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Dyadic Diaphony of Digital Nets Over $\mathbb{Z}_2$

By

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Dedicated to Prof. H. Niederreiter on the occasion of his 60th birthday

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Abstract. The dyadic diaphony, introduced by Hellekalek and Leeb, is a quantitative measure for the irregularity of distribution of point sets in the unit-cube. In this paper we study the dyadic diaphony of digital nets over $\mathbb{Z}_2$. We prove an upper bound for the dyadic diaphony of nets and show that the convergence order is best possible. This follows from a relation between the dyadic diaphony and the $\mathscr{L}_2$ discrepancy. In order to investigate the case where the number of points is small compared to the dimension we introduce the limiting dyadic diaphony, which is defined as the limiting case where the dimension tends to infinity. We obtain a tight upper and lower bound and we compare this result with the limiting dyadic diaphony of a random sample.

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1. Introduction

Many applications, notably numerical integration (see for example [19]), require well distributed point sets in the unit-cube. What we mean by well distributed can be defined in several ways, all of which have their own significance. Geometrical concepts of measuring the irregularity of distribution comprise the $\mathscr{L}_2$ discrepancy and the star-discrepancy (see for example [7], [14], [18]). Other papers consider measures specifically related to numerical integration [2], [3], [13], [19], [24]. In such cases the worst-case error, that is, the worst approximation of the integral of functions from the unit-ball of some function space, has frequently been considered and analyzed. Furthermore, several connections between those seemingly different concepts have been pointed out, see [13], [24].

The dyadic diaphony (see Definition 1) considered here is another measure for the distribution of a point set. This measure is based on a function, depending on a point set in the unit-cube, whose function value tells us about the distribution

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property of a point set. Introduced by Hellekalek and Leeb [12], it has been shown that this is indeed a valid measure, in the sense that the function value tends to zero if and only if the point set is uniformly distributed in the unit-cube.

In [5] several connections between the dyadic diaphony, discrepancy and numerical integration of functions from a certain function space have been pointed out. Moreover, it was shown in [5] how the dyadic diaphony can be understood geometrically.

In this paper we study the dyadic diaphony of digital nets over $\mathbb{Z}_2$. Digital nets are well known for their excellent distribution properties. This is also reflected by our results here, unless the number of points is small compared to the dimension. More details are given in the following outline of the paper.

The definitions of dyadic diaphony and digital nets are given in the following section. In Section 3 we prove an upper bound for the dyadic diaphony of digital (t,m,s)-nets over $\mathbb{Z}_2$ (Theorem 1). From this result we obtain for any dimension $s \geqslant 1$ the existence of digital nets with dyadic diaphony of order $(\log N)^{(s-1)/2} N^{-1}$ (Corollary 1) and in Theorem 2 we show that this order is best possible for any point set.

Though we have shown the existence of digital nets with the best possible convergence order, there remains a gap in the applicability of our upper bound. Namely, the bound is bigger than one if the number of points is small compared to the dimension, though it is known that the dyadic diaphony lies always between zero and one. For practical purposes it is often the case that the number of points is rather small compared to the dimension (and therefore it is a very important research topic), hence there is a need to investigate this case separately. This point is pursued in Section 4, where we introduce the limiting dyadic diaphony. The limiting dyadic diaphony is the limiting case of the dyadic diaphony as the dimension tends to infinity. This approach is similar to the limiting discrepancy introduced in [24]. Sloan and Woźniakowski [24] introduced the concept of strong tractability, meaning that the limiting discrepancy is finite and decays polynomially with the number of points. This has subsequently been the topic of much research. However, in our setting the notion of strong tractability is not useful, as the limiting dyadic diaphony is always between zero and one. Still, the limiting dyadic diaphony is useful to analyze the convergence rate for small point sets, as obviously any finite number of points is small compared to infinity. We calculate the limiting dyadic diaphony for digital nets and the expected value of the dyadic diaphony of a random sample. The comparison shows that in terms of the limiting dyadic diaphony both point sets perform equally well.

In Section 5 we discuss the previous results. We show that point sets with almost best possible limiting dyadic diaphony can be constructed in practice using a component-by-component algorithm. Further we prove that a convergence order of $O((\log N)^{(s-1)/2} N^{-1})$ can only be achieved if N is exponentially large in the dimension.

2. Preliminaries

Throughout this paper let $\mathbb{N}_0$ denote the set of non-negative integers. For $k \in \mathbb{N}_0$ with base 2 representation $k = \kappa_{a-1} 2^{a-1} + \cdots + \kappa_1 2 + \kappa_0$, with

$\kappa_i \in \{0,1\}$, we define the Walsh function $\text{wal}_k : \mathbb{R} \longrightarrow \{-1,1\}$ periodic with period 1, by

$$\text{wal}_k(x) := (-1)^{\xi_1\kappa_0+\cdots+\xi_a\kappa_{a-1}},$$

for $x \in [0,1)$ with base 2 representation $x = \xi_1/2 + \xi_2/2^2 + \cdots$ (unique in the sense that infinitely many of the ξ_i must be zero). For dimension $s \geqslant 2$, $x_1,\ldots,x_s \in [0,1)$ and $k_1,\ldots,k_s \in \mathbb{N}_0$ we define $\text{wal}_{k_1,\ldots,k_s} : [0,1)^s \longrightarrow \{-1,1\}$ by

$$\text{wal}_{k_1,\ldots,k_s}(x_1,\ldots,x_s) := \prod_{j=1}^{s} \text{wal}_{k_j}(x_j).$$

For vectors $\boldsymbol{k} = (k_1,\ldots,k_s) \in \mathbb{N}_0^s$ and $\boldsymbol{x} = (x_1,\ldots,x_s) \in [0,1)^s$ we write

$$\text{wal}_{\boldsymbol{k}}(\boldsymbol{x}) := \text{wal}_{k_1,\ldots,k_s}(x_1,\ldots,x_s).$$

It is clear from the definitions that Walsh functions are piecewise constant. It can be shown that for any integer $s \geqslant 1$ the system $\{\text{wal}_{k_1,\ldots,k_s} : k_1,\ldots,k_s \geqslant 0\}$ is a complete orthonormal system in $L_2([0,1)^s)$, see for example [1], [17] or [21, Satz 1]. More information on Walsh functions can be found in [1], [21], [22], [25].

Throughout the paper let $P_{N,s} = \{\boldsymbol{x}_0,\ldots,\boldsymbol{x}_{N-1}\}$ denote a point set in the s-dimensional unit-cube $[0,1)^s$. For $\boldsymbol{k} \in \mathbb{N}_0^s$ we define

$$S_N(\boldsymbol{k}, P_{N,s}) := \frac{1}{N}\sum_{n=0}^{N-1} \text{wal}_{\boldsymbol{k}}(\boldsymbol{x}_n).$$

Now we can give the definition of the dyadic diaphony (see Hellekalek and Leeb, [12]).

