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On a New Continued Fraction Expansion with Non-Decreasing Partial Quotients

By

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Abstract. We investigate metric properties of the digits occurring in a new continued fraction expansion with non-decreasing partial quotients, the so-called Engel continued fraction (ECF) expansion.

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1. Introduction

Recently, Hartono, Kraaikamp and Schweiger [6] introduced a new continued fraction algorithm with non-decreasing partial quotients, called the *Engel continued fraction* (ECF) expansion. Basic and ergodic properties of this expansion were studied. The name of this new continued fraction expansion is 'borrowed' from the classical *Engel Series expansion*, although it should be stressed that the ECF-expansion of a number $x \in (0,1)$, and the corresponding Engel Series expansion of x hardly have anything in common, except that for both types of expansions the sequences of digits are non-decreasing sequences of positive integers. To illustrate this point a little bit further, recall that the Engel Series expansion of any $x \in (0,1)$ is generated by the map $S : [0,1) \to [0,1)$, given by

$$S(x) := \left(\left[\frac{1}{x}\right] + 1\right)\left(x - \frac{1}{[\frac{1}{x}] + 1}\right), \quad x \neq 0; \ \ S(0) := 0,$$

where $[\xi]$ is the largest integer not exceeding ξ, see also Figure 1. Note that the Engel Series map S is in fact equal to the Lüroth Series expansion map, normalized by $[1/x]$ (see e.g. Section 2.2 in [1] for more information on the Lüroth Series expansion).

Using S, one can find a (unique) series expansion of every $x \in (0,1)$, given by

$$x = \frac{1}{q_1(x)} + \frac{1}{q_1(x)q_2(x)} + \cdots + \frac{1}{q_1(x)q_2(x)\cdots q_n(x)} + \cdots,$$

where $q_n(x) = [1/S^{n-1}(x)] + 1$, $n \geqslant 1$. In fact, it was Sierpiński [14] in 1911 who first studied these series expansions. Note however, that the Engel Series expansion can be also written as an *ascending* continued fraction

$$x = \cfrac{1 + \cfrac{1 + \cfrac{1+\cdots}{q_3(x)}}{q_2(x)}}{q_1(x)},$$

and as a continued fraction expansion of x it has been studied as early as 1202 by Fibonacci in his classical book *Liber abaci* [3].

The metric properties of the Engel Series expansion have been studied by Erdös, Rényi and Szüsz [2] and Rényi [10]. Schweiger [11] showed that S is ergodic and Thaler [15] found a whole family of σ-finite, infinite invariant measures for S. Fractal properties of exceptional sets related to the Engel Series expansion have been discussed by Liu and the second author, see [16, 9]. Further information on the Engel series can be found in Galambos [4, 5] and Schweiger [12].

In [6], Hartono, Kraaikamp and Schweiger introduced and studied a variation of the classical regular continued fraction (RCF) expansion. As is well-known, for every $x \in (0,1)$, the RCF-expansion of x can be obtained using the so-called *Gauss-map* $T : [0,1) \to [0,1)$, defined by

$$T(x) = \frac{1}{x} - \left[\frac{1}{x}\right], \quad x \neq 0;\ \ T(0) = 0.$$

Now – as the Engel Series expansion map is equal to the Lüroth Series map divided by $[1/x]$ – the Engel continued fraction (ECF) map $T_E : [0,1) \to [0,1)$ is given by

$$T_E(x) := \frac{T(x)}{\left[\frac{1}{x}\right]}, \quad x \neq 0;\ \ T_E(0) := 0, \tag{1}$$

hence the name, ECF (see also Figure 1).

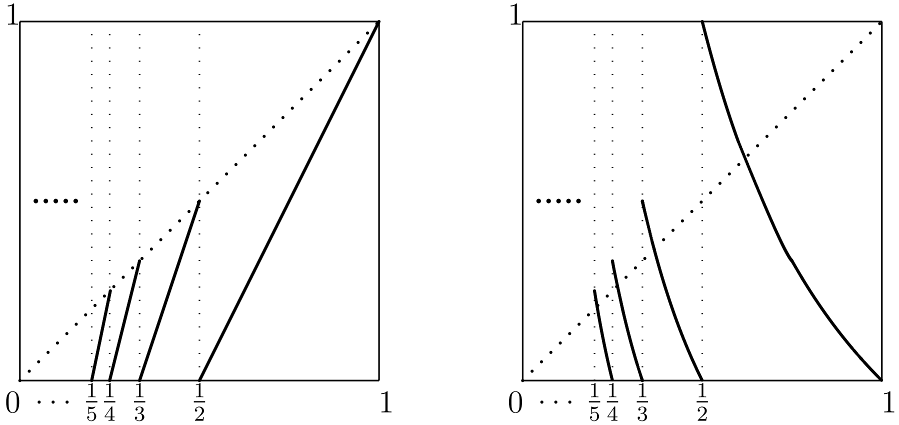

Figure 1. The Engel Series expansion map S (left), and the ECF-map T_E (right)

