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The Singularity Spectrum $f(\alpha)$ of Some Moran Fractals*

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Abstract. We show that the multifractal decomposition behaves as expected for a family of sets E known as homogeneous Moran fractals associated with the Fibonacci sequence ω, using probability measures $\mu(\omega)$ associated with the Fibonacci sequence ω. For each value of a parameter $\alpha \in (\alpha_{\min}, \alpha_{\max})$, we define 'multifractal components' E_α of E, and show that they are fractals in the sense of Taylor. We give the explicit formula for the dimension of E_α. Also our method can be used for the Moran fractals associated with some more general sequences.

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1. Introduction

Multifractal analysis has been proved to be a very useful technique in the analysis of singular measures, both in theory and applications; see, for example, [6, 7, 10] and references therein. Under certain circumstances, a measure μ gives rise to sets of points where μ has local density of exponent α, with the dimensions of these sets indicating the distribution of the singularities of the measure.

To be more precise, for a finite measure μ on $\mathbb{R}^d$, its pointwise dimension at x is defined as follows:

$$\dim_{\mathrm{loc}}\mu(x) = \lim_{r\to 0} \frac{\log \mu(B(x,r))}{\log 2r},$$

if this limit exists. For $\alpha > 0$, define

$$K_\alpha = \left\{ x \in K : \lim_{r\to 0} \log \mu(B(x,r)) / \log 2r = \alpha \right\} \tag{1.1}$$

where $B(x,r)$ is the closed ball with center x and radius r. The set K_α may be thought of as the set where the 'local dimension' of K equals α or as a 'multifractal component' of K.

The main problem in multifractal analysis is to estimate the size of K_α. This is done by calculating the functions $f_\mu(\alpha) = \dim K_\alpha$; $F_\mu(\alpha) = \mathrm{Dim} K_\alpha$ for $\alpha \geqslant 0$, where dim and Dim denote the Hausdorff dimension and Packing dimension, respectively. These functions are generally known as the "multifractal spectrum"

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of μ, or "the singularity spectrum" of μ. Heuristic arguments, using techniques of Statistical Mechanics (see [10] for example), show that the singularity spectrum should be finite on a compact interval, noted by Dom (μ), and is expected to the Legendre transform conjugate of the L^q-spectrum τ associated with μ (see definition below), that is, for all $\alpha \in \mathrm{Dom}(\mu)$,

$$f_\mu(\alpha) = \inf\{\alpha q + \tau(q); q \in \mathbb{R}\} =: \tau^*(\alpha). \tag{1.2}$$

The multifractal analysis of a probability measure is concerned with rigorous arguments insuring that the Legendre transform formula (1.2) holds. The rigorous arguments for multifractal formalisms have been established for Gibbs measures (see, [1, 3, 14, 15]) and graph directed self-similar measures (see [2, 4]). The aim of the present paper is to discuss multifractal structures of a particular type of fractal called a Moran fractal. It should be pointed out that the Moran fractal discussed in this paper is quite different from that in [2]. The Moran fractals in [2] are constructed by an iterative procedure using a given fixed number of similarity ratios. In our case, the contraction ratios and the number of ratios may be different at each step, and the measure associated with this kind of structure is neither Gibbs nor self-similar. We cannot do most of the work on a symbolic space and then transfer the results to subsets of $\mathbb{R}^d$, as done in the ordinary way. Our proofs are for a class of homogeneous Moran fractals associated with Fibonacci sequence; Using the arguments of this paper it is easy to extend the results to a larger class of homogeneous Moran fractals associated with the sequences of which the frequency of the letter exists. We have restricted to the special case in order to get to the heart of the problem without introducing unnecessary technical details.

2. Definitions and Results

This section contains some basic definitions and the main results of this paper. The proofs will be given in Section 3.

2.1. Moran set. Let $\{n_k\}_{k \geqslant 1}$ be a sequence of positive integers and $\{c_k\}_{k \geqslant 1}$ be a sequence of positive numbers satisfying that

$$n_k \geqslant 2, \qquad 0 < c_k < 1, \quad n_k c_k \leqslant 1 \quad \textit{for } k \geqslant 1.$$

Define $D_0 = \phi$, and for any $k \geqslant 1$, set

$$D_{m,k} = \{(i_m, i_{m+1}, \ldots, i_k); \quad 1 \leqslant i_j \leqslant n_j, \quad m \leqslant j \leqslant k\}$$

and $D_k = D_{1,k}$. Define $D = \bigcup_{k \geqslant 0} D_k$. If $\sigma = (\sigma_1, \ldots, \sigma_k) \in D_k$, $\tau = (\tau_1, \ldots, \tau_m) \in D_{k+1,m}$, let $\sigma * \tau = (\sigma_1, \ldots, \sigma_k, \tau_1, \ldots, \tau_m)$.

Definition 2.1. Suppose J is a closed interval of length 1. The collection $\mathscr{F} = \{J_\sigma, \sigma \in D\}$ of closed subintervals of J is called having homogeneous Moran structure, if it satisfies the following conditions:

(i) $j_\phi = j$;

(ii) For all $k \geqslant 0$ and $\sigma \in D_k$, $J_{\sigma*1}, J_{\sigma*2}, \ldots, J_{\sigma*n_{k+1}}$ are subintervals of J_σ, and satisfy that $J^\circ_{\sigma*i} \cap J^\circ_{\sigma*j} = \phi (i \neq j)$, where A° denotes the interior of A.

(iii) For any $k \geqslant 1$, $\sigma \in D_{k-1}$, $c_k = \frac{|J_{\sigma*j}|}{|J_\sigma|}$, $1 \leqslant j \leqslant n_k$ where $|A|$ denotes the diameter of A.

Suppose that $\mathscr{F}$ is a collection of closed subintervals of J having homogeneous Moran structure, we call $E(\mathscr{F}) = \bigcap_{k \geqslant 1} \bigcup_{\sigma \in D_k} J_\sigma$ a homogeneous Moran set determined by $\mathscr{F}$. we often use $M(J, \{n_k\}, \{c_k\})$ to denote the collection of homogeneous Moran sets determined by J, $\{n_k\}$ and $\{c_k\}$. A more general Moran fractal structure was proposed in [9].

Remark 2.1. If $\lim_{n\to\infty} \sup_{\sigma \in D_n} |J_\sigma| > 0$, then E contains interior points. Thus the measure and dimension properties will be trivial. We assume therefore

$$\lim_{n\to\infty} \sup_{\sigma \in D_n} |J_\sigma| = 0.$$

Let $A = \{a, b\}$ be a two-letter alphabet, and A^* the free monoid generated by A. Consider the following homomorphism on A^*, $F : a \to ab$, $b \to a$, we see that $F^n(a) = F^{n-1}(a)F^{n-2}(a)$, thus $F^n(a)$, as $n \to \infty$, will define an infinite sequence

$$\omega = \lim_{n\to\infty} F^n(a) = s_1 s_2 s_3 \cdots s_n \cdots \in \{a, b\}^{\mathbb{N}},$$

which is called the Fibonacci sequence.