Definition 1. The dyadic diaphony $F_N(P_{N,s})$ of a point set $P_{N,s} = \{\boldsymbol{x}_0,\ldots,\boldsymbol{x}_{N-1}\}$ in $[0,1)^s$ is defined by

$$F_N(P_{N,s}) := \left(\frac{1}{3^s - 1} \sum_{\substack{\boldsymbol{k} \in \mathbb{N}_0^s \\ \boldsymbol{k} \neq \boldsymbol{0}}} \rho(\boldsymbol{k}) |S_N(\boldsymbol{k}, P_{N,s})|^2 \right)^{1/2},$$

where for an integer vector $\boldsymbol{k} = (k_1,\ldots,k_s) \in \mathbb{N}_0^s$, $\rho(\boldsymbol{k}) = \prod_{i=1}^s \rho(k_i)$ and for $k \in \mathbb{N}_0$,

$$\rho(k) := \begin{cases} 1 & \text{if } k = 0, \\ 2^{-2g} & \text{if } 2^g \leqslant k < 2^{g+1} \quad \text{with } g \in \mathbb{N}_0. \end{cases}$$

We note that Grozdanov and Stoilova [8] generalized the dyadic diaphony to a so-called b-adic diaphony, $b \geqslant 2$, and quite recently Grozdanov, Nikolova and Stoilova [10] introduced a so-called generalized b-adic diaphony. For a very general definition see [9].

In [12], Hellekalek and Leeb proved that for any point set $P_{N,s}$ we have $0 \leqslant F_N(P_{N,s}) \leqslant 1$. Further they showed that a sequence $P_s = (\boldsymbol{x}_n)_{n \geqslant 0}$ is uniformly distributed modulo 1 if and only if $\lim_{N\to\infty} F_N(P_{N,s}) = 0$, where $P_{N,s}$

consists of the first N points of the sequence P_s. Further they showed that for any dimension $s \geqslant 1$ and for any point set $P_{N,s}$ the dyadic diaphony $F_N(P_{N,s})$ can be computed with $O(sN^2)$ operations. If $P'_{2^{gs},s}$ is the regular grid, that is,

$$P'_{2^{gs},s} = \{(a_1/2^g, \dots, a_s/2^g) : 0 \leqslant a_i < 2^g,\ a_i \in \mathbb{Z},\ 1 \leqslant i \leqslant s\}$$

then

$$F^2_{2^{gs}}(P'_{2^{gs},s}) = \frac{1}{3^s - 1}((1 + 2^{-2g+1})^s - 1)$$

(see [12, Proposition 3.4]).

Currently the most effective constructions of point sets with good distribution properties are based on the concept of (t, m, s)-nets in a base b. For a definition of such nets see [19]. In practice all concrete constructions of (t, m, s)-nets in a base b are based on a general construction scheme which is the concept of digital point sets. Here in this paper we only deal with the case $b = 2$, i.e., we only consider (t, m, s)-nets in base 2 and hence we introduce the digital construction only for this special case. For a general definition see for example [15], [16] or [19]. In the following let $\mathbb{Z}_2$ denote the finite field with two elements.

Definition 2. Let $s \geqslant 1$, $m \geqslant 1$ and $0 \leqslant t \leqslant m$ be integers. Choose s $m \times m$ matrices $C_1, \dots, C_s$ over $\mathbb{Z}_2$ with the following property: for any integers $d_1, \dots, d_s \geqslant 0$ with $d_1 + \cdots + d_s = m - t$ the system of the

first d_1 rows of C_1, together with the
⋮
first d_{s-1} rows of C_{s-1}, together with the
first d_s rows of C_s

is linearly independent over $\mathbb{Z}_2$.

Consider the following construction principle for point sets consisting of 2^m points in $[0, 1)^s$: represent n, $0 \leqslant n < 2^m$, in base 2, $n = n_0 + n_1 2 + \cdots + n_{m-1} 2^{m-1}$, and multiply the matrix C_j, $1 \leqslant j \leqslant s$, with the vector $\vec{n} = (n_0, \dots, n_{m-1})^T$ of digits of n in $\mathbb{Z}_2$,

$$C_j \vec{n} =: (y_1^{(j)}, \dots, y_m^{(j)})^T.$$

Now we set

$$x_n^{(j)} := \frac{y_1^{(j)}}{2} + \cdots + \frac{y_m^{(j)}}{2^m}$$

and

$$\boldsymbol{x}_n = (x_n^{(1)}, \dots, x_n^{(s)}).$$

The point set $P^{\mathrm{net}}_{2^m,s} = \{\boldsymbol{x}_0, \dots, \boldsymbol{x}_{2^m-1}\}$ is called a digital (t, m, s)-net over $\mathbb{Z}_2$ and the matrices $C_1, \dots, C_s$ are called the generator matrices of the digital net.

Note that any digital (t, m, s)-net over $\mathbb{Z}_2$ is a (t, m, s)-net in base 2 as shown by Niederreiter [19]. Further it follows from Definition 2 that any d-dimensional projection of a digital (t, m, s)-net over $\mathbb{Z}_2$ is a digital (t, m, d)-net over $\mathbb{Z}_2$.

3. On the Convergence Rate of the Dyadic Diaphony of Digital Nets over $\mathbb{Z}_2$

In this section we investigate the dyadic diaphony of digital nets constructed over $\mathbb{Z}_2$. We show that $F_N(P_{N,s}^{\mathrm{net}}) = O((\log N)^{\frac{s-1}{2}} N^{-1})$ and that this is best possible for any point set. (We remark that the generalized spectral test, which is another measure of uniform distribution closely related to the dyadic diaphony, of digital nets was computed by Hellekalek [11].)

We introduce some notation. For $m \times m$ matrices $C_1, \ldots, C_s$ over $\mathbb{Z}_2$ we define the set

$$\mathscr{D} := \{\boldsymbol{k} \in \{0, \ldots, 2^m - 1\}^s \setminus \{\boldsymbol{0}\} : C_1^T \vec{k}_1 + \cdots + C_s^T \vec{k}_s = \vec{0}\},$$

where $\boldsymbol{k} = (k_1, \ldots, k_s)$ and where for $k \in \{0, \ldots, 2^m - 1\}$ with $k = \kappa_0 + \kappa_1 2 + \cdots + \kappa_{m-1} 2^{m-1}$ we write $\vec{k} = (\kappa_0, \ldots, \kappa_{m-1})^T \in \mathbb{Z}_2^m$.