For any $x \in (0,1)$, the ECF-map generates a new continued fraction expansion of x of the form

$$x = \cfrac{1}{b_1(x) + \cfrac{b_1(x)}{b_2(x) + \ddots + \cfrac{b_{n-1}(x)}{b_n(x) + \cdots}}}, \quad b_n(x) \in \mathbb{N}, \ b_n(x) \leqslant b_{n+1}(x), \qquad (2)$$

where $b_n(x) = [1/T_E^{n-1}(x)]$, $n \geqslant 1$. We denote the continued fraction in the right-hand side of (2) – the ECF-expansion of x – by $[[0; b_1, b_2, \ldots, b_n, \ldots]]$. Note that for $x \in (0,1)$ one has that $q_1(x) = b_1(x) + 1$, but that in general no apparent relation exists between $q_n(x)$ and $b_n(x)$ for $n \geqslant 2$.

Hartono, Kraaikamp and Schweiger [6] studied the ergodic properties of T_E associated to this new continued fraction expansion. They showed that T_E has no finite invariant measure equivalent to the Lebesgue measure λ, but that T_E has infinitely many σ-finite, infinite invariant measures. Also it is shown that T_E is ergodic with respect to λ.

Clearly, as S is a piecewise linear map, and T_E is a continued fraction map (see Figure 1), the metric properties of the ECF-expansion are quite different from those of the Engel Series expansion. For example, in [10] Rényi introduced the random variables $\varepsilon_k = \varepsilon_k(x)$ $(k = 2, 3, \ldots)$, defined as the number of times that the integer k occurs in the non-decreasing sequence of digits $(q_n(x))_{n \geqslant 1}$. It is shown that the ε_k are independent variables and that the distribution of ε_k is given by $P(\varepsilon_k = r) = (k-1)/k^{r+1}$ $(r = 0, 1, \ldots)$. From this Rényi obtains most of the – at that time – known and several new metric results for Engel's series. Since T_E is a continued fraction map, the random variables ε_k, now defined for the ECF, are not independent, and Rényi's elegant approach cannot be applied here. See also the recent paper by Schweiger [13], where it is shown that algorithms – similar to the Engel Series expansion or the ECF – 'producing' monotonically non-decreasing sequences of digits can have very different metric behavior.

The aim of this paper is to derive metric properties of the digits $(b_n(x))_{n \geqslant 1}$ occurring in the Engel continued fraction expansion. We also give the Hausdorff dimensions of different kinds of exceptional sets on which the metric properties fail to hold.

This paper is organized as follows. In Section 2, we recall the basic properties of the Engel continued fraction expansions. Section 3 is devoted to study the metric properties of the digits. Also some fractal properties of exceptional sets are mentioned in this section.

2. The Engel Continued Fraction (ECF) Expansions

In this section we recall some basic properties of the ECF. Let $x \in (0,1)$, and define

$$b_1(x) := \left[\frac{1}{x}\right],$$
$$b_n(x) := b_1(T_E^{n-1}(x)), \quad n \geqslant 2, \ T_E^{n-1}(x) \neq 0.$$

From definition (1) of T_E, it follows that

$$x = \frac{1}{b_1(x) + b_1(x)T_E(x)} = \cdots = \cfrac{1}{b_1(x) + \cfrac{b_1(x)}{b_2(x) + \cdots + \cfrac{b_{n-1}(x)}{b_n(x) + b_n(x)T_E^n(x)}}},$$

where $T_E^0(x) = x$ and $T_E^n(x) = T_E(T_E^{n-1}(x))$ for $n \geqslant 1$ and

$$1 \leqslant b_1(x) \leqslant b_2(x) \leqslant \cdots \leqslant b_n(x) \leqslant \cdots$$

As usual the convergents are obtained via finite truncation,

$$\frac{P_n(x)}{Q_n(x)} := \cfrac{1}{b_1(x) + \cfrac{b_1(x)}{b_2(x) + \cdots + \cfrac{b_{n-1}(x)}{b_n(x)}}},$$

where $P_0(x) := 0$, $Q_0(x) := 1$. We have (see [6], Proposition 2.1)

$$P_n(x) = b_n(x)P_{n-1}(x) + b_{n-1}(x)P_{n-2}(x) \quad \text{for } n \geqslant 2, \tag{3}$$

$$Q_n(x) = b_n(x)Q_{n-1}(x) + b_{n-1}(x)Q_{n-2}(x) \quad \text{for } n \geqslant 2. \tag{4}$$