Let $F^n(a) = s_1 s_2 \cdots s_{|F^n(a)|}$, $s_i \in A$, where $|F^n(a)|$ denotes the length of the word $F^n(a)$. For any $n \geqslant 1$, write $\omega_n = \omega|_n = s_1 s_2 \cdots s_n$, $|\omega_n| = n$. We denote by $|\omega_n|_a$ the number of the occurrence of the letter a in ω_n, and $|\omega_n|_b$ the number of occurrence of b. Then $|\omega_n|_a + |\omega_n|_b = n$.

By [16], we know that $\lim_{n\to\infty} \frac{|\omega_n|_a}{n} = \eta$, where $\eta^2 + \eta = 1$, For more details on substitutive sequence and related properties, we refer to [16].

Let $0 < r_a < \frac{1}{2}$, $0 < r_b < \frac{1}{3}$, $r_a, r_b \in \mathbb{R}$. In the Moran construction above, let

$$|J| = 1, \quad n_k = \begin{cases} 2, & \text{if } s_k = a \\ 3, & \text{if } s_k = b \end{cases};$$

$$c_{k_j} = c_k = \begin{cases} r_a, & \text{if } s_k = a \\ r_b, & \text{if } s_k = b \end{cases}, \quad 1 \leqslant j \leqslant n_k.$$

Assume that $\forall k \geqslant 1$, $\forall \sigma \in D_k$ and $1 \leqslant j \leqslant n_{k+1}$, for the $k+1$-order fundamental element $J_{\sigma * j} \subset J_\sigma$, $d(J_{\sigma * i}, J_{\sigma * j}) \geqslant \Delta_k |J_\sigma|$ for all $i \neq j$, where $\{\Delta_k\}$ is a sequence of positive reals. Let $\Delta = \inf \Delta_k$.

Then we construct the homogeneous Moran set relating to the Fibonacci sequence and denote it by $E(\omega) = (J, \{n_k\}, \{c_k\})$. By [9], we have that $\dim_H E = \liminf_{n\to\infty} d_n$, $\dim_P E = \limsup_{n\to\infty} d_n$, where d_n fulfills $\prod_{i=1}^k n_i c_i^{d_k} = 1$. In our setting

$$d_k = \frac{-|\omega_k|_a \log 2 - |\omega_k|_b \log 3}{|\omega_k|_a \log r_a + |\omega_k|_b \log r_b},$$

and

$$\dim_H E = \dim_P E = \lim_{k\to\infty} d_k = \frac{-\log 2 - \eta \log 3}{\log r_a + \eta \log r_b},$$

where $\eta^2 + \eta = 1$.

By the construction of $E(\omega)$, we have $\forall \sigma \in D_k$, $|J_\sigma| = r_a^{|\omega_k|_a} r_b^{|\omega_k|_b}$.

2.2. Measure. Let $P_a = (P_{a_1}, P_{a_2})$, $P_b = (P_{b_1}, P_{b_2}, P_{b_3})$ be probability vectors, i.e. $P_{a_i} > 0$, $P_{b_i} > 0$, and $\sum_{i=1}^{2} P_{a_i} = 1$, $\sum_{i=1}^{3} P_{b_i} = 1$. For any $k \geqslant 1$, $\sigma \in D_k$, from Section 2.1, we know $\sigma = \sigma_1\sigma_2\cdots\sigma_k$ where

$$\sigma_k \in \begin{cases} \{1,2\}, & \text{if } s_k = a \\ \{1,2,3\}, & \text{if } s_k = b \end{cases}.$$

For $\sigma = \sigma_1\sigma_2\cdots\sigma_k$, define $\sigma(a)$ as follows: let $\omega_k = s_1s_2\cdots s_k$, $e_1 < e_2 < \cdots < e_{|\omega_k|_a}$ be the occurrences of the letter a in ω_k, then $\sigma(a) = \sigma_{e_1}\sigma_{e_2}\cdots\sigma_{e_{|\omega_k|_a}}$. Similarly, let $\delta_1 < \delta_2 < \cdots < \delta_{|\omega_k|_b}$ be the occurrences of the letter b in ω_k, then $\sigma(b) = \sigma_{\delta_1}\sigma_{\delta_2}\cdots\sigma_{\delta_{|\omega_k|_b}}$.

Now define

$$P_{\sigma(a)} = P_{\sigma_{e_1}}P_{\sigma_{e_2}}\cdots P_{\sigma_{e_{|\omega_k|_a}}},$$

$$P_{\sigma(b)} = P_{\sigma_{\delta_1}}P_{\sigma_{\delta_2}}\cdots P_{\sigma_{\delta_{|\omega_k|_b}}}.$$

It is obvious that $\Sigma_{\sigma\in D_k} P_{\sigma(a)}P_{\sigma(b)} = 1$.

Let μ be a mass distribution on E, such that for any $\sigma \in D_k$,

$$\mu(J_\sigma) = P_{\sigma(a)}P_{\sigma(b)},$$

since μ is relating with ω, we denote it by $\mu(\omega)$.

2.3. The multifractal dimension function. Let us briefly recall the notations and the main results proved by Olsen [12]. In the sequel, μ is a Borel probability measure on $\mathbb{R}^d$. Let K be a nonempty subset of $\mathbb{R}^d$. For any q, $t \in \mathbb{R}$ and $\delta > 0$, we introduce the quantities

$$\bar{\mathscr{H}}^{q,t}_{\mu,\delta}(K) = \inf\left\{\sum_i \mu(B(x_i,r_i))^q(2r_i)^t; \{B(x_i,r_i)\} \text{ is a centered } \delta\text{-covering of } K\right\},$$

$$\bar{\mathscr{H}}^{q,t}_{\mu}(K) = \sup_{\delta>0} \bar{\mathscr{H}}^{q,t}_{\mu,\delta}(K),$$

$$\bar{\mathscr{P}}^{q,t}_{\mu,\delta}(K) = \sup\left\{\sum_i \mu(B(x_i,r_i))^q(2r_i)^t; \{B(x_i,r_i)\} \text{ is a centered } \delta\text{-packing of } K\right\},$$

$$\bar{\mathscr{P}}^{q,t}_{\mu}(K) = \inf_{\delta>0} \bar{\mathscr{P}}^{q,t}_{\mu,\delta}(K).$$

The function $\bar{\mathscr{H}}^{q,t}_{\mu}$ is σ-subadditive but not increasing and the function $\bar{\mathscr{P}}^{q,t}_{\mu}$ is increasing but not σ-subadditive. That is the reason why Olsen introduced the following modifications of $\bar{\mathscr{H}}^{q,t}_{\mu}$ and $\bar{\mathscr{P}}^{q,t}_{\mu}$:

$$\mathscr{H}^{q,t}_{\mu}(K) = \sup_{F\subset K} \bar{\mathscr{H}}^{q,t}_{\mu}(F), \qquad \mathscr{P}^{q,t}_{\mu}(K) = \inf_{K\subset\cup K_i} \sum_i \bar{\mathscr{P}}^{q,t}_{\mu}(K_i).$$

The functions $\mathscr{H}^{q,t}_{\mu}$ and $\mathscr{P}^{q,t}_{\mu}$ are outer measures (in the Carathéodory sense) for which Borel sets are measurable. They are multifractal extensions of the Hausdorff measure $\mathscr{H}^t$ and the packing measure $\mathscr{P}^t$, for more details on the measures $\mathscr{H}^t$ and $\mathscr{P}^t$, see [5].