Proposition 1. *Let $P_{2^m,s}^{\mathrm{net}} = \{\boldsymbol{x}_0, \ldots, \boldsymbol{x}_{2^m-1}\}$ be a digital (t, m, s)-net over $\mathbb{Z}_2$ generated by the regular $m \times m$ matrices $C_1, \ldots, C_s$ over $\mathbb{Z}_2$. Then for the dyadic diaphony of $P_{2^m,s}^{\mathrm{net}}$ we have*

$$\frac{1}{3^s - 1} \sum_{\boldsymbol{k} \in \mathscr{D}} \rho(\boldsymbol{k}) + \frac{1}{3^s - 1} \left(\left(1 + \frac{2}{2^{2m}} \right)^s - 1 \right) \leqslant F_{2^m}^2 (P_{2^m,s}^{\mathrm{net}})$$

$$\leqslant \frac{4^s}{3^s - 1} \sum_{\boldsymbol{k} \in \mathscr{D}} \rho(\boldsymbol{k}) + \frac{1}{2^{2m}} \frac{2s}{3^s - 1} \left(2\zeta(2) \left(3 + \frac{4\zeta(2)}{2^m} \right)^{s-1} + \left(1 + \frac{2}{2^{2m}} \right)^{s-1} \right).$$

For the proof of Proposition 1 we need two lemmas.

Lemma 1. *Let $\{\boldsymbol{x}_0, \ldots, \boldsymbol{x}_{2^m-1}\}$ be a digital net over $\mathbb{Z}_2$ generated by the $m \times m$ matrices $C_1, \ldots, C_s$ over $\mathbb{Z}_2$. Then for all integers $0 \leqslant k_1, \ldots, k_s < 2^m$ we have*

$$\sum_{n=0}^{2^m-1} \mathrm{wal}_{k_1, \ldots, k_s}(\boldsymbol{x}_n) = \begin{cases} 2^m, & \text{if } C_1^T \vec{k}_1 + \cdots + C_s^T \vec{k}_s = \vec{0}, \\ 0, & \text{otherwise}, \end{cases}$$

where for $0 \leqslant k < 2^m$ with $k = \kappa_0 + \kappa_1 2 + \cdots + \kappa_{m-1} 2^{m-1}$ we write $\vec{k} = (\kappa_0, \ldots, \kappa_{m-1})^T \in \mathbb{Z}_2^m$ and $\vec{0}$ denotes the zero vector in $\mathbb{Z}_2^m$.

Proof. See [3, Lemma 2]. □

Lemma 2. *We have*

$$\sum_{\boldsymbol{l} \in \mathbb{N}_0^s} \rho(\boldsymbol{l}) = 3^s \quad \text{and} \quad \sum_{\boldsymbol{l} \in \mathbb{N}_0^s} \rho(2^m \boldsymbol{l}) = \left(1 + \frac{2}{2^{2m}} \right)^s.$$

Proof. We have

$$\sum_{\boldsymbol{l} \in \mathbb{N}_0^s} \rho(2^m \boldsymbol{l}) = \left(\sum_{l=0}^{\infty} \rho(2^m l) \right)^s = \left(1 + \sum_{g=0}^{\infty} \sum_{l=2^g}^{2^{g+1}-1} \frac{1}{2^{2(g+m)}} \right)^s = \left(1 + \frac{2}{2^{2m}} \right)^s.$$

The first equality follows now by setting $m = 0$ in the equation above. □

We are ready to prove Proposition 1.

Proof. The proof is based on an technique introduced by Niederreiter [19]. First we need some notation: for a non-negative integer k with base 2 representation $k = \sum_{i=0}^{\infty} \kappa_i 2^i$ we write $\mathrm{tr}_m(k) := \kappa_0 + \kappa_1 2 + \cdots + \kappa_{m-1} 2^{m-1}$ and

$$\vec{\mathrm{tr}}_m(k) := (\kappa_0, \ldots, \kappa_{m-1})^T \in \mathbb{Z}_2^m.$$

Since $P^{\mathrm{net}}_{2^m,s}$ is a digital (t,m,s)-net over $\mathbb{Z}_2$, using Lemma 1 we obtain

$$(3^s - 1) F_N^2(P^{\mathrm{net}}_{2^m,s}) = \sum_{\substack{\boldsymbol{k} \in \mathbb{N}_0^s \setminus \{\boldsymbol{0}\} \\ C_1^T \vec{\mathrm{tr}}_m(k_1) + \cdots + C_s^T \vec{\mathrm{tr}}_m(k_s) = \vec{0}}} \rho(\boldsymbol{k}) =: \Sigma.$$

For $\boldsymbol{k} \in \mathbb{N}_0^s$, $\boldsymbol{k} \neq \boldsymbol{0}$, we consider two cases:

1. Assume $\boldsymbol{k} = 2^m \boldsymbol{l}$ with $\boldsymbol{l} \in \mathbb{N}_0^s$, $\boldsymbol{l} \neq \boldsymbol{0}$. In this case we have $\mathrm{tr}_m(k_j) = 0$ for $1 \leqslant j \leqslant s$ and the condition

$$C_1^T \vec{\mathrm{tr}}_m(k_1) + \cdots + C_s^T \vec{\mathrm{tr}}_m(k_s) = \vec{0}$$

is trivially fulfilled for any choice of $C_1, \ldots, C_s$.

2. Assume $\boldsymbol{k} = \boldsymbol{k}^* + 2^m \boldsymbol{l}$ with $\boldsymbol{l} \in \mathbb{N}_0^s$, $\boldsymbol{k}^* = (k_1^*, \ldots, k_s^*) \neq \boldsymbol{0}$ and $0 \leqslant k_j^* < 2^m$ for all $1 \leqslant j \leqslant s$. In this case we have $\vec{\mathrm{tr}}_m(k_j) = \vec{k}_j^*$ for all $1 \leqslant j \leqslant s$ and our condition becomes

$$C_1^T \vec{k}_1^* + \cdots + C_s^T \vec{k}_s^* = \vec{0}. \tag{1}$$

Hence we obtain

$$\Sigma = \sum_{\boldsymbol{l} \in \mathbb{N}_0^s \setminus \{\boldsymbol{0}\}} \rho(2^m \boldsymbol{l}) + \sum_{\boldsymbol{k}^* \in \mathscr{D}} \sum_{\boldsymbol{l} \in \mathbb{N}_0^s} \rho(\boldsymbol{k}^* + 2^m \boldsymbol{l}) =: \Sigma_1 + \Sigma_2. \tag{2}$$

From Lemma 2 we find

$$\Sigma_1 = -1 + \left(1 + \frac{2}{2^{2m}}\right)^s. \tag{3}$$

We consider

$$\Sigma_2 = \sum_{\boldsymbol{k} \in \mathscr{D}} \prod_{i=1}^{s} \left(\sum_{l=0}^{\infty} \rho(k_i + 2^m l) \right).$$