From (3) and (4), one has

$$P_n(x)Q_{n-1}(x) - P_{n-1}(x)Q_n(x) = (-1)^{n-1}\prod_{j=1}^{n-1} b_j(x) \quad \text{for } n \geqslant 2. \tag{5}$$

For any $x \in (0,1)$, let $\left\{\frac{P_n(x)}{Q_n(x)},\ n \geqslant 1\right\}$ be the sequence of ECF-convergents of x, then $\lim_{n\to\infty} \frac{P_n(x)}{Q_n(x)} = x$ and for any $n \geqslant 1$,

$$x = \frac{P_n(x) + b_n(x)T_E^n(x)P_{n-1}(x)}{Q_n(x) + b_n(x)T_E^n(x)Q_{n-1}(x)}. \tag{6}$$

For any $n \geqslant 1$ and $b_1, b_2, \ldots, b_n \in \mathbf{N}$ with $b_1 \leqslant b_2 \leqslant \cdots \leqslant b_n$, we define the cylinder sets $B(b_1, b_2, \ldots, b_n)$ by

$$B(b_1, b_2, \ldots, b_n) = \{x \in (0,1) :\ b_1(x) = b_1,\ b_2(x) = b_2, \ldots, b_n(x) = b_n\}.$$

The following results have been obtained in [6] and will play key roles in the sequel.

Lemma 1. *For any $n \geqslant 1$ and $b_1, b_2, \ldots, b_n \in \mathbf{N}$ with $b_1 \leqslant b_2 \leqslant \cdots \leqslant b_n$, we have*

$$\lambda(B(b_1, b_2, \ldots, b_n)) = \frac{\prod_{j=1}^{n-1} b_j}{Q_n(Q_n + Q_{n-1})},$$

where Q_n is obtained by (4) *recursively.*

Lemma 2. *For any $n \geqslant 1$ and $b_1, b_2, \ldots, b_n \in \mathbf{N}$ with $b_1 \leqslant b_2 \leqslant \cdots \leqslant b_n$,*

$$\sum_{b_1 \leqslant \cdots \leqslant b_n} \frac{\lambda(B(b_1, b_2, \ldots, b_n))}{b_n + 1} \leqslant \left(\frac{313}{324}\right)^n.$$

For any $n \geqslant 1$ and $x \in (0,1)$, setting

$$z_n(x) = b_n(x) T_E^n(x).$$

Lemma 3. *Let* $\gamma = \frac{313}{324}$, *then*

$$\lambda(\{x \in (0,1) : z_n(x) < t\}) = t(1 + O(\gamma^n)).$$

3. Metric Properties of $\{b_n(x), n \geqslant 1\}$

In this section, we will study the metric properties of the digits occurring in the ECF. We shall make use of the following more general result obtained by Galambos [4].

Lemma 4. *Let* $X_1, X_2, \ldots, X_n, \ldots$ *be random variables defined on a given probability space and assume that*

(i) *the* X*'s are uniformly bounded from below, i.e., there is a fixed real number* M *such that*

$$X_i \geqslant M \quad (i = 1, 2, \ldots);$$

(ii) $\lim_{n\to\infty} \mathbf{E} X_n = 0$;
(iii) *the variance*

$$\mathbf{Var}\left(\sum_{i=1}^{n} X_i\right) = O(n^t) \quad (0 < t < 2).$$

Then

$$\mathbf{P}\left(\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} X_i = 0\right) = 1.$$

Erdös, Rényi and Szüsz showed in [2] that for the Engel Series expansion for λ-almost all x

$$\lim_{n\to\infty} q_n^{1/n}(x) = e,$$

see also [5, p. 101]. For the ECF a similar result holds.

Theorem 5. *For* λ*-almost all* $x \in (0,1)$,

$$\lim_{n\to\infty} b_n^{1/n}(x) = e.$$

Proof. For any $x \in (0,1)$ and $n \geqslant 1$, setting

$$R_n(x) = \frac{b_{n+1}(x)}{b_n(x)} = \frac{b_{n+1}(x) T_E^n(x)}{z_n(x)},$$

where $\{z_n(x), n \geqslant 1\}$ are defined in Section 2, and

$$X_n(x) = \log R_n(x) - 1.$$

Now we check that $\{X_n(x), n \geqslant 1\}$ satisfies (i), (ii), (iii) in Lemma 4.

It is clear that $X_n(x) \geqslant -1$ for any $n \geqslant 1$, thus (i) is satisfied.

Since by the algorithm of ECF,

$$|b_{n+1}(x)T_E^n(x) - 1| \leqslant \frac{1}{b_{n+1}(x)+1},$$

we have

$$\log R_n(x) = -\log z_n(x) + O\left(\frac{1}{b_{n+1}(x)}\right) \tag{7}$$

with a uniform constant in $O(\cdot)$.