The measures $\mathscr{H}_\mu^{q,t}$, $\mathscr{P}_\mu^{q,t}$ assign in the usual way a dimension to each subset K of $\mathbb{R}^d$. They are respectively denoted by $\dim_\mu^q(K)$, $\mathrm{Dim}_\mu^q(K)$ and characterized by:

$$\mathscr{H}_\mu^{q,t}(K) = \begin{cases} \infty, & \text{for } t < \dim_\mu^q(K), \\ 0, & \text{for } t > \dim_\mu^q(K), \end{cases}$$

$$\mathscr{P}_\mu^{q,t}(K) = \begin{cases} \infty, & \text{for } t < \mathrm{Dim}_\mu^q(K), \\ 0, & \text{for } t > \mathrm{Dim}_\mu^q(K). \end{cases}$$

The number $\dim_\mu^q(K)$ is a multifractal extension of the Hausdorff dimension $\dim(K)$ of K whereas the number $\mathrm{Dim}_\mu^q(K)$ is a multifractal extension of the packing dimension $\mathrm{Dim}(K)$. More precisely, we have the equalities

$$\dim(K) = \dim_\mu^0(K), \qquad \mathrm{Dim}(K) = \mathrm{Dim}_\mu^0(K).$$

We can also remark that $\dim_\mu^q(K) \leqslant \mathrm{Dim}_\mu^q(K)$. Then we are able to define the multifractal dimension functions b_μ, $B_\mu : \mathbb{R} \to [-\infty, +\infty]$ by

$$b_\mu(q) = \dim_\mu^q(\operatorname{supp}\mu), \qquad B_\mu(q) = \mathrm{Dim}_\mu^q(\operatorname{supp}\mu).$$

These functions satisfy the following properties:

Proposition 2.1 [12]. *Let μ be a probability measure on $\mathbb{R}^d$. Then*

(i) $b_\mu(1) = B_\mu(1) = 0$;
(ii) $b_\mu(0) = \dim(\operatorname{supp}\mu)$, $B_\mu(0) = \mathrm{Dim}(\operatorname{supp}\mu)$;
(iii) $b_\mu \leqslant B_\mu$;
(iv) *b_μ is decreasing and B_μ is convex and decreasing.*

The functions b_μ and B_μ are related to the multifractal spectrum of the measure μ. More precisely, if $f^*(x) = \inf_y(xy + f(y))$ denotes the Legendre transform of the function f, Olsen rigorously proved the following statement.

Proposition 2.2 [12]. *Let μ be a probability measure on $\mathbb{R}^d$. Define $\underline{\alpha} = \sup_{q>0}\frac{-b(q)}{q}$ and $\bar{\alpha} = \inf_{q<0}\frac{-b(q)}{q}$. For all $\alpha \in (\underline{\alpha}, \bar{\alpha})$, we have*

$$\dim\{x \in \operatorname{supp}\mu; \dim_{\mathrm{loc}}\mu(x) = \alpha\} \leqslant b_\mu^*(\alpha).$$

$$\mathrm{Dim}\{x \in \operatorname{supp}\mu; \dim_{\mathrm{loc}}\mu(x) = \alpha\} \leqslant B_\mu^*(\alpha).$$

It is more difficult to obtain a minoration for the dimensions of the sets described in the proposition. Nasr et al. in [8] give a new sufficient condition for a valid multifractal formalism as follows.

Theorem 2.1 [8]. *Let μ be a probability measure on $\mathbb{R}^d$ and $q \in \mathbb{R}$, suppose that $B'_\mu(q)$ exists, if $\bar{\mathscr{H}}_\mu^{q,B_\mu(q)}(\operatorname{supp}\mu) > 0$, then*

$$\dim(K(-B'_\mu(q))) = \mathrm{Dim}K(-B'_\mu(q)) = B^*(-B'_\mu(q)) = b^*(-B'_\mu(q)).$$

2.4. Main results. From now on, we assume that $E(\omega)$ is a Moran fractal defined in section 2.1, and $\mu(\omega)$ is a probability measure introduced in section

2.2. The notations $D_k, J_\sigma, P_{\sigma(a)}, P_{\sigma(b)}$ are as above, in the following, we denote $E(\omega)$ by E, and $\mu(\omega)$ by μ for simplicity. Let

$$E(\alpha) = \left\{ x \in \operatorname{supp}\mu; \lim_{r\to 0} \frac{\log(\mu(B(x,r)))}{\log(2r)} = \alpha \right\}, \quad \alpha \geqslant 0;$$

$$f_\mu(\alpha) = \dim_H E(\alpha);$$

$$F_\mu(\alpha) = \operatorname{Dim} E(\alpha), b_\mu(q) = \dim_\mu^q(\operatorname{supp}\mu(\omega)), \qquad B_\mu(q) = \operatorname{Dim}_\mu^q(\operatorname{supp}\mu(\omega)).$$

Now we define an auxiliary function $\beta(q)$ as follows: For each $q \in \mathbb{R}$ and $k \geqslant 1$, there is a unique number $\beta_k(q)$ such that

$$\sum_{\sigma \in D_k} (P_{\sigma(a)} P_{\sigma(b)})^q |J_\sigma|^{\beta_k(q)} = 1. \tag{2.1}$$

By a simple calculation, we get

$$\beta_k(q) = \frac{-\log\left(\sum_1^2 P_{a_i}^q\right) - \frac{k-|\omega_k|_a}{|\omega_k|_a} \log\left(\sum_1^3 P_{b_i}^q\right)}{\log r_a + \frac{k-|\omega_k|_a}{|\omega_k|_a} \log r_b}.$$

Clearly, for any $k \geqslant 1$, $\beta_k(0) = d_k$ and $\beta_k(1) = 0$. Also, by implicit differentiation

$$\beta_k'(q) = \frac{-\sum_{\sigma \in D_k} (P_{\sigma(a)} P_{\sigma(b)})^q (r_a^{|\omega_k|_a} r_b^{|\omega_k|_b})^{\beta_k(q)} \log(P_{\sigma(a)} P_{\sigma(b)})}{\sum_{\sigma \in D_k} (P_{\sigma(a)} P_{\sigma(b)})^q (r_a^{|\omega_k|_a} r_b^{|\omega_k|_b})^{\beta_k(q)} \log(r_a^{|\omega_k|_a} r_b^{|\omega_k|_b})} \tag{2.2}$$

and

$$\beta_k''(q) = \frac{-\sum_{\sigma \in D_k} (P_{\sigma(a)} P_{\sigma(b)})^q (r_a^{|\omega_k|_a} r_b^{|\omega_k|_b})^{\beta_k(q)} [\log(P_{\sigma(a)} P_{\sigma(b)}) + \beta_k'(q) \log(r_a^{|\omega_k|_a} r_b^{|\omega_k|_b})]^2}{\sum_{\sigma \in D_k} (P_{\sigma(a)} P_{\sigma(b)})^q (r_a^{|\omega_k|_a} r_b^{|\omega_k|_b})^{\beta_k(q)} \log(r_a^{|\omega_k|_a} r_b^{|\omega_k|_b})} \tag{2.3}$$

