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Semi-Classical Analysis of a Dirac Equation without Adiabatic Decoupling

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Abstract. We study adiabatic decoupling for Dirac equation with some scaling which yields that the mass appears with a coefficient ε^α where ε is the semi-classical parameter and $\alpha > 0$. Therefore, the system presents an avoided crossing. The scale $\alpha = 1/2$ is critical: adiabatic decoupling holds for $\alpha \in (0, 1/2)$ while for $\alpha \geqslant 1/2$, there is energy transfer at leading order between the two modes. We describe this transfer in terms of two-scale Wigner measures by means of Landau-Zener formula which takes into account the change of polarization of the measures after the crossing.

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1. Introduction

Let us consider the Dirac equation

$$\left(\sum_{\mu=0}^{3} \gamma^\mu (i\hbar\partial_\mu + e\tilde{A}_\mu) - Mc \right) \phi = 0, \tag{1}$$

where ϕ denotes the "spinorfield", $\phi = \phi(y) \in \mathbf{C}^4$, $y = (y_0, y') \in \mathbf{R}^4$ with y_0 the time variable and $y' = (y_1, y_2, y_3)$ the space variable and where γ^μ, $0 \leqslant \mu \leqslant 3$, are the 4×4 Dirac matrices which satisfy

$$(\gamma^0)^* = \gamma^0, \quad (\gamma^k)^* = -\gamma^k, \quad (\gamma^0\gamma^k)^* = \gamma^0\gamma^k, \quad 1 \leqslant k \leqslant 3 \tag{2}$$

$$\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 0, \quad \mu \neq \nu, \quad (\gamma^0)^2 = \mathrm{Id}, \quad (\gamma^k)^2 = -\mathrm{Id}. \tag{3}$$

We use the standard representation of Dirac matrix: the Dirac-Pauli one (see [32] p. 36). In this representation, the 4×4 matrices γ^μ are functions of the 2×2 Pauli matrices σ_k, $1 \leqslant k \leqslant 3$. More precisely, we have

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{4}$$

and

$$\gamma^0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad \gamma^k = \begin{pmatrix} 0 & \sigma_k \\ -\sigma_k & 0 \end{pmatrix}, \qquad \gamma^5 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{5}$$

The functions $\tilde{A}_\mu = \tilde{A}_\mu(y)$ for $0 \leqslant \mu \leqslant 3$ are the components of the electromagnetic potential, in particular $\tilde{A}_0$ is the electric potential and $\tilde{A} = (\tilde{A}_1, \tilde{A}_2, \tilde{A}_3)$ is the magnetic potential vector. Hence the electric field is $E = \partial_{y_0}\tilde{A} - \nabla_{y'}\tilde{A}_0$ and the magnetic field $B = \mathrm{curl}_{y'}\tilde{A}$. The physical constants M and e are the electron's rest mass and its charge, c is the velocity of light and $\hbar$ the Planck constant. The matrix

$$P = \sum_{\mu=1}^{3} \gamma^0\gamma^\mu(\hbar\xi - e\tilde{A}_\mu) + \gamma^0 Mc$$

has two eigenvalues $l^\pm = \pm\sqrt{M^2c^2 + |\hbar\xi - e\tilde{A}_\mu|^2}$ which are of multiplicity 2.

Our aim is to study adiabatic decoupling for this equation in some special situation which is slightly different than the usual one. Several works have been devoted to the study of Dirac equation with slowly varying external potentials. One supposes that $\tilde{A} = A(\varepsilon y)$ and one studies ϕ on the macroscopic scale, i.e. one studies the evolution as ε goes to 0 of $\psi^\varepsilon(t,x) = \phi(y)$. Thus, one analyzes the system

$$i\hbar\varepsilon\partial_t\psi^\varepsilon = \sum_{1}^{3} \gamma^0\gamma^\mu \left(\frac{\hbar\varepsilon}{i}\partial_\mu - eA_\mu(t,x) \right)\psi^\varepsilon - eA_0(t,x)\psi^\varepsilon + \gamma^0 Mc\psi^\varepsilon.$$

Any solution of this system can be decomposed on the two modes associated to each eigenvalue $l^\pm$ and these two components evolve independently. Such a phenomenon is called adiabatic decoupling (from "*adiabatos*" = impassable). The reader can refer to [31] where he will find a presentation of adiabatic theory and its application to Dirac equation with slowly variable coefficients or to [14] where adiabatic decoupling is proved in terms of Wigner measures.

We focus here also on the case of slowly varying potentials but we scale the size of the electromagnetic fields. More precisely, we set

$$\tilde{A}(y) = \lambda A(\varepsilon' y), \qquad \varepsilon' y = (t,x) = (t, x_1, x_2, x_3) \in \mathbf{R} \times \mathbf{R}^3,$$

and we suppose that $|A| \sim 1$, $\varepsilon' \ll 1$ and $\lambda \geqslant 1$. We study ϕ on the macroscopic scale: we set $\psi(t,x) = \phi(y)$. Then, ψ satisfies

$$i\frac{\hbar\varepsilon'}{\lambda}\partial_t\psi = \sum_{1}^{3} \gamma^0\gamma^\mu \left(\frac{\hbar\varepsilon'}{i\lambda}\partial_\mu - eA_\mu(t,x) \right)\psi - eA_0(t,x)\psi + \gamma^0 \frac{cM}{\lambda}\psi.$$

We set $\varepsilon = \frac{\hbar\varepsilon'}{e\lambda}$, thus $\varepsilon \ll 1$. We suppose that $\frac{Mc}{e} \sim 1$ and we set

$$\frac{Mc}{e\lambda} = m\varepsilon^\alpha, \qquad \alpha > 0, \quad m \sim 1.$$

We study the asymptotic behavior as ε goes to 0. In particular, the