If $k_i = 0$, then we have

$$\sum_{l=0}^{\infty} \rho(k_i + 2^m l) = \sum_{l=0}^{\infty} \rho(2^m l) = 1 + \frac{2}{2^{2m}}.$$

Assume that $k_i \neq 0$. Then we have

$$\begin{aligned} \sum_{l=0}^{\infty} \rho(k_i + 2^m l) &= \sum_{l=0}^{\infty} 2^{-2\lfloor \log_2(k_i + 2^m l) \rfloor} \\ &\leqslant \sum_{l=0}^{\infty} \frac{4}{(k_i + 2^m l)^2} \end{aligned}$$

$$\leqslant \frac{4}{(k_i)^2} + \sum_{l=1}^{\infty} \frac{4}{2^{2m} l^2}$$
$$\leqslant 4\rho(k_i) + \frac{4\zeta(2)}{2^{2m}}.$$

Hence in both cases we have

$$\sum_{l=0}^{\infty} \rho(k_i + 2^m l) \leqslant 4\rho(k_i) + \frac{4\zeta(2)}{2^{2m}}.$$

Now

$$\begin{aligned}\Sigma_2 &\leqslant \sum_{\boldsymbol{k} \in \mathscr{D}} \prod_{i=1}^{s} \left(4\rho(k_i) + \frac{4\zeta(2)}{2^{2m}} \right) \\ &= 4^s \sum_{\boldsymbol{k} \in \mathscr{D}} \rho(\boldsymbol{k}) + \frac{4^s \zeta(2)^s}{2^{2sm}} (\#\mathscr{D}) \\ &\quad + \sum_{j=1}^{s-1} \left(\frac{4\zeta(2)}{2^{2m}} \right)^{s-j} \sum_{1 \leqslant i_1 < \cdots < i_j \leqslant s} \sum_{\boldsymbol{k} \in \mathscr{D}} \rho(k_{i_1}) \cdots \rho(k_{i_j}).\end{aligned}$$

Let $1 \leqslant j \leqslant s-1$ and $1 \leqslant i_1 < \cdots < i_j \leqslant s$ be fixed. Then

$$\sum_{\boldsymbol{k} \in \mathscr{D}} \rho(k_{i_1}) \cdots \rho(k_{i_j}) = \sum_{h_1, \ldots, h_j = 0}^{2^m - 1} \rho(h_1) \cdots \rho(h_j) \ \#\{\boldsymbol{k} \in \mathscr{D} : k_{i_d} = h_d, 1 \leqslant d \leqslant j\}.$$

If in the system

$$C_1^T \vec{k}_1 + \cdots + C_s^T \vec{k}_s = \vec{0} \tag{4}$$

the values of $\vec{k}_{i_1}, \ldots, \vec{k}_{i_j}$ are prescribed and $s - j - 1$ of the remaining $\vec{k}$'s are chosen arbitrarily, then for the remaining $\vec{k}$ there is exactly one possible choice such that the equation in (4) holds. This follows since $C_1, \ldots, C_s$ are regular. Hence,

$$\#\{\boldsymbol{k} \in \mathscr{D} \ : \ k_{i_d} = h_d, \ 1 \leqslant d \leqslant j\} \leqslant 2^{m(s-j-1)}.$$

Thus with Lemma 2 we get

$$\sum_{\boldsymbol{k} \in \mathscr{D}} \rho(k_{i_1}) \cdots \rho(k_{i_j}) \leqslant 2^{m(s-j-1)} \sum_{h_1, \ldots, h_j = 0}^{2^m - 1} \rho(h_1) \cdots \rho(h_j) \leqslant 2^{m-(s-j-1)} 3^j.$$

In the same way as above it follows that $\#\mathscr{D} = 2^{m(s-1)}$. Therefore we obtain

$$\begin{aligned}\Sigma_2 &\leqslant 4^s \sum_{\boldsymbol{k} \in \mathscr{D}} \rho(\boldsymbol{k}) + \frac{4^s \zeta(2)^s}{2^{ms+m}} + \sum_{j=1}^{s-1} \binom{s}{j} \left(\frac{4\zeta(2)}{2^{2m}} \right)^{s-j} 2^{m(s-j-1)} 3^j \\ &= 4^s \sum_{\boldsymbol{k} \in \mathscr{D}} \rho(\boldsymbol{k}) + \frac{1}{2^m} \left(3 + \frac{4\zeta(2)}{2^m} \right)^s - \frac{3^s}{2^m}.\end{aligned}$$

We get

$$\begin{aligned}\Sigma &\leqslant 4^s \sum_{\boldsymbol{k} \in \mathscr{D}} \rho(\boldsymbol{k}) + \frac{1}{2^m}\left(3+\frac{4\zeta(2)}{2^m}\right)^s - \frac{3^s}{2^m} - 1 + \left(1+\frac{2}{2^{2m}}\right)^s \\ &\leqslant 4^s \sum_{\boldsymbol{k} \in \mathscr{D}} \rho(\boldsymbol{k}) + \frac{1}{2^m}\frac{4\zeta(2)s}{2^m}\left(3+\frac{4\zeta(2)}{2^m}\right)^{s-1} + \frac{2s}{2^{2m}}\left(1+\frac{2}{2^{2m}}\right)^{s-1}.\end{aligned}$$

The upper bound follows. Finally from (2) and (3) we get

$$\Sigma \geqslant -1 + \left(1+\frac{2}{2^{2m}}\right)^s + \sum_{\boldsymbol{k} \in \mathscr{D}} \rho(\boldsymbol{k})$$

such that also the lower bound is proved. □

Lemma 3. *Let $C_1, \ldots, C_s$ be the generator matrices of a digital (t,m,s)-net over $\mathbb{Z}_2$. For $\mathfrak{v} = \{v_1, \ldots, v_e\} \subseteq \{1, \ldots, s\}$, $\mathfrak{v} \neq \emptyset$, let*

$$\mathscr{B}(\mathfrak{v}) = \sum_{\substack{k_1,\ldots,k_e=1 \\ C_{v_1}^T \vec{k}_1 + \cdots + C_{v_e}^T \vec{k}_e = \vec{0}}}^{2^m-1} \prod_{j=1}^{e} \rho(k_j).$$

Then we have

$$\mathscr{B}(\mathfrak{v}) \leqslant \frac{2^{2t}}{2^{2m}}\left(\frac{16}{3}\right)^{|\mathfrak{v}|}\left(m-t+\frac{1}{8}\right)^{|\mathfrak{v}|-1}.$$

Proof. The result follows from [4, Lemma 7] by noting that for $k \neq 0$ we have $\rho(k) = 6\psi(k)$, where ψ is defined in [4]. □

In order to obtain an upper bound on the dyadic diaphony we need to establish an upper bound on $\sum_{\boldsymbol{k} \in \mathscr{D}} \rho(\boldsymbol{k})$. This is done in the following proposition.