Thus $\beta_k'(q) < 0$ for all q and $\beta_k(q)$ is a strictly decreasing function. Also, note that $\beta_k''(q) \geqslant 0$. As a matter of fact, either $\beta_k''(q) > 0$ for all q or else $P_{\sigma(a)} P_{\sigma(b)} = (r_a^{|\omega_k|_a} r_b^{|\omega_k|_b})^{d_k}$ and $\beta_k(q) = -d_k q + d_k$ for all q. This follows from the fact that if $\beta_k''(q_0) = 0$ for some q_0, then from Eq. (2.3), $\beta_k'(q_0) = -\log(P_{\sigma(a)} P_{\sigma(b)}) / \log(r_a^{|\omega_k|_a} r_b^{|\omega_k|_b})$, for all $\sigma \in D_k$. Therefore, $P_{\sigma(b)} P_{\sigma(a)} = {r_a^{|\omega_k|_a} r_b^{|\omega_k|_b}}^{-\beta_k'(q_0)}$ for all $\sigma \in D_k$ and $k \geqslant 1$. This implies $P_{\sigma(b)} P_{\sigma(a)} = (r_a^{|\omega_k|_a} r_b^{|\omega_k|_b})^{d_k}$ for all $\sigma \in D_k$ and $k \geqslant 1$. In this case, and only in this case, $\beta_k(q) = -d_k q + d_k$ for all q.

Proposition 2.3. *For all $q \in \mathbb{R}$, for all $k \geqslant 1$, $\beta_k(q)$ defined by* (2.1) *fulfills the following*:

(i) *$\beta_k(q)$ is decreasing, and* $\lim_{q \to \mp\infty} \beta_k(q) = \pm\infty$;

(ii) *Either $P_{\sigma(a)} P_{\sigma(b)} = (r_a^{|\omega_k|_a} r_b^{|\omega_k|_b})^{d_k}$, $\forall \sigma \in D_k$, $k \geqslant 1$ and $\beta_k(q) = -q d_k + d_k$, or $\beta_k''(q) > 0$ for all q.*

(iii) *For all $k \geqslant 1$, $\beta_k(0) = d_k$, $\beta_k(1) = 0$.*

Our auxiliary function is

$$\beta(q) = \lim_{k\to\infty} \beta_k(q) = \frac{-\log\left(\sum_{i=1}^{2} P_{a_i}{}^q\right) - \eta \log\left(\sum_{j=1}^{3} P_{b_j}{}^q\right)}{\log r_a + \eta \log r_b}, \tag{2.4}$$

where $\eta^2 + \eta = 1$.

Also, by implicit differentiation

$$\beta'(q) = \frac{-1}{\log r_a + \eta \log r_b}\left[\frac{\sum_{i=1}^{2} P_{a_i}{}^q \log P_{a_i}}{\sum_{i=1}^{2} P_{a_i}{}^q} + \eta \frac{\sum_{j=1}^{3} P_{b_j}{}^q \log P_{b_j}}{\sum_{j=1}^{3} P_{b_j}{}^q}\right] \tag{2.5}$$

and

$$\beta''(q) = \frac{-1}{\log r_a + \eta \log r_b}\left[\frac{(P_{a_1}P_{a_2})^q\left(\log \frac{P_{a_1}}{P_{a_2}}\right)^2}{\left(\sum_{i=1}^{2} P_{a_i}{}^q\right)^2} + \eta \frac{\sum_{i,j=1, i\neq j}^{3} (P_{b_j}P_{b_i})^q\left(\log (P_{b_j}/P_{b_i})\right)^2}{\left(\sum_{j=1}^{3} P_{b_j}{}^q\right)^2}\right] \tag{2.6}$$

Thus, $\beta'(q) < 0$ for all q, so that $\beta(q)$ is a strictly decreasing function. Also, note that $\beta''(q) \geqslant 0$. As a matter of fact, either $\beta''(q) > 0$ for all q, or $P_{a_1} = P_{a_2}$, $P_{b_1} = P_{b_2} = P_{b_3}$ and $\beta(q) = -dq + d$ for all q. This follows from the fact if $\beta''(q_0) = 0$ for some q_0 then from Eq. (2.6), $P_{a_1} = P_{a_2}$, and $P_{b_1} = P_{b_2} = P_{b_3}$. This implies $\beta(q) = -dq + d$. In this case, and only in this case, $\beta(q) = -dq + d$ for all q.

Proposition 2.4. *For all* $q \in \mathbb{R}$, *for all* $k \geqslant 1$, $\beta(q)$ *defined by* (2.4) *fulfills the following*:

(i) $\beta(q)$ *is strictly decreasing and* $\lim_{q\to\mp\infty} \beta(q) = \pm\infty$;
(ii) *Either* $P_{a_1} = P_{a_2}$, $P_{b_1} = P_{b_2} = P_{b_3}$ *and* $\beta(q) = -dq + d$, *or* $\beta''(q) > 0$ *for all* q;
(iii) $\beta(0) = \lim_{k\to\infty} d_k = d$, $\beta(1) = 0$.

Remark 2.1. By a simple calculation, we get: $P_{a_1} = P_{a_2} = \frac{1}{2}$ and $P_{b_i} = \frac{1}{3}$, $i = 1, 2, 3$. $\Leftrightarrow P_{\sigma(a)}P_{\sigma(b)} = (r_a^{|\omega_k|_a} r_b^{|\omega_k|_b})^{d_k}$ for all $\sigma \in D_k$ and $k \geqslant 1$.

In this paper, we get two main theorems:

Theorem A. *Assume* $\Delta > 0$, *then for all* $q \in \mathbb{R}$, $b_\mu(q) = B_\mu(q) = \beta(q)$.

Theorem B. *Assume that* $\Delta > 0$ *and let* $\beta(q)$ *be the function in* (2.4). *Then there exist numbers* $0 \leqslant \underline{\alpha} \leqslant \bar{\alpha}$ *such that*

$$f_\mu(\alpha) = F_\mu(\alpha) = \begin{cases} \beta^*(\alpha), & \text{for } \alpha \in (\underline{\alpha}, \bar{\alpha}) \\ 0, & \text{for } \alpha \notin (\underline{\alpha}, \bar{\alpha}). \end{cases}$$

The complete proofs of Theorem *A* and Theorem *B* are given in Section 3.

3. Proof of the Theorems

In this section, we shall give the proofs of Theorem *A* and Theorem *B*. At first, we prove some auxiliary results.