Let $F_n(t)$ be the distribution function of $z_n(x)$, by Lemma 3, we have

$$\mathbf{E}(-\log z_n(x)) = \int_0^1 -\log t \; dF_n(t) = 1 + O(\gamma^n).$$

Thus

$$\mathbf{E}(\log R_n(x)) = 1 + O(\gamma^n) + \mathbf{E}\left(O\left(\frac{1}{b_{n+1}(x)}\right)\right).$$

By Lemma 2 and the Borel-Cantelli lemma, we know $b_n(x) \to \infty$ a.s., thus

$$\lim_{n\to\infty} \mathbf{E}X_n(x) = 0, \tag{8}$$

and

$$\mathbf{E}\left(\sum_{i=1}^{n} X_i(x)\right) = o(n). \tag{9}$$

Now we estimate $\mathbf{E}\big(\sum_{i=1}^{n} X_i(x)\big)^2$.

$$\begin{aligned}
\mathbf{E}X_n^2(x) &= \mathbf{E}(\log^2 R_n(x) - 2\log R_n(x) + 1) \\
&= \mathbf{E}\Bigg(\log^2 z_n(x) - 2\log z_n(x) O\left(\frac{1}{b_{n+1}(x)}\right) \\
&\qquad + O\left(\frac{1}{b_{n+1}^2(x)}\right) - 2\log R_n(x) + 1\Bigg).
\end{aligned}$$

Since

$$\mathbf{E}(\log^2 z_n(x)) = \int_0^1 \log^2 t \; dF_n(t) = 2 + O(\gamma^n),$$

$$\mathbf{E}\left(-2\log z_n(x) O\left(\frac{1}{b_{n+1}(x)}\right)\right) = O(\mathbf{E}\log z_n(x)) = O(1),$$

we have

$$\mathbf{E}X_n^2(x) = O(1). \tag{10}$$

Now

$$\begin{aligned}
&\mathbf{E}(X_1(x)+X_2(x)+\cdots+X_n(x))^2\\
&\quad\leqslant 2\mathbf{E}(X_1(x)+X_2(x)+\cdots+X_p(x))^2+\mathbf{E}(X_{p+1}(x)+\cdots+X_n(x))^2\\
&\quad= O(p^2)+2\sum_{k=p+1}^{n}\mathbf{E}X_k^2(x)+4\sum_{p<k<m\leqslant n}\mathbf{E}(X_k(x)X_m(x)),
\end{aligned}$$

where $p=[n^{1/4}]$. For any $p<k<m\leqslant n$, by Lemma 1 and (4), we have

$$\begin{aligned}
&\mathbf{E}(X_k(x)X_m(x))\\
&\quad=\mathbf{E}\left(\left(\log\frac{b_{k+1}(x)}{b_k(x)}-1\right)\left(\log\frac{b_{m+1}(x)}{b_m(x)}-1\right)\right)\\
&\quad=\sum_{b_1\leqslant\cdots\leqslant b_m\leqslant b_{m+1}}\left(\left(\log\frac{b_{k+1}}{b_k}-1\right)\left(\log\frac{b_{m+1}}{b_m}-1\right)\right)\lambda(B(b_1,\ldots,b_m,b_{m+1}))\\
&\quad=\sum_{b_1\leqslant\cdots\leqslant b_m}\left(\log\frac{b_{k+1}}{b_k}-1\right)\lambda(B(b_1,\ldots,b_m))\\
&\qquad\cdot\sum_{b_{m+1}\geqslant b_m}\left(\log\frac{b_{m+1}}{b_m}-1\right)\frac{\lambda(B(b_1,\ldots,b_m,b_{m+1}))}{\lambda(B(b_1,\ldots,b_m))}\\
&\quad=\sum_{b_1\leqslant\cdots\leqslant b_m}\left(\log\frac{b_{k+1}}{b_k}-1\right)\lambda(B(b_1,\ldots,b_m))\\
&\qquad\cdot\sum_{b_{m+1}\geqslant b_m}\left(\log\frac{b_{m+1}}{b_m}-1\right)\frac{b_m(1+y)}{(b_{m+1}+b_my)(b_{m+1}+1+b_my)},
\end{aligned}$$

where $y=\frac{Q_{m-1}}{Q_m}$. Since

$$\begin{aligned}
&\sum_{b_{m+1}\geqslant b_m}\left(\log\frac{b_{m+1}}{b_m}-1\right)\frac{b_m(1+y)}{(b_{m+1}+b_my)(b_{m+1}+1+b_my)}\\
&\qquad\leqslant\sum_{b_{m+1}\geqslant b_m}\left(\log\frac{b_{m+1}}{b_m}-1\right)\frac{4b_m}{b_{m+1}^2}\\
&\qquad\leqslant\sum_{j=1}^{\infty}\sum_{jb_m\leqslant b_{m+1}\leqslant(j+1)b_m}(\log(j+1)-1)\frac{4}{jb_{m+1}}\\
&\qquad\leqslant\sum_{j=1}^{\infty}(\log(j+1)-1)\frac{4}{j^2}:=M_1,
\end{aligned}$$