Proposition 2. *Let $C_1, \ldots, C_s$ be the generator matrices of a digital (t,m,s)-net over $\mathbb{Z}_2$ with $t < m$. Then we have*

$$\sum_{\boldsymbol{k} \in \mathscr{D}} \rho(\boldsymbol{k}) \leqslant \frac{2^{2t}}{2^{2m}} 7^s (m-t)^{s-1}.$$

Proof. With Lemma 3 we have

$$\sum_{\boldsymbol{k} \in \mathscr{D}} \rho(\boldsymbol{k}) = \sum_{\substack{k_1,\ldots,k_s=0 \\ (k_1,\ldots,k_s) \neq (0,\ldots,0) \\ C_1^T \vec{k}_1 + \cdots + C_s^T \vec{k}_s = \vec{0}}}^{2^m-1} \prod_{i=1}^{s} \rho(k_i) = \sum_{\substack{\mathfrak{v} \subseteq \{1,\ldots,s\} \\ \mathfrak{v} \neq \emptyset}} \mathscr{B}(\mathfrak{v}) \tag{5}$$

$$\leqslant \frac{2^{2t}}{2^{2m}} \frac{1}{m-t} \sum_{\substack{\mathfrak{v} \subseteq \{1,\ldots,s\} \\ \mathfrak{v} \neq \emptyset}} \left(\frac{16}{3}\right)^{|\mathfrak{v}|}\left(m-t+\frac{1}{8}\right)^{|\mathfrak{v}|}$$

$$\leqslant \frac{2^{2t}}{2^{2m}} \frac{1}{m-t} \left(1 + \frac{16}{3}\left(m - t + \frac{1}{8}\right)\right)^s$$

$$\leqslant \frac{2^{2t}}{2^{2m}} 7^s (m-t)^{s-1}. \qquad (6)$$

This is the desired result. □

The following theorem follows now easily from Propositions 1 and 2.

Theorem 1. *Let $P^{\text{net}}_{2^m,s}$ be a digital (t,m,s)-net over $\mathbb{Z}_2$ with $t < m$ and with regular generator matrices $C_1, \ldots, C_s$. Then we have*

$$F^2_{2^m}(P^{\text{net}}_{2^m,s}) \leqslant c(s) 2^{2t} \frac{(m-t)^{s-1}}{2^{2m}},$$

where $c(s) > 0$ only depends on the dimension s.

As the upper bound also depends on the t-value of the digital net we consider (t,s)-sequences in the following. A (t,s)-sequence in base 2 is a sequence of points $(\boldsymbol{x}_n)_{n \geqslant 0}$ such that for all $m > t$ and $l \geqslant 0$ we have that $\{\boldsymbol{x}_n : l2^m \leqslant n < (l+1)2^m\}$ is a (t,m,s)-net in base 2. A digital (t,s)-sequence over $\mathbb{Z}_2$ is obtained by using $\mathbb{N} \times \mathbb{N}$ generator matrices $C_1, \ldots, C_s$ over $\mathbb{Z}_2$, see [19] for more information.

From [20] it follows that for every dimension s there exists a digital (t,s)-sequence over $\mathbb{Z}_2$ such that $t \leqslant 5s$. Thus it follows that for all $s \geqslant 1$ and $m > 5s$ there is a digital $(5s,m,s)$-net over $\mathbb{Z}_2$. (Note that if there is a digital (t,m,s)-net then it follows that also a digital $(t+1,m,s)$-net exists.) As we can now increase m without changing the t-value we obtain the following corollary from Theorem 1.

Corollary 1. *For any dimension $s \geqslant 1$ and $m > 5s$ there exists a digital net $P^{\text{net}}_{N,s}$ over $\mathbb{Z}_2$ consisting of $N = 2^m$ points such that*

$$F_N(P^{\text{net}}_{N,s}) \leqslant c'(s) \frac{(\log N)^{\frac{s-1}{2}}}{N},$$

where the constant $c'(s) > 0$ only depends on the dimension s.

In the following we also prove a lower bound on the dyadic diaphony which shows that the convergence rate shown in Corollary 1 is best possible. This is done using Roth's lower bound on the $\mathscr{L}_2$ discrepancy, which is another measure for the distribution properties of a point set. In the proof below we use the generalized notion of weighted $\mathscr{L}_2$ discrepancy, which was introduced in [24]. In the following let D denote the index set $D = \{1, 2, \ldots, s\}$ and let $\gamma = (\gamma_1, \gamma_2, \ldots)$ be a sequence of non-negative real numbers. For $u \subseteq D$ let $|u|$ be the cardinality of u and for a vector $\boldsymbol{x} \in [0,1)^s$ let $\boldsymbol{x}_u$ denote the vector from $[0,1)^{|u|}$ containing all components of $\boldsymbol{x}$ whose indices are in u. Further let $\gamma_u = \prod_{j \in u} \gamma_j$, $d\boldsymbol{x}_u = \prod_{j \in u} dx_j$ and let $(\boldsymbol{x}_u, 1)$ be the vector from $[0,1)^s$ with all components whose indices are not in u replaced by 1. Then the weighted $\mathscr{L}_2$ discrepancy of a point set $P_{N,s} = \{\boldsymbol{x}_0, \ldots, \boldsymbol{x}_{N-1}\}$ is defined as

$$\mathscr{L}_{2,\gamma}(P_{N,s}) = \left(\sum_{\substack{u \subseteq D \\ u \neq \emptyset}} \gamma_u \int_{[0,1]^{|u|}} \Delta((\boldsymbol{x}_u, 1))^2 d\boldsymbol{x}_u \right)^{1/2},$$

where

$$\Delta(t_1,\dots,t_s)=\frac{A_N([0,t_1)\times\cdots\times[0,t_s))}{N}-t_1\cdots t_s,$$

where $0\leqslant t_j\leqslant 1$ and $A_N([0,t_1)\times\cdots\times[0,t_s))$ denotes the number of indices n with $\boldsymbol{x}_n\in[0,t_1)\times\cdots\times[0,t_s)$. We can see from the definition of the weighted $\mathscr{L}_2$ discrepancy that the weights $\gamma_u=\prod_{j\in u}\gamma_j$ modify the importance of different projections (see [6], [24] for more information on weights).