3.1. Propositions.

Proposition 3.1. *Given $q \in \mathbb{R}$, there exists a probability measure ν_q supported by* $\operatorname{supp}\mu$ *such that for any $k \geqslant 1$ and $\sigma_0 \in D_k$,*

$$\nu_q(J_{\sigma_0}) = \frac{\mu(J_{\sigma_0})^q |J_{\sigma_0}|^{\beta(q)}}{\sum_{\sigma \in D_k} \mu(J_\sigma)^q |J_\sigma|^{\beta(q)}}.$$

Proof. Take a sequence of probability measures $\{\nu_m; m \geqslant 1\}$ supported by $\operatorname{supp}\mu$ such that for any $\sigma_0 \in D_m$,

$$\nu_m(J_{\sigma_0}) = \frac{\mu(J_{\sigma_0})^q |J_{\sigma_0}|^{\beta(q)}}{\sum_{\sigma \in D_m} \mu(J_\sigma)^q |J_\sigma|^{\beta(q)}}. \tag{3.1}$$

More precisely, we can construct ν_m as follows:

First, we distribute the unit mass among the rank-m basic intervals according to (3.1). Inductively, suppose we have already distributed mass of proportion $\nu_m(J_\sigma)$ to a basic interval $J_\sigma(\sigma \in D_n, n \geqslant m)$, then we distribute the mass concentrated on J_σ evenly to each of its $n+1$ subintervals, i.e.,

$$\nu_m(J_{\sigma * j}) = \begin{cases} \frac{P_{a_j}^q}{\sum_{i=1}^2 P_{a_i}^q} \nu_m(J_\sigma), & 1 \leqslant j \leqslant 2, \quad s_{n+1} = a, \\ \frac{P_{b_j}^q}{\sum_{i=1}^3 P_{b_i}^q} \nu_m(J_\sigma), & 1 \leqslant j \leqslant 3, \quad s_{n+1} = b. \end{cases}$$

Repeating the above procedure, we get the desired measure. Now fix some $m \geqslant 1$, for any $k < m$ and $\sigma_0 \in D_k$, we get

$$\nu_m(J_{\sigma_0}) = \sum_{\sigma \in D_{k+1,m}} \nu_m(J_{\sigma_0 * \sigma}),$$

combining with (3.1), we have

$$\sum_{\sigma \in D_m} |J_\sigma|^q \mu(J_\sigma)^q \nu_m(J_{\sigma_0}) = \sum_{\sigma \in D_{k+1,m}} \mu(J_{\sigma_0 * \sigma})^q |J_{\sigma_0 * \sigma}|^{\beta(q)}. \tag{3.2}$$

For any $\sigma_1 \in D_k$, by the definition of μ

$$\mu(J_{\sigma_1 * \sigma}) / \mu(J_{\sigma_1}) = \mu(J_{\sigma_0 * \sigma}) / \mu(J_{\sigma_0}),$$

thus by (3.2)

$$\mu(J_{\sigma_1})^q \left(\sum_{\sigma \in D_m} |J_\sigma|^{\beta(q)} \mu(J_\sigma)^q \right) \nu_m(J_{\sigma_0}) = \mu(J_{\sigma_0})^q \sum_{\sigma \in D_{k+1,m}} \mu(J_{\sigma_1 * \sigma})^q |J_{\sigma_1 * \sigma}|^{\beta(q)},$$

this gives

$$\left(\sum_{\sigma_1 \in D_k} \mu(J_{\sigma_1})^q |J_{\sigma_1}|^{\beta(q)} \right) \left(\sum_{\sigma \in D_m} |J_\sigma|^{\beta(q)} \mu(J_\sigma)^q \right) \nu_m(J_{\sigma_0})$$
$$= \mu(J_{\sigma_0})^q |J_{\sigma_0}|^{\beta(q)} \sum_{\sigma_1 \in D_k, \sigma \in D_{k+1,m}} \mu(J_{\sigma_1 * \sigma})^q |J_{\sigma_1 * \sigma}|^{\beta(q)}.$$

Observing that

$$\sum_{\sigma\in D_m}\mu(J_\sigma)^q|J_\sigma|^{\beta(q)}=\sum_{\sigma_1\in D_k,\sigma\in D_{k+1,m}}\mu(J_{\sigma_1*\sigma})^q|J_{\sigma_1*\sigma}|^{\beta(q)},$$

one gets

$$\nu_m(J_{\sigma_0})=\mu(J_{\sigma_0})^q|J_{\sigma_0}|^{\beta(q)}\Big/\sum_{\sigma\in D_k}\mu(J_\sigma)^q|J_\sigma|^{\beta(q)}.$$

To summarize, we get a sequence of probability measures $\{\nu_m\}_{m\geqslant 1}$ which are supported by $\operatorname{supp}\mu$, and satisfy (3.1) for any $k\leqslant m$ and $\sigma_0\in D_k$.

Now Helly's theorem [13] enables us to extract a subsequence $\{\nu_{m_n}\}_{n=1}^{\infty}$ converging weakly to a limit measure ν_q. By the properties of weak convergence, we have for any $k\geqslant 1$ and $\sigma_0\in D_k$,

$$\nu_q(J_{\sigma_0})=\frac{\mu(J_{\sigma_0})^q|J_{\sigma_0}|^{\beta(q)}}{\sum_{\sigma\in D_k}\mu(J_\sigma)^q|J_\sigma|^{\beta(q)}}.$$

Finally, for any $x\notin\operatorname{supp}\mu$, since $\operatorname{supp}\mu$ is a closed set, there exists an open interval U containing x and separated for $\operatorname{supp}\mu$, thus $\nu_q(U)\leqslant\underline{\lim}_{n\to\infty}\nu_{m_n}(U)=0$, which asserts that ν_q is supported by $\operatorname{supp}\mu$. □

If $\sigma=\sigma_1\sigma_2\cdots\sigma_n\in D_n$ $\left(\text{indeed, } \sigma_k\in\begin{cases}\{1,2\}, & s_k=a\\ \{1,2,3\}, & s_k=b\end{cases}\right)$ and $k\in\{1,2,\ldots,n\}$ then we write $\sigma|_k=\sigma_1\sigma_2\cdots\sigma_k$, put $p_0=\min_{\substack{i=1,2\\ j=1,2,3}}\{p_{a_i},p_{b_j}\}$ and $r_0=\max\{r_a,r_b\}$.

Proposition 3.2. *Suppose* $\Delta>0$. *We have* $\overline{\lim}_{r\to 0}\big(\sup_{x\in E}\frac{\mu B(x,cr)}{\mu B(x,r)}\big)<\infty$ *for any* $c>1$.

Proof. Let $c>1$ and $r>0$. For $x\in J_\sigma\subset\mathscr{F}$, $\sigma\in D$, choose k, $l\in\mathbb{N}$ such that

$$|J_{\sigma|k+1}|\leqslant r<|J_{\sigma|k}|,\tag{3.2}$$

$$\Delta|J_{\sigma|\ell+1}|\leqslant cr<\Delta|J_{\sigma|\ell}|.\tag{3.3}$$

Then we have

$$J_{\sigma|k+1}(x)\subseteq B(x,r),$$

$$E\cap B(x,cr)\subseteq J_{\sigma|\ell+1}(x).$$

[Indeed, it is obvious that $J_{\sigma|k+1}(x)\subseteq B(x,r)$. Now let $y\in E\cap B(x,cr)$ and assume $y\in J_{\sigma|\ell+1}(x)$. Then there exists $j<\ell+1$ such that $\tau|_j=\sigma|_j$ and $\tau_{j+1}\neq\sigma_{j+1}$, and

$$|y-x|\geqslant d(J_{\sigma|j+1},J_{\tau|j+1})\geqslant\Delta_k|J_{\sigma|j}|\geqslant\Delta|J_{\sigma|\ell}|>cr,$$

which is a contradiction since $y\in B(x,cr)$.]