we have

$$\begin{aligned}
&\mathbf{E}(X_k(x)X_m(x))\\
&\leqslant M_1 \sum_{b_1 \leqslant \cdots \leqslant b_m} \left(\log \frac{b_{k+1}}{b_k} - 1\right)\lambda(B(b_1,\ldots,b_m))\\
&= M_1 \sum_{b_1 \leqslant \cdots \leqslant b_{k+1}} \left(\log \frac{b_{k+1}}{b_k} - 1\right)\lambda(B(b_1,\ldots,b_{k+1}))\\
&= M_1\mathbf{E}(X_k(x))\\
&= O(\gamma^k) + \mathbf{E}O\left(\frac{1}{b_{k+1}(x)}\right)\\
&= O(\gamma^p) + \mathbf{E}O\left(\frac{1}{b_p(x)}\right).
\end{aligned}$$

It follows

$$\sum_{p<k<m\leqslant n} \mathbf{E}(X_k(x)X_m(x)) \leqslant n^2\left(O(\gamma^{n^{1/4}}) + O\left(\mathbf{E}\frac{1}{b_{n^{1/4}}(x)}\right)\right).$$

For any $q \geqslant 1$, by Lemma 2,

$$\begin{aligned}
\mathbf{E}\left(\frac{1}{b_q(x)}\right) &= \sum_{b_1 \leqslant \cdots \leqslant b_q} \frac{\lambda(B(b_1,b_2,\ldots,b_q))}{b_q}\\
&\leqslant 2 \sum_{b_1 \leqslant \cdots \leqslant b_q} \frac{\lambda(B(b_1,b_2,\ldots,b_q))}{b_q+1} \leqslant 2\gamma^q,
\end{aligned}$$

thus

$$\sum_{p<k<m\leqslant n} \mathbf{E}(X_k(x)X_m(x)) \leqslant n^2\big(O(\gamma^{n^{1/4}})\big). \tag{11}$$

By (10) and (11), we have

$$\mathbf{E}(X_1(x) + X_2(x) + \cdots + X_n(x))^2 = O(n). \tag{12}$$

By (8) and (12), $\{X_n(x), n \geqslant 1\}$ satisfies (ii), (iii) in Lemma 4. By Lemma 4, we have for λ-almost all $x \in (0,1)$,

$$\lim_{n\to\infty} b_n^{1/n}(x) = e.$$

□

Theorem 5 has a number of corollaries, reminiscent of classical results by Lévy and Khintchine for the regular continued fraction expansion (RCF). Khintchine [7] showed in 1935 that for λ-almost all $x \in (0,1)$, with RCF-expansion $x = [0; a_1, a_2, \ldots]$,

$$\lim_{n\to\infty} (a_1 \cdot a_2 \cdot \cdots \cdot a_n)^{1/n} = \prod_{k=1}^{\infty} \left(1 + \frac{1}{k(k+1)}\right)^{\frac{\log k}{\log 2}} = 2.6854\ldots\ .$$

An immediate corollary of Theorem 5 is the following 'Khintchine-type' result for the ECF.

Theorem 6. *For λ-almost all $x \in (0,1)$,*

$$\lim_{n\to\infty} (b_1 \cdot b_2 \cdot \cdots \cdot b_n)^{1/n^2} = \sqrt{e} = 1.64872\ldots .$$

In 1929 Paul Lévy [8] showed, that for λ-almost all $x \in (0,1)$ one has that

$$\lim_{n\to\infty} \frac{1}{n} \log \left| x - \frac{p_n(x)}{q_n(x)} \right| = \frac{-\pi^2}{6\log 2} = -2.3731\ldots,$$

where $(p_n(x)/q_n(x))_{n \geqslant 1}$ is the sequence of RCF-convergents of x. For the ECF we have a similar result for the convergent rate.

Theorem 7. *For λ-almost all $x \in (0,1)$,*

$$\lim_{n\to\infty} \frac{1}{n^2} \log \left| x - \frac{P_n(x)}{Q_n(x)} \right| = -\frac{1}{2}.$$

Proof. By (5) and (6),

$$\begin{aligned} \left| x - \frac{P_n(x)}{Q_n(x)} \right| &= \left| \frac{P_n(x) + b_n(x)T_E^n(x)P_{n-1}(x)}{Q_n(x) + b_n(x)T_E^n(x)Q_{n-1}(x)} - \frac{P_n(x)}{Q_n(x)} \right| \\ &= \frac{T_E^n(x) \prod_{j=1}^n b_j(x)}{Q_n(x)(Q_n(x) + b_n(x)T_E^n(x)Q_{n-1}(x))}. \end{aligned}$$