Theorem 2. *For any dimension $s\geqslant 1$ there exists a constant $\bar{c}(s)>0$, depending only on the dimension s, such that for any point set $P_{N,s}$ consisting of N points in $[0,1)^s$ we have*

$$F_N(P_{N,s})\geqslant\bar{c}(s)\frac{(\log N)^{\frac{s-1}{2}}}{N}.$$

Proof. In [2] it was shown that the expected value of the weighted $\mathscr{L}_2$ discrepancy of a point set $\widetilde{P}_{N,s}$, which is randomized by a digital shift in base 2, is given by

$$\mathbb{E}(\mathscr{L}^2_{2,(12)}(\widetilde{P}_{N,s}))=\sum_{\substack{\boldsymbol{k}\in\mathbb{N}_0^s\\ \boldsymbol{k}\neq\boldsymbol{0}}} r(\gamma,\boldsymbol{k})\left|\frac{1}{N}\sum_{h=0}^{N-1}\mathrm{wal}_k(\boldsymbol{x}_h)\right|^2,$$

where $\boldsymbol{k}=(k_1,\dots,k_s)\in\mathbb{N}_0^s$, $r(\gamma,\boldsymbol{k})=\prod_{j=1}^s r(\gamma_j,k_j)$ and for $k=0$ we have $r(\gamma_j,0)=1+\gamma_j/3$ and for $k\geqslant 1$ we have $r(\gamma_j,k)=\gamma_j\rho(k)/12$. Hence for $\gamma_j=12$ we have $r(\gamma_j,0)=5=5\rho(0)$ and for $k\geqslant 1$ we have $r(12,k)=\rho(k)$. Therefore we have

$$r((12),\boldsymbol{k})=\prod_{i=1}^s r(12,k_i)\leqslant 5^s\prod_{i=1}^s\rho(k_i)=5^s\rho(\boldsymbol{k}).$$

Hence from the definition of dyadic diaphony we obtain the inequality

$$\mathbb{E}(\mathscr{L}^2_{2,(12)}(\widetilde{P}_{N,s}))\leqslant 5^s(3^s-1)F^2_{2,N}(P_{N,s}). \tag{7}$$

Roth [23] proved that for any dimension $s\geqslant 1$ there exists a constant $\hat{c}(s)>0$ such that for any point set consisting of N points in the s-dimensional unit-cube $[0,1)^s$ the classical $\mathscr{L}_2$ discrepancy of a point set satisfies

$$\mathscr{L}_2^2(P_{N,s})\geqslant\hat{c}(s)\frac{(\log N)^{s-1}}{N^2}.$$

Here we just note that the weights only change the constant $\hat{c}(s)$, but do not change the convergence rate of the bound (see [4], [24] for more information). Hence, for any point set $P_{N,s}$ consisting of N points in the s-dimensional unit-cube there is a constant $\tilde{c}(s)$, depending only on the dimension s, such that

$$\mathscr{L}^2_{2,(12)}(P_{N,s})\geqslant\tilde{c}(s)\frac{(\log N)^{s-1}}{N^2}.$$

Now from (7) it follows that there is a constant $\bar{c}(s)$, depending only on the dimension, such that

$$F_{2,N}^2(P_{N,s}) \geqslant \bar{c}^2(s)\frac{(\log N)^{s-1}}{N^2}.$$

The result follows. □

By keeping track of the constant $c'(s)$ in Corollary 1 we observe that $c'(s)$ is growing exponentially with the dimension s. Hence, though we obtain the best possible convergence of $O((\log N)^{(s-1)/2}N^{-1})$ of the dyadic diaphony of digital nets, the upper bound in Corollary 1 is only smaller than 1 if N is exponentially large in the dimension s. Recall that for any point set $P_{N,s}$ consisting of N elements we have $F_N(P_{N,s}) \leqslant 1$, hence only if N is large compared to the dimension s we can observe the best possible convergence order. One might improve the constant $c'(s)$ so that it actually decays with the dimension (this was done for the $\mathscr{L}_2$ discrepancy, see [4]), but still the $\log N$ factor grows exponentially with the dimension, causing the bound to be greater than one in many practical cases. Note that for high dimensional problems choosing a number of points N exponentially large in the dimension is often not feasible.

This problem is addressed in the next section where we consider the case when N is small compared to the dimension.

4. The Limiting Dyadic Diaphony

In many applications we require high dimensional point sets which are well distributed. In order to investigate the behaviour of such point sets in very high dimensions we introduce the concept of limiting dyadic diaphony (see also [24], where a limiting discrepancy was introduced). Let P_N be a point set in $[0,1)^\infty$ with N elements and let $P_{N,s}$ be the projection of P_N to the first s coordinates. Then we have

$$\begin{aligned} F_N^2(P_{N,s}) &= \frac{1}{3^s-1}\left(\frac{1}{N^2}\sum_{l,n=0}^{N-1}\sum_{\substack{\boldsymbol{k}\in\mathbb{N}_0^s\\ \boldsymbol{k}\neq\boldsymbol{0}}}\rho(\boldsymbol{k})\mathrm{wal}_{\boldsymbol{k}}(\boldsymbol{x}_l)\mathrm{wal}_{\boldsymbol{k}}(\boldsymbol{x}_n)\right)\\ &= -\frac{1}{3^s-1}+\frac{1}{3^s-1}\frac{1}{N^2}\sum_{l,n=0}^{N-1}\prod_{j=1}^{s}\left(\sum_{k=0}^{\infty}\rho(k)\mathrm{wal}_k(x_{l,j})\mathrm{wal}_k(x_{n,j})\right)\\ &= -\frac{1}{3^s-1}+\frac{3^s}{3^s-1}\frac{1}{N^2}\sum_{l,n=0}^{N-1}\prod_{j=1}^{s}\left(\frac{1}{3}+\frac{1}{3}\sum_{k=1}^{\infty}\rho(k)\mathrm{wal}_k(x_{l,j})\mathrm{wal}_k(x_{n,j})\right). \end{aligned} \tag{8}$$

Note that

$$0 \leqslant \frac{1}{3}+\frac{1}{3}\sum_{k=1}^{\infty}\rho(k)\mathrm{wal}_k(x_{l,j})\mathrm{wal}_k(x_{n,j}) \leqslant 1, \tag{9}$$

which follows from [5, Theorem 1] and [3, Eq. (7)]. Hence it follows that

$$\frac{3^s}{3^s-1}\frac{1}{N^2}\sum_{l,n=0}^{N-1}\prod_{j=1}^{s}\left(\frac{1}{3}+\frac{1}{3}\sum_{k=1}^{\infty}\rho(k)\mathrm{wal}_k(x_{l,j})\mathrm{wal}_k(x_{n,j})\right)$$

is monotonically decreasing with increasing dimension. As it is also bounded from below by zero it follows that

$$\lim_{s\to\infty}\frac{3^s}{3^s-1}\frac{1}{N^2}\sum_{l,n=0}^{N-1}\prod_{j=1}^{s}\left(\frac{1}{3}+\frac{1}{3}\sum_{k=1}^{\infty}\rho(k)\mathrm{wal}_k(x_{l,j})\mathrm{wal}_k(x_{n,j})\right)$$

exists and hence also $\lim_{s\to\infty}F_N(P_{N,s})$ exists for any point set P_N in $[0,1)^{\infty}$.