Since $c>1$ and $\Delta<1$, $k\geqslant \ell$. Equations (3.2) and (3.3) imply that

$$\frac{1}{c}=\frac{r}{cr}\leqslant\frac{|J_{\sigma|k}|}{\Delta|J_{\sigma|\ell+1}|}=\frac{r_a^{|w_k|_a}r_b^{|w_k|_b}}{\Delta r_a^{|w_{\ell+1}|_a}r_b^{|w_{\ell+1}|_b}}=\frac{r_a^{|w_k|_a-|w_{\ell+1}|_a}\cdot r_b^{|w_k|_b-|w_{\ell+1}|_b}}{\Delta}$$
$$\leqslant\frac{r_0^{k-\ell-1}}{\Delta},$$

which yields

$$k-\ell\leqslant 1+\frac{\log(\Delta/c)}{\log r_0}\overset{\wedge}{=}\Delta(c).$$

Now the definition of μ implies that

$$\frac{\mu B(x,cr)}{\mu B(x,r)}\leqslant\frac{\mu(J_{\sigma|\ell+1})}{\mu(J_{\sigma|k+1})}\leqslant\frac{1}{p_0^{k-\ell}}\leqslant\frac{1}{p_0^{\Delta(x)}}\overset{\wedge}{=}p_0(c),$$

thus

$$\varlimsup_{r\to 0}\left(\sup_{x\in E}\frac{\mu B(x,cr)}{\mu B(x,r)}\right)\leqslant p_0<\infty.$$

Proposition 3.3. *If $\Delta>0$ then there exist $c_h>0$ and $c_p>0$ such that for any $q\in\mathbb{R}$*

$$c_h\varliminf_{n\to\infty}\sum_{\sigma\in D_n}\mu(J_\sigma)^q|J_\sigma|^{\beta(q)}$$
$$\leqslant\mathscr{H}_\mu^{q,\beta(q)}(E)\leqslant\mathscr{P}_\mu(E)^{q,\beta(q)}\leqslant\bar{\mathscr{P}}_\mu^{q,\beta(q)}(E)$$
$$\leqslant c_p\varlimsup_{n\to\infty}\sum_{\sigma\in D_n}\mu(J_\sigma)^q|J_\sigma|^{\beta(q)}.$$

Proof. Write $\bar{h}=\varlimsup_{n\to\infty}\sum_{\sigma\in D_n}\mu(J_\sigma)^q|J_\sigma|^{\beta(q)}$, $\underline{h}=\varliminf_{n\to\infty}\sum_{\sigma\in D_n}\mu(J_\sigma)^q|J_\sigma|^{\beta(q)}$. Since $\mathscr{H}_\mu^{q,\beta(q)}(E)\leqslant\mathscr{P}_\mu^{q,\beta(q)}(E)\leqslant\bar{\mathscr{P}}_\mu^{q,\beta(q)}(E)$ (see [12]), in order to prove the proposition, it suffices to verity that

(i) $0<\underline{h}<\infty\Rightarrow\mathscr{H}_\mu^{q,\beta(q)}(E)\geqslant c_h\cdot\underline{h}$

(ii) $\underline{h}=\infty\Rightarrow\mathscr{H}_\mu^{q,\beta(q)}(E)=\infty$,

(iii) $\bar{h}=0\Rightarrow\bar{\mathscr{P}}_\mu^{q,\beta(q)}(E)=0$,

(iv) $0<\bar{h}<\infty\Rightarrow\bar{\mathscr{P}}_\mu^{q,\beta(q)}(E)\leqslant c_p\bar{h}$.

(i) If $0<\underline{h}<\infty$, then when n is sufficiently large, we have

$$\sum_{\sigma\in D_n}\mu(J_\sigma)^q|J_\sigma|^{\beta(q)}>\frac{h}{2}.\tag{3.4}$$

Let $\delta>0$ and $\{B(x_i,r_i)\}_{i\in\mathbb{N}}$ be a centered δ-covering of E. For each i choose $\sigma(i)\in D_n$, for any $n\geqslant 1$ such that $x_i\in J_{\sigma(i)}$. For each $i\in\mathbb{N}$ choose $k_i,\ell_i\in\mathbb{N}$ such that

$$|J_{\sigma(i)|k_i+1}|\leqslant r_i<|J_{\sigma(i)|k_i}|,$$

$$\Delta|J_{\sigma(i)|\ell_i+1}|\leqslant r_i<\Delta|J_{\sigma(i)|\ell_i}|.$$

Notice that

$$J_{\sigma(i)|k_i+1}(x_i) \subseteq B(x_i, r_i), \qquad E \cap B(x_i, r_i) \subseteq J_{\sigma(i)|\ell_i+1}(x),$$

by Proposition 3.2, there exists a probability measure ν_q supported by E, such that

$$\nu_q(J_{\sigma(i)}) = \frac{\mu(J_{\sigma(i)})^q |J_{\sigma(i)}|^{\beta(q)}}{\sum_{\sigma \in D_{|\sigma(i)|}} \mu(J_\sigma)^q |J_\sigma|^{\beta(q)}}. \tag{3.5}$$

Then we have

$$\begin{aligned}
\nu_q(E) &\leqslant \sum_i \nu_q B(x_i, r_i) \leqslant \sum_i \nu_q(J_{\sigma(i)|\ell_i+1}(x_i)) \\
&= \sum_i \frac{\mu(J_{\sigma(i)|\ell_i+1}(x_i))^q |J_{\sigma(i)|\ell_i+1}(x_i)|^{\beta(q)}}{\sum_{\sigma \in D_{\ell_i+1}} \mu(J_\sigma)^q |J_\sigma|^{\beta(q)}} \\
&< 2/\underline{h} \sum_i \mu(J_{\sigma(i)|\ell_i+1}(x_i))^q |J_{\sigma(i)|\ell_i+1}(x_i)|^{\beta(q)}.
\end{aligned} \tag{3.6}$$

If $\beta(q) \geqslant 0$, then $|J_{\sigma(i)|\ell_i+1}(x_i)|^{\beta(q)} \leqslant (2\Delta)^{\beta(q)} (2r_i)^{\beta(q)}$. If $\beta(q) < 0$, then

$$|J_{\sigma(i)|\ell_i+1}(x_i)| = \begin{cases} r_a |J_{\sigma(i)|\ell_i}|, & s_{\ell_i+1} = a, \\ r_b |J_{\sigma(i)|\ell_i}|, & s_{\ell_i+1} = b, \end{cases}$$

implies that $|J_{\sigma(i)|\ell_i+1}| \geqslant \min\{r_a, r_b\} \cdot |J_{\sigma(i)|\ell_i}|$, thus

$$2r_i \leqslant 2\Delta |J_{\sigma(i)|\ell_i}| \leqslant \frac{2\Delta}{\min\{r_a, r_b\}} |J_{\sigma(i)|\ell_i+1}|,$$

that is $|J_{\sigma(i)|\ell_i+1}|^{\beta(q)} \leqslant \left(\frac{\min\{r_a, r_b\}}{2\Delta}\right)^{\beta(q)} (2r_i)^{\beta(q)}$. In all cases

$$|J_{\sigma(i)|\ell_i+1}|^{\beta(q)} \leqslant k_1 (2r_i)^{\beta(q)}, \tag{3.7}$$

where k_1 is a suitable constant.