Since

$$\frac{1}{b_{n+1}(x)+1} < T_E^n(x) \leqslant \frac{1}{b_{n+1}(x)},$$

we have

$$\frac{\prod_{j=1}^n b_j(x)}{2Q_n^2(x)b_{n+1}(x)} \leqslant \left| x - \frac{P_n(x)}{Q_n(x)} \right| \leqslant \frac{\prod_{j=1}^n b_j(x)}{Q_n^2(x)b_{n+1}(x)}.$$

Notice that

$$b_n(x)Q_{n-1}(x) \leqslant Q_n(x) \leqslant 2b_n(x)Q_{n-1}(x), \tag{13}$$

we have

$$2^{-(2n+1)} \left(\prod_{j=1}^{n+1} b_j(x) \right)^{-1} \leqslant \left| x - \frac{P_n(x)}{Q_n(x)} \right| \leqslant \left(\prod_{j=1}^{n+1} b_j(x) \right)^{-1}.$$

By Theorem 6, we have for λ-almost all $x \in (0,1)$,

$$\lim_{n\to\infty} \frac{1}{n^2} \sum_{j=1}^{n} \log b_j(x) = \frac{1}{2},$$

thus for λ-almost all $x \in (0,1)$,

$$\lim_{n\to\infty} \frac{1}{n^2} \log \left| x - \frac{P_n(x)}{Q_n(x)} \right| = -\frac{1}{2}.$$

□

In view of Theorem 5 it is natural to consider for $\alpha \geqslant 1$ the sets $A(\alpha)$, defined by

$$A(\alpha) = \left\{ x \in (0,1) : \lim_{n\to\infty} b_n^{1/n}(x) = \alpha \right\}.$$

In [9] and [16] similar sets $A(\alpha)$ were introduced for the Engel Series expansion, and it was shown in [9] that for any $\alpha \geqslant 1$, the Hausdorff dimension $\dim_H A(\alpha)$ equals 1, thus settling a question raised by Galambos in [5]. This result can be generalized to the ECF, i.e., for the ECF one has that

$$\dim_H A(\alpha) = 1, \quad \text{for any } \alpha \geqslant 1.$$

There are two other classical results on RCF by Lévy [8], for which similar results hold for the ECF. Lévy showed that for λ-almost all $x \in (0,1)$

$$\lim_{n\to\infty} \frac{1}{n} \log q_n = \frac{\pi^2}{12 \log 2}$$

and

$$\lim_{n\to\infty} \frac{1}{n} \log \left(\lambda(\Delta_n(x)) \right) = \frac{-\pi^2}{6 \log 2},$$

where

$$\Delta_n(x) = \{ y \in (0,1) : a_1(y) = a_1(x), \ldots, a_n(y) = a_n(x) \}.$$

The following 'Lévy-type' theorem on ECF is a direct consequence of (13), Theorems 5 and 6, and the fact that $Q_0(x) = 1$.

Theorem 8. *For λ-almost all $x \in (0,1)$,*

$$\lim_{n\to\infty} \frac{1}{n^2} \log Q_n(x) = \frac{1}{2}$$

and

$$\lim_{n\to\infty} \frac{1}{n^2} \log \left(\lambda(B_n(x)) \right) = -\frac{1}{2}.$$

Here $B_n(x)$ is the abbreviation of $B(b_1, b_2, \ldots, b_n)$, if the ECF-expansion of x is given by $x = [[0; b_1, b_2, \ldots]]$.

We consider again the random variables $R_n(x)$, defined by

$$R_n(x) = \frac{b_{n+1}(x)}{b_n(x)}, \quad n = 1, 2, \ldots .$$

We have the following result.

Theorem 9. *The sequence* $\frac{1}{n\log n}\sum_{j=1}^{n}\frac{b_{j+1}(x)}{b_j(x)}$, $n \geqslant 1$, *converges in probability to* 1. *That is to say, for any fixed* $\varepsilon > 0$,

$$\lim_{n\to\infty}\lambda\left\{x\in(0,1):\left|\frac{1}{n\log n}\sum_{j=1}^{n}\frac{b_{j+1}(x)}{b_j(x)}-1\right|>\varepsilon\right\}=0.$$

Proof. Fix $n \geqslant 1$. For any $1 \leqslant k \leqslant n$, define

$$U_k(x)=\begin{cases}\frac{b_{k+1}(x)}{b_k(x)} & \text{if } \frac{b_{k+1}(x)}{b_k(x)}\leqslant n\log n\\ 0 & \text{otherwise,}\end{cases}$$

$$V_k(x)=\begin{cases}0 & \text{if } \frac{b_{k+1}(x)}{b_k(x)}\leqslant n\log n\\ \frac{b_{k+1}(x)}{b_k(x)} & \text{otherwise.}\end{cases}$$

