1$ hold for all $w=1,\ldots,\ell$ (here, $\alpha_\ell=0$). Since $\Lambda_w-\Lambda_\ell=\alpha_w(r_1\log v+s_1\log u)$ is always positive for $w<\ell$, we may assume (up to replacing (r_1,s_1) by $(-r_1,-s_1)$) that $r_1\log v+s_1\log u>0$, and now the fact that $\Lambda_1 - \Lambda_\ell > \cdots > \Lambda_\ell - \Lambda_\ell = 0$ implies that $\alpha_1 > \cdots > \alpha_\ell = 0$. Thus, with the polynomial of one variable

$$g(Z) = \sum_{i=1}^{\ell} \gamma_i Z^{\alpha_i} \tag{45}$$

we get that the relation

$$\sum_{w=1}^{\ell} \gamma_w X^{i_w - i_\ell} Y^{j_w - j_\ell} = \sum_{i=1}^{\ell} \gamma_i Z^{\alpha_i} \tag{46}$$

holds with $Z = X^{r_1} Y^{s_1}$. Let $\delta_1, \ldots, \delta_{\ell'}$ be all the distinct roots of the polynomial shown at (45) of multiplicities $\eta_1, \ldots, \eta_{\ell'}$, respectively. Relation (35) now implies that

$$\begin{aligned} |\gamma_1| \prod_{w=1}^{\ell'} |v^{r_1} u^{s_1} - \delta_w|^{\eta_w} &= |g(v^{r_1} u^{s_1})| \\ &= \left| \sum_{w=1}^{\ell} \gamma_w \exp(\Lambda_w - \Lambda_\ell) \right| \\ &\leqslant \sum_{w=\ell+1}^{t} |\gamma_w| \exp(\Lambda_w - \Lambda_\ell) \\ &= O_\varepsilon\left(\frac{1}{\exp(\Lambda_\ell - \Lambda_{\ell+1})} \right). \end{aligned} \tag{47}$$

We now apply the same argument as before. The only chance that the left hand side of (47) can be small is when $u^{r_1} v^{s_1}$ is very close to one of the roots δ_w of $g(Z)$. Since a number can't be simultaneously close to two distinct fixed numbers, we may assume that $v^{r_1} u^{s_1}$ is very close to δ_1. But then

$$|v^{r_1} u^{s_1} - \delta_w| \gg_\varepsilon 1 \quad \text{for } w \neq 1, \tag{48}$$

while for the presumably small non-zero factor $|u^{r_1} v^{s_1} - \delta_1|$ we use, as before, a lower bound for the corresponding linear form in logarithms to conclude that

$$|u^{r_1} v^{s_1} - \delta_1| \gg_\varepsilon \exp(-c_7 BH), \tag{49}$$

with some maybe larger c_7 than at the case $t = 3$ which incorporates the maximal height of all the roots δ_w of $g(Z)$. As before, (47), (48) and (49) lead to the conclusion that

$$0 < (i_\ell - i_{\ell+1}) \log v + (j_\ell - j_{\ell+1}) \log u < c_7 B \ell H + O_\varepsilon(1), \tag{50}$$

where the factor $\ell \geqslant \eta_1$ accounts for the multiplicity of δ_1. As before, we can argue once again that when B is sufficiently small and H is sufficiently large, then (50) implies that $j_\ell - j_{\ell+1}$ cannot be zero, and therefore, from (50), we infer that

$$\left| \frac{i_\ell - i_{\ell+1}}{j_\ell - j_{\ell+1}} + \frac{\log u}{\log v} \right| < \frac{3 c_7 B t}{\varepsilon} + O_\varepsilon\left(\frac{1}{H} \right). \tag{51}$$

But obviously, the fact that $v^{r_1}u^{s_1}$ was close to δ_1; i.e., $|v^{r_1}u^{s_1}-\delta_1| = O_\varepsilon(1)$, implies that

$$\left|\frac{r_1}{s_1}+\frac{\log u}{\log v}\right| = O_\varepsilon\left(\frac{1}{H}\right). \tag{52}$$

But now (51) and (52) show that if B is chosen in such a way that it is smaller than some constant (computable in terms of the numbers γ_i and ε), and if H is large, then

$$\frac{i_\ell - i_{\ell+1}}{j_\ell - j_{\ell+1}} = \frac{r_1}{s_1},$$

therefore the point $(i_{\ell+1}, j_{\ell+1})$ belongs to the line passing through all the points (i_w, j_w) with $1 \leqslant w \leqslant \ell$. This concludes the proof of the claim. □

We now point out how the conclusion of Theorem 2.1 follows from the above Claim.