If $q < 0$, $E \cap B(x_i, r_i) \subseteq J_{\sigma(i)|\ell_i+1}$ implies that

$$\mu(J_{\sigma(i)|\ell_i+1})^q \leqslant \mu(B(x_i, r_i))^q. \tag{3.8}$$

If $q \geqslant 0$, $J_{\sigma(i)|\ell_i+1} \subseteq B(x_i, r_i/\Delta)$ implies that

$$\mu(J_{\sigma(i)|\ell_i+1})^q \leqslant \left(\frac{\mu B\left(x_i, \frac{r_i}{\Delta}\right)}{\mu B(x_i, r_i)} \right)^q \mu B(x_i, r_i)^q \leqslant m^q \mu B(x_i, r_i)^q, \tag{3.9}$$

where $m \triangleq \sup\limits_{\substack{x \in E \\ r > 0}} \frac{\mu B(x, \Delta^{-1} r)}{\mu B(x, r)} < \infty$ (by Proposition 3.2). It follows from (3.5)–(3.9) that

$$\nu_q(E) \leqslant \frac{k_2}{\underline{h}} \sum_i \mu B(x_i, r_i)^q (2r_i)^{\beta(q)}$$

for a suitable constant k_2. Hence

$$c_h \cdot \underline{h} \triangleq \frac{\underline{h}}{k_2} = \frac{\underline{h}}{k_2} \nu_q(E) \leqslant \bar{\mathscr{H}}^{q,\beta(q)}_{\mu,\delta}(E) \leqslant \bar{\mathscr{H}}^{q,\beta(q)}_{\mu}(E) \leqslant \mathscr{H}^{q,\beta(q)}_{\mu}(E).$$

(ii) $\underline{h} = \infty$. For any $\varepsilon > 0$, we have for sufficiently large n,

$$\sum_{\sigma \in D_n} |J_\sigma|^{\beta(q)} \cdot \mu(J_\sigma)^q > \varepsilon^{-1},$$

by an argument analogous to that of (i), we get

$$\nu_q(E) \leqslant \frac{\varepsilon k_2}{2} \sum_i \mu B(x_i, r_i)^q (2r_i)^{\beta(q)}$$

for any $\delta > 0$ and $\{B(x_i, r_i)\}_i$ a centered δ-covering of E. Hence

$$\mathscr{H}_\mu^{q,\beta(q)}(E) \geqslant \bar{\mathscr{H}}_\mu^{q,\beta(q)}(E) \geqslant \bar{\mathscr{H}}_{\mu,\delta}^{q,\beta(q)}(E) \geqslant \nu_q(E) \frac{2}{k_2 \varepsilon} = \frac{2}{k_2 \cdot \varepsilon}$$

this implies $\mathscr{H}_\mu^{q,\beta(q)}(E) = \infty$.

(iii) $\bar{h} = 0$. For any $\varepsilon > 0$,

$$\sum_{\sigma \in D_n} \mu(J_\sigma)^q |J_\sigma|^{\beta(q)} < \varepsilon \tag{3.10}$$

holds for all sufficiently large n. Let $\delta > 0$ and $\{B(x_i, r_i)\}_i$ be a centered δ-packing of E. For each $i \in \mathbb{N}$, choose $\sigma(i) \in D_n$, $\forall n$ such that $x_i \in J_{\sigma(i)}$, and integers k_i, $\ell_i \in \mathbb{N}$ such that

$$|J_{\sigma(i)|k_i+1}| \leqslant r_i < |J_{\sigma(i)|k_i}|,$$
$$\Delta |J_{\sigma(i)|\ell_i+1}| \leqslant r_i < \Delta |J_{\sigma(i)|\ell_i}|.$$

Since

$$J_{\sigma(i)|k_i+1}(x_i) \subseteq B(x_i, r_i), \qquad E \cap B(x_i, r_i) \subseteq J_{\sigma(i)|\ell_i+1}(x),$$

by Proposition 3.2, there exists a probability measure ν_q supported by E such that

$$\nu_q(J_{\sigma(i)}) = \frac{\mu(J_{\sigma|i})^q |J_{\sigma(i)}|^{\beta(q)}}{\sum_{\sigma \in D_{|\sigma(i)|}} \mu(J_\sigma)^q |J_\sigma|^{\beta(q)}}. \tag{3.11}$$

If $\beta(q) \leqslant 0$, we have

$$(2r_i)^{\beta(q)} \leqslant 2^{\beta(q)} |J_{\sigma(i)|k_i+1}|^{\beta(q)}.$$

If $\beta(q) > 0$,

$$|J_{\sigma(i)|k_i+1}| = \begin{cases} r_a |J_{\sigma(i)|k_i}|, & s_{k_i} + 1 = a, \\ r_b |J_{\sigma(i)|k_i}|, & s_{k_i} + 1 = b \end{cases}$$

implies $|J_{\sigma(i)|k_i+1}| \geqslant \min\{r_a, r_b\} |J_{\sigma(i)|k_i}|$, i.e.

$$(2r_i)^{\beta(q)} \leqslant \left(\frac{2}{\min\{r_a, r_b\}} \right)^{\beta(q)} |J_{\sigma(i)|k_i+1}|^{\beta(q)}.$$

Thus in both cases, we always have

$$(2r_i)^{\beta(q)} \leqslant K_1 |J_{\sigma(i)|k_i+1}|^{\beta(q)}, \tag{3.12}$$

where K_1 is a suitable constant.

If $q \leqslant 0$, we have

$$\mu(B(x_i, r_i))^q \leqslant \mu(J_{\sigma(i)|k_i+1})^q.$$

If $q > 0$, the proof of Proposition 3.2 implies that

$$\begin{aligned}\mu(B(x_i, r_i))^q &\leqslant \mu(J_{\sigma(i)|\ell_i+1})^q \\ &= \left(\frac{\mu(J_{\sigma(i)|\ell_i+1})}{\mu(J_{\sigma(i)|k_i+1})}\right)^q \mu(J_{\sigma(i)|k_i+1})^q \\ &\leqslant (p_0(c))^q \mu(J_{\sigma(i)|k_i+1})^q.\end{aligned}$$

In both cases, we have

$$\mu(B(x_i, r_i))^q \leqslant K_2 \mu(J_{\sigma(i)|k_i+1})^q, \tag{3.13}$$

where K_2 is a suitable constant.