Then

$$\begin{aligned}\lambda&\left\{x\in(0,1):\left|\frac{1}{n\log n}\sum_{k=1}^{n}\frac{b_{k+1}(x)}{b_k(x)}-1\right|>\varepsilon\right\}\\ &\leqslant\lambda\left\{x\in(0,1):\left|\frac{1}{n\log n}\sum_{k=1}^{n}U_k(x)-1\right|>\varepsilon\right\}\\ &\quad+\lambda\{x\in(0,1):\sum_{k=1}^{n}V_k(x)\neq 0\}\\ &=:\lambda(A_n)+\lambda(B_n).\end{aligned}$$

By Lemma 1,

$$\begin{aligned}\lambda&\left\{x\in(0,1):\frac{b_{k+1}(x)}{b_k(x)}>n\log n\right\}\\ &=\sum_{b_1\leqslant\cdots\leqslant b_k}\ \sum_{b_{k+1}\geqslant n\log n\cdot b_k}\lambda(B(b_1,\ldots,b_k,b_{k+1}))\\ &=\sum_{b_1\leqslant\cdots\leqslant b_k}\lambda(B(b_1,\ldots,b_k))\sum_{b_{k+1}\geqslant n\log n\cdot b_k}\frac{b_k(1+y)}{(b_{k+1}+b_ky)(b_{k+1}+1+b_ky)},\end{aligned}$$

where $y=\frac{Q_{k-1}}{Q_k}$. Since

$$\begin{aligned}&\sum_{b_{k+1}\geqslant n\log n\cdot b_k}\frac{b_k(1+y)}{(b_{k+1}+b_ky)(b_{k+1}+1+b_ky)}\\ &\qquad\leqslant\sum_{b_{k+1}\geqslant n\log n\cdot b_k}\frac{2b_k}{b_{k+1}^2}=O\left(\frac{1}{n\log n}\right),\end{aligned}$$

we have

$$\lambda(B_n)\leqslant\sum_{k=1}^{n}\lambda\left\{x\in(0,1):\frac{b_{k+1}(x)}{b_k(x)}>n\log n\right\}=O\left(\frac{1}{n\log n}\right).\tag{14}$$

For any $1 \leqslant k \leqslant n$,

$$\begin{aligned}\mathbf{E}U_k(x) &= \sum_{b_1 \leqslant \cdots \leqslant b_k} \sum_{b_k \leqslant b_{k+1} \leqslant n \log n \cdot b_k} \frac{b_{k+1}}{b_k} \lambda(B(b_1, \ldots, b_k, b_{k+1})) \\ &= \sum_{b_1 \leqslant \cdots \leqslant b_k} \lambda(B(b_1, \ldots, b_k)) \\ &\quad \times \sum_{b_k \leqslant b_{k+1} \leqslant n \log n \cdot b_k} \frac{b_{k+1}}{b_k} \cdot \frac{b_k(1+y)}{(b_{k+1}+b_k y)(b_{k+1}+1+b_k y)}.\end{aligned}$$

Since

$$\begin{aligned}&\sum_{b_k \leqslant b_{k+1} \leqslant n \log n \cdot b_k} \frac{b_{k+1}}{b_k} \cdot \frac{b_k(1+y)}{(b_{k+1}+b_k y)(b_{k+1}+1+b_k y)} \\ &\quad \approx \sum_{b_k \leqslant b_{k+1} \leqslant n \log n \cdot b_k} \frac{b_{k+1}}{b_k} \cdot \frac{b_k}{b_{k+1}^2} = \sum_{b_k \leqslant b_{k+1} \leqslant n \log n \cdot b_k} \frac{1}{b_{k+1}} \\ &\quad \approx \log n,\end{aligned}$$

where $f_k \approx g_k$ denotes there exist positive constants C_1, C_2 such that $C_1 f_k \leqslant g_k \leqslant C_2 f_k$. Thus

$$\mathbf{E}U_k(x) \approx \log n. \tag{15}$$

We claim that

$$\lim_{n\to\infty} \frac{\mathbf{E}U_n(x)}{\log n} = 1. \tag{16}$$

In fact,

$$\begin{aligned}\mathbf{E}U_n(x) &= \sum_{b_1 \leqslant \cdots \leqslant b_n} \sum_{b_n \leqslant b_{n+1} \leqslant n \log n \cdot b_n} \frac{b_{n+1}}{b_n} \lambda(B(b_1, \ldots, b_k, b_{n+1})) \\ &= \sum_{b_1 \leqslant \cdots \leqslant b_n} \lambda(B(b_1, \ldots, b_n)) \\ &\quad \times \sum_{b_n \leqslant b_{n+1} \leqslant n \log n \cdot b_n} \frac{b_{n+1}}{b_n} \cdot \frac{b_n(1+y)}{(b_{n+1}+b_n y)(b_{n+1}+1+b_n y)}.\end{aligned}$$