Assume that (35) has infinitely many solutions. Then there must exist a pair of indices (i,j) (which gives the slope of the Newton polygon), and some root K of $g(Z)$, where now g is the one variable polynomial associated to the polynomial

$$\sum_{(r,s)\,\in\, D'} \gamma_{r,s}X^rY^s$$

in the same way as indicated at (45)–(46), such that infinitely many of the pairs (u,v) satisfying (35) satisfy also

$$u^iv^j = K, \tag{53}$$

with a fixed value of the non-zero rational number K. Suppose that there are pairs (u,v) with $H = H(u)$ arbitrarily large, satisfying (53) and not satisfying any of the conditions of Theorem 2.1. Since H can be arbitrarily large, we conclude that one of the numbers i and j is positive and the other is negative. Assume that $i > 0$ and $j < 0$, replace j by $-j$, and rewrite the above relation as

$$\frac{u^i}{v^j} = K. \tag{54}$$

Since condition 2 does not hold, it follows that $K \neq 1$. We may assume that $\gcd(i,j) = 1$ for if $d = \gcd(i,j)$, then any pair (u,v) satisfying (54) satisfies also $u^{i/d}/v^{j/d} = K'$, where $K' = K^{1/d}$. Assuming $\gcd(i,j) = 1$, fix some pair of positive rational numbers (u_1, v_1) with $u_1^i/v_1^j = K$. Then, for each pair (u,v) satisfying (54) we have $(u/u_1)^i = (v/v_1)^j$. So, since $\gcd(i,j) = 1$ for each pair of rationals (u,v) with (54), there is a non-zero rational number ρ such that $u = \rho^j u_1, v = \rho^i v_1$. Write $\rho = x/y, u_1 = X_1/Y_1, v_1 = X_2/Y_2$, where in each quotient the numerator and denominator are coprime positive integers. Assume that (u,v) satisfies (54) and that it does not satisfy conditions 1, 2, or 3 of Theorem 2.1. Let $\mathscr{D} = \gcd(u-1, v-1)$, i.e., the greatest common divisor of the numerators of $u-1$ and $v-1$. Then $\mathscr{D}$ divides $\gcd(x^jX_1 - y^jY_1, x^iX_2 - y^iY_2)$. This implies that $\mathscr{D}$ divides $x^{ij}(X_1^iY_2^j - X_2^jY_1^i)$. The greatest common divisor of x^j and $\mathscr{D}$ must divide

Y_1 since x and y are coprime. So, in fact, $\mathscr{D}$ divides $Y_1^i(X_1^iY_2^j - X_2^jY_1^i)$. This last number is non-zero, since otherwise $u_1^i = v_1^j$, contradicting that $K \neq 1$. Hence, $\mathscr{D} = \gcd(u-1, v-1)$ has an upper bound independent of u and v. But this contradicts our assumption that condition 1 of Theorem 2.1 is not satisfied for pairs (u, v) with H arbitrarily large.

Clearly, the contradiction must have come from the fact that we have assumed infinitely many solutions (u, v) for equation (35).

To recapitulate, we have shown that we can choose K_1, K_2 and K_3 such that *all but finitely many pairs* $(u, v) \in \mathscr{L}_\varepsilon$ satisfy at least one of the conditions 1, 2 or 3 from Theorem 2.1 with these constants. We may now clearly increase K_1 in such a way that all these finitely many exceptional pairs (u, v) in $\mathscr{L}_\varepsilon$ satisfy 1 with this new K_1, and Theorem 2.1 is now proved.

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