It follows from (3.10)–(3.13) that

$$\begin{aligned}\sum_i \mu(B(x_i, r_i))^q (2r_i)^{\beta(q)} &\leqslant K_1 K_2 \sum_i \mu(J_{\sigma(i)|k_i+1})^q |J_{\sigma(i)|k_i+1}|^{\beta(q)} \\ &= K_1 K_2 \sum_i \left[\frac{\mu(J_{\sigma(i)|k_i+1})^q |J_{\sigma(i)|k_i+1}|^{\beta(q)}}{\sum_{\sigma \in D_{k_i+1}} \mu(J_\sigma)^q |J_\sigma|^{\beta(q)}}\right. \\ &\quad \left. \cdot \sum_{\sigma \in D_{k_i+1}} \mu(J_\sigma)^q |J_\sigma|^{\beta(q)}\right] \\ &< K_1 K_2 \cdot \varepsilon \sum_i \nu_q(J_{\sigma(i)|k_i+1}) \leqslant K_1 K_2 \cdot \varepsilon \sum_i \nu_q(B(x_i, r_i)) \\ &= K_1 K_2 \cdot \varepsilon \cdot \nu_q\left(\bigcup_i B(x_i, r_i)\right) \leqslant K_1 K_2 \cdot \varepsilon.\end{aligned}$$

Thus

$$\bar{\mathscr{P}}_{\mu,\delta}^{q,\beta(q)}(E) \leqslant K_1 K_2 \varepsilon$$

for any $\varepsilon > 0$ and $\delta > 0$, which clearly implies that $\bar{\mathscr{P}}_{\mu}^{q,\beta(q)}(E) = 0$.

(iv) $0 < \bar{h} < \infty$. When n is large enough, we have

$$\sum_{\sigma \in D_n} |J_\sigma|^{\beta(q)} \mu(J_\sigma)^q < 2\bar{h},$$

using a similar argument as that in (i). we show

$$\sum_i \mu(B(x_i, r_i))^q (2r_i)^{\beta(q)} \leqslant K_1 K_2 \cdot 2\bar{h}$$

for any $\delta > 0$ and any centered δ-packing $\{B(x_i, r_i)\}_i$ of E. Hence

$$\bar{\mathscr{P}}_{\mu}^{q,\beta(q)}(E) \leqslant K_1 K_2 \cdot 2\bar{h} \stackrel{\wedge}{=} c_p \cdot \bar{h}.$$

$\square$

3.2. Proof of Theorems.

Proof of Theorem A. We only give the proof for $b_\mu(q)=\beta(q)$, the proof of $B_\mu(q)=\beta(q)$ is similar. For any $t\geqslant\beta(q)=\lim_{k\to\infty}\beta_k(q)$, there exist infinitely many k such that $t>\beta_k(q)$. For any $F\subset\operatorname{supp}\mu$, we have for sufficiently large k $(\sigma\in D_k, |J_\sigma|\leqslant\delta)$

$$\bar{\mathscr{H}}^{q,t}_{\mu,\delta}(F)\leqslant\sum\mu(J_\sigma)^q|J_\sigma|^t\leqslant\sum\mu(J_\sigma)^q|J_\sigma|^{\beta_k(q)}=1,$$

then we obtain $\mathscr{H}^{q,t}_\mu(\operatorname{supp}\mu)\leqslant 1$ and $b_\mu(q)\leqslant\beta(q)$.

For any $t<\beta(q)$, if $\mathscr{H}^{q,t}_\mu(\operatorname{supp}\mu)=\infty$, we have $b_\mu(q)\geqslant\beta(q)$. If $\mathscr{H}^{q,t}_\mu(\operatorname{supp}\mu)<\infty$, we have $\mathscr{H}^{q,\beta(q)}_\mu(\operatorname{supp}\mu)<\infty$. Since $\mathscr{H}^{q,\beta(q)}_\mu(\operatorname{supp}\mu)\leqslant\mathscr{H}^{q,t}_\mu(\operatorname{supp}\mu)$ and one has $t<\beta_k(q)$ for k large enough, we have

$$\sum_{\sigma\in D_k}\mu(J_\sigma)^q|J_\sigma|^{\beta(q)}>\sum_{\sigma\in D_k}\mu(J_\sigma)^q|J_\sigma|^{\beta_k(q)}=1,$$

which implies

$$\varliminf_{k\to\infty}\sum_{\sigma\in D_k}\mu(J_\sigma)^q|J_\sigma|^{\beta(q)}>0.$$

Now by Proposition 3.3, we get $\mathscr{H}^{q,\beta(q)}_\mu(\operatorname{supp}\mu)>0$, this implies $b_\mu(q)\geqslant\beta(q)$.

Proof of Theorem B. Since for any $q\in\mathbb{R}$, $\beta'(q)$ exists, then $\forall\alpha\in(\bar{\alpha},\underline{\alpha})$, there exists a unique q_α such that $-\alpha=\frac{d\beta(q)}{dq}$, whereupon $\beta^*(\alpha)=-\beta'(q_\alpha)q_\alpha+\beta(q_\alpha)=\alpha q_\alpha+\beta(q_\alpha)$. If $\mathscr{H}^{q_\alpha,\beta(q_\alpha)}_\mu(\operatorname{supp}\mu)=\infty>0$, by Theorem 2.1 [8], we have

$$f_\mu(\alpha)=F_\mu(\alpha)=\beta^*(\alpha).$$

If $\mathscr{H}^{q_\alpha,\beta(q_\alpha)}_\mu(\operatorname{supp}\mu)<\infty$, by Proposition 3.3, we have

$$\mathscr{H}^{q_\alpha,\beta(q_\alpha)}_\mu(\operatorname{supp}\mu)\geqslant c_h\varliminf_{n\to\infty}\sum_{\sigma\in D_n}\mu(J_\sigma)^{q_\alpha}|J_\sigma|^{\beta(q_\alpha)}.$$

Now if $\varliminf_{n\to\infty}\sum_{\sigma\in D_n}\mu(J_\sigma)^{q_\alpha}|J_\sigma|^{\beta(q_\alpha)}>0$, by Theorem 2.1 of [8], we have $f_\mu(\alpha)=F_\alpha(\alpha)=\beta^*(\alpha)$. By a direct calculation, we can get $\beta(q)-\beta_n(q)=O(\frac{1}{n})$, by (2.1), we have $\sum_{\sigma\in D_n}\mu(J_\sigma)^{q_\alpha}|J_\sigma|^{\beta(q_\alpha)}=|J_\sigma|^{\beta(q_\alpha)-\beta_n(q_\alpha)}\geqslant(\min\{r_a,r_b\})^{n(\beta(q_\alpha)-\beta_n(q_\alpha))}$, therefore

$$\varliminf_{n\to\infty}\sum_{\sigma\in D_n}\mu(J_\sigma)^{q_\alpha}|J_\sigma|^{\beta(q_\alpha)}>0.$$

If $\alpha\notin(\underline{\alpha},\bar{\alpha})$, by Lemma 4.4 of [2], we have $E(\alpha)=\phi$, this completes the proof.

Remark. Let $A=\{a_1,a_2,\ldots,a_m\}$. $w=s_1s_2\cdots s_k\cdots$ is a sequence over A, $s_i\in A$. If for any $a_i\in A$, $\lim_{k\to\infty}\frac{|s_1s_2\cdots s_k|_{a_i}}{k}=\eta_i>0$, then we say that the sequence w has the frequency vector $\eta=(\eta_1,\eta_2,\ldots,\eta_m)$.

For the Moran fractals associated with this kind of sequences, our method in this paper can be employed to give the similar result as in the Fibonacci case.

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