Since

$$\begin{aligned}&\sum_{b_n \leqslant b_{n+1} \leqslant n \log n \cdot b_n} \frac{b_{n+1}}{b_n} \cdot \frac{b_n(1+y)}{(b_{n+1}+b_n y)(b_{n+1}+1+b_n y)} \\ &\quad = \sum_{b_n \leqslant b_{n+1} \leqslant n \log n \cdot b_n} \frac{b_{n+1}}{b_n} \cdot \frac{b_n}{b_{n+1}(b_{n+1}+1)}\end{aligned}$$

$$+ \sum_{b_n \leqslant b_{n+1} \leqslant n \log n \cdot b_n} \frac{b_{n+1}}{b_n} \left(\frac{b_n(1+y)}{(b_{n+1}+b_n y)(b_{n+1}+1+b_n y)} - \frac{b_n}{b_{n+1}(b_{n+1}+1)} \right)$$

$$= \sum_{b_n \leqslant b_{n+1} \leqslant n \log n \cdot b_n} \frac{1}{b_{n+1}+1} + \sum_{b_n \leqslant b_{n+1} \leqslant n \log n \cdot b_n} \frac{b_{n+1}}{b_n} O\left(\frac{b_n y}{b_{n+1}^2}\right)$$

$$= \sum_{b_n \leqslant b_{n+1} \leqslant n \log n \cdot b_n} \frac{1}{b_{n+1}+1} + O(1) \frac{1}{b_n+1} \log n,$$

by Lemma 2, we have

$$\lim_{n \to \infty} \frac{\mathbf{E} U_n(x)}{\log n} = 1,$$

thus (16) holds.

Notice that $1 \leqslant k \leqslant n$.

$$\begin{aligned}
\mathbf{E} U_k^2(x) &= \sum_{b_1 \leqslant \cdots \leqslant b_k} \sum_{b_k \leqslant b_{k+1} \leqslant n \log n \cdot b_k} \frac{b_{k+1}^2}{b_k^2} \lambda(B(b_1, \ldots, b_k, b_{k+1})) \\
&= \sum_{b_1 \leqslant \cdots \leqslant b_k} \lambda(B(b_1, \ldots, b_k)) \\
&\quad \times \sum_{b_k \leqslant b_{k+1} \leqslant n \log n \cdot b_k} \frac{b_{k+1}^2}{b_k^2} \cdot \frac{b_k(1+y)}{(b_{k+1}+b_k y)(b_{k+1}+1+b_k y)} \\
&\approx \sum_{b_1 \leqslant \cdots \leqslant b_k} \lambda(B(b_1, \ldots, b_k)) \times \sum_{b_k \leqslant b_{k+1} \leqslant n \log n \cdot b_k} \frac{b_{k+1}^2}{b_k^2} \cdot \frac{b_k}{b_{k+1}^2} \\
&\approx n \log n,
\end{aligned}$$

by Chebyshev's inequality and (15), we get

$$\begin{aligned}
&\lambda\left\{ x \in (0,1) : \left| \sum_{k=1}^{n} (U_k(x) - \mathbf{E}(U_k(x))) \right| > \varepsilon \sum_{k=1}^{n} \mathbf{E}(U_k(x)) \right\} \\
&\quad \leqslant \frac{\mathbf{Var}\left(\sum_{k=1}^n U_k(x) \right)}{\left(\varepsilon \sum_{k=1}^n \mathbf{E}(U_k(x)) \right)^2} \leqslant \frac{\mathbf{E}\left(\sum_{k=1}^n U_k(x) \right)^2}{\left(\varepsilon \sum_{k=1}^n \mathbf{E}(U_k(x)) \right)^2} \leqslant \frac{n \sum_{k=1}^n \mathbf{E}(U_k^2(x))}{\left(\varepsilon \sum_{k=1}^n \mathbf{E}(U_k(x)) \right)^2} \\
&\quad = O\left(\frac{\log n}{\log^2(n \log n)} \right).
\end{aligned}$$

That is to say, $\frac{1}{\sum_{k=1}^{n} \mathbf{E}(U_k(x))} \sum_{k=1}^{n} U_k(x)$ converges in probability to 1. Since $\lim_{n\to\infty} \frac{\mathbf{E}U_n(x)}{\log n} = 1$ by (16), we get

$$\lim_{n\to\infty} \frac{1}{n \log n} \sum_{k=1}^{n} \mathbf{E}(U_k(x)) = 1.$$

Thus $\mathbf{P}(A_n) \to 0$ as $n \to \infty$. □

Remark 10. By (15), we know $\mathbf{E}R_n(x) = \infty$ for any $n \geqslant 1$.

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