Definition 3. The limiting dyadic diaphony $F_{N,\lim}(P_N)$ of a point set P_N in $[0,1)^{\infty}$ is defined by

$$F_{N,\lim}(P_N):=\lim_{s\to\infty}F_N(P_{N,s}),$$

where $P_{N,s}$ is the projection of P_N to the first s coordinates.

Recall that $0\leqslant F_N(P_{N,s})\leqslant 1$, hence we also have $0\leqslant F_{N,\lim}(P_N)\leqslant 1$. In the following we consider two choices of point sets in $[0,1)^{\infty}$.

From [5, Theorem 1] together with [2, Theorem 4.4] it follows that there exists a digital net $P_{2^m,s}^{\mathrm{net}}$, extensible in the dimension s, such that

$$F_{2^m}^2(P_{2^m,s}^{\mathrm{net}})\leqslant\frac{3^s}{3^s-1}\frac{1}{2^m-1}\quad\text{for all } s\geqslant 1.$$

(In [2] it is also shown that such a digital net can be found by computer search using a component-by-component algorithm, see [2, Algorithm 4.3], where $\gamma_j=1$ for all $j\geqslant 1$.) Hence for the infinite dimensional digital net $P_{2^m}^{\mathrm{net}}$ we obtain

$$F_{2^m,\lim}(P_{2^m}^{\mathrm{net}})\leqslant\frac{1}{\sqrt{2^m-1}}.$$

Note that in this case we can only obtain a convergence rate of $O(2^{-m/2})$. This is indeed best possible, as it follows from (8) and (9) that

$$\begin{aligned}F_N^2(P_{N,s})&=-\frac{1}{3^s-1}+\frac{3^s}{3^s-1}\frac{1}{N^2}\sum_{l,n=0}^{N-1}\prod_{j=1}^{s}\left(\frac{1}{3}+\frac{1}{3}\sum_{k=1}^{\infty}\rho(k)\mathrm{wal}_k(x_{l,j})\mathrm{wal}_k(x_{n,j})\right)\\&\geqslant-\frac{1}{3^s-1}+\frac{3^s}{3^s-1}\frac{1}{N^2}\sum_{n=0}^{N-1}\prod_{j=1}^{s}\left(\frac{1}{3}+\frac{1}{3}\sum_{k=1}^{\infty}\rho(k)\mathrm{wal}_k(x_{n,j})\mathrm{wal}_k(x_{n,j})\right)\\&=-\frac{1}{3^s-1}+\frac{3^s}{3^s-1}\frac{1}{N^2}\sum_{n=0}^{N-1}\prod_{j=1}^{s}\left(\frac{1}{3}+\frac{1}{3}\sum_{k=1}^{\infty}\rho(k)\right)\\&=\frac{1}{3^s-1}\left(-1+\frac{3^s}{N}\right).\end{aligned}\tag{10}$$

Therefore we obtain the following theorem.

Theorem 3. *For any point set P_N in $[0,1)^\infty$ consisting of N elements we have*

$$F_{N,\lim}(P_N) \geqslant \frac{1}{\sqrt{N}}.$$

Hence we also obtain the following theorem.

Theorem 4. *For a digital net $P_{2^m}^{\text{net}}$ constructed by a component-by-component algorithm we have*

$$\frac{1}{\sqrt{2^m}} \leqslant F_{2^m,\lim}(P_{2^m}^{\text{net}}) \leqslant \frac{1}{\sqrt{2^m-1}}.$$

In the following we also calculate the expected value of the limiting dyadic diaphony of a random sample $P_N^{\text{rand}} = \{\boldsymbol{x}_0, \ldots, \boldsymbol{x}_{N-1}\}$. As $\int_0^1 \text{wal}_0(x)\text{d}x = 1$ and $\int_0^1 \text{wal}_k(x)\text{d}x = 0$ for $k>0$ we obtain

$$\begin{aligned}\int_{[0,1)^{Ns}} |S_N(\boldsymbol{k}, P_N^{\text{rand}})|^2 \,\text{d}\boldsymbol{x}_0 \cdots \text{d}\boldsymbol{x}_{N-1} &= \frac{1}{N^2}\sum_{n,m=0}^{N-1}\int_{[0,1)^{2s}} \text{wal}_{\boldsymbol{k}}(\boldsymbol{x}_n)\text{wal}_{\boldsymbol{k}}(\boldsymbol{x}_m)\text{d}\boldsymbol{x}_n\text{d}\boldsymbol{x}_m \\ &= \frac{1}{N^2}\sum_{n=0}^{N-1} 1 = \frac{1}{N}.\end{aligned}$$

Therefore we obtain from Definition 1 and Lemma 2 that

$$\mathbb{E}(F_N^2(P_{N,s}^{\text{rand}})) = \int_{[0,1)^{Ns}} F_N^2(P_{N,s}^{\text{rand}})\text{d}\boldsymbol{x}_1 \cdots \text{d}\boldsymbol{x}_N = \frac{1}{N}. \tag{11}$$

Let $P_{N,s}$ be a point set with N elements in $[0,1)^s$. By adding k zeros in the remaining coordinates of each point we obtain a point set $P_{N,s+k}$ in $[0,1)^{s+k}$ and for $k=\infty$ we obtain a point set P_N in $[0,1)^\infty$. It can easily be verified that

$$(3^{s+1}-1)F_{N,s+1}^2(P_{N,s+1}) = 3(3^s-1)F_{N,s}^2(P_{N,s}) + 2$$

and therefore

$$(3^{s+k}-1)F_{N,s+k}^2(P_{N,s+k}) = 3^k(3^s-1)F_{N,s}^2(P_{N,s}) + 3^k - 1 \quad \text{for all } k \geqslant 0.$$

Hence the limiting dyadic diaphony for such a point set is given by

$$F_{N,\lim}^2(P_N) = \frac{3^s-1}{3^s}F_{N,s}^2(P_{N,s}) + \frac{1}{3^s}. \tag{12}$$

If we consider a point set $P_{N,s,0}^{\text{rand}}$ where the first s coordinates of each point are chosen randomly and the remaining coordinates are chosen to be zero, we obtain, by using (11), that the mean square limiting dyadic diaphony of this point set is given by

$$\mathbb{E}(F_{N,\lim}^2(P_{N,s,0}^{\text{rand}})) = \frac{3^s-1}{3^s}\frac{1}{N} + \frac{1}{3^s}.$$

Hence, by choosing all coordinates randomly, we obtain

$$\frac{1}{\sqrt{N}} \geqslant \sqrt{\mathbb{E}(F_{N,\lim}^2(P_N^{\text{rand}}))} \geqslant \mathbb{E}(F_{N,\lim}(P_N^{\text{rand}})) \geqslant \frac{1}{\sqrt{N}},$$

where the rightmost inequality follows from Theorem 3. We have shown the following theorem.

Theorem 5. *The expected value of the limiting dyadic diaphony of a random sample P_N^{rand} is given by*

$$\mathbb{E}(F_{N,\mathrm{lim}}(P_N^{\mathrm{rand}})) = \frac{1}{\sqrt{N}}.$$

5. Discussion

It follows from Theorem 5 that, in terms of the limiting dyadic diaphony, digital nets do not perform better than an average random sample. This may not come as a surprise as for very high dimensions we necessarily obtain a quality parameter $t = m$ for any digital net. On the other hand, Theorem 4 (using a component-by-component construction algorithm presented in [2]) yields a constructive approach to a point set with a limiting dyadic diaphony almost as small as possible. (Let $\varepsilon > 0$. Then, in order to obtain a limiting dyadic diaphony smaller than $(N-1)^{-1/2} + \varepsilon$, it is enough to construct only $s = \lceil -2\log_3 \varepsilon \rceil$ many dimensions as the rest of the coordinates can be chosen to be zero. This follows from (12). For example, for $N = 2^{30} \approx 10^9$ it would be enough to construct 28 dimensions to obtain a limiting dyadic diaphony of at most $1.01 \cdot (2^{30}-1)^{-1/2}$ and 32 dimensions to obtain a limiting dyadic diaphony of at most $1.001 \cdot (2^{30}-1)^{-1/2}$. This is feasible using a component-by-component algorithm.

It follows from (11) that for a finite dimensional random sample $P_{N,s}^{\mathrm{rand}}$ we have

$$\sqrt{\mathbb{E}(F_{N,s}^2(P_{N,s}^{\mathrm{rand}}))} = \frac{1}{\sqrt{N}}.$$

Thus it follows that

$$\sqrt{\mathbb{E}(F_{N,s}^2(P_{N,s}^{\mathrm{rand}}))} = \sqrt{\mathbb{E}(F_{N,\mathrm{lim}}^2(P_N^{\mathrm{rand}}))} \quad \text{for all } s \geqslant 1,$$

where $P_{N,s}$ is again the projection of P_N to the first s coordinates. Hence, the root mean square dyadic diaphony of a random sample is independent of the dimension. Corollary 1 on the other hand shows that for digital nets with a large enough number of points the dyadic diaphony converges faster than the limiting dyadic diaphony.

What we mean by a large enough number of points will be made more precise in the following. From (10) it follows that

$$F_N^2(P_{N,s}) \geqslant \frac{1}{3^s-1}\left(-1 + \frac{3^s}{N}\right)$$

for any point set $P_{N,s}$. Hence, for example for $0 < N < 3^s/2$ it follows that

$$F_N(P_{N,s}) \geqslant \sqrt{\frac{1}{2N}}.$$

Thus only if N is exponentially large in the dimension we can observe the convergence rate shown in Corollary 1.

References

[1] Chrestenson HE (1955) A class of generalized Walsh functions. Pacific J Math **5**: 17–31
[2] Dick J, Kuo F, Pillichshammer F, Sloan IH (2004) Construction algorithms for polynomial lattice rules for multivariate integration. Math Comp (to appear)
[3] Dick J, Pillichshammer F (2003) Multivariate integration in weighted Hilbert spaces based on Walsh functions and weighted Sobolev spaces. J Complexity (to appear)
[4] Dick J, Pillichshammer F (2003) On the mean square weighted L_2 discrepancy of randomized digital (t, m, s)-nets over $\mathbb{Z}_2$. Acta Arithm (to appear)
[5] Dick J, Pillichshammer F (2004) Diaphony, discrepancy, spectral test and worst-case error (submitted)
[6] Dick J, Sloan IH, Wang X, Woźniakowksi H (2004) Liberating the weights. J Complexity **20**(5): 593–623
[7] Drmota M, Tichy RF (1997) Sequences, discrepancies and applications. Lect Notes Math **1651**. Berlin Heidelberg New York: Springer
[8] Grozdanov V, Stoilova S (2001) On the theory of b-adic diaphony. C R Acad Bulg Sci **54**(3): 31–34
[9] Grozdanov V, Stoilova S (2004) The general diaphony. C R Acad Bulg Sci **57**(1): 13–18
[10] Grozdanov V, Nikolova E, Stoilova S (2003) Generalized b-adic diaphony. C R Acad Bulg Sci **56**(4): 23–30
[11] Hellekalek P (2002) Digital (t, m, s)-nets and the spectral test. Acta Arith **105**: 197–204
[12] Hellekalek P, Leeb H (1997) Dyadic diaphony. Acta Arith **80**: 187–196
[13] Hickernell FJ (1998) A generalized discrepancy and quadrature error bound. Math Comp **67**: 299–322
[14] Kuipers L, Niederreiter H (1974) Uniform Distribution of Sequences. New York: Wiley
[15] Larcher G (1998) Digital point sets: analysis and application. In: Hellekalek P, Larcher G (eds) Random and Quasi-Random Point Sets. Lect Notes Statistics **138**: 167–222. Berlin Heidelberg New York: Springer
[16] Larcher G, Niederreiter H, Schmid WCh (1996) Digital nets and sequences constructed over finite rings and their application to quasi-Monte Carlo integration. Monatsh Math **121**: 231–253
[17] Niederdrenk K (1982) Die endliche Fourier- und Walshtransformation mit einer Einführung in die Bildverarbeitung. Braunschweig: Vieweg
[18] Matoušek J (1999) Geometric Discrepancy. Berlin Heidelberg New York: Springer
[19] Niederreiter H (1992) Random Number Generation and Quasi-Monte Carlo Methods. CBMS-NSF Series in Appl Math **63** Philadelphia: SIAM
[20] Niederreiter H, Xing CP (2001) Rational Points on Curves over Finite Fields. London Math Soc Lect Notes Series **285**. Cambridge: Univ Press
[21] Pirsic G (1995) Schnell konvergierende Walshreihen über Gruppen. Master's Thesis, University of Salzburg (available at http://www.ricam.oeaw.ac.at/people/page/pirsic/)
[22] Rivlin TJ, Saff EB (2000) Joseph L. Walsh Selected Papers. Berlin Heidelberg New York: Springer
[23] Roth KF (1959) On irregularities of distribution. Mathematika **1**: 73–79
[24] Sloan IH, Woźniakowski H (1998) When are quasi-Monte Carlo algorithms efficient for high dimensional integrals? J Complexity **14**: 1–33
[25] Walsh JL (1923) A closed set of normal orthogonal functions. Amer J Math **55**: 5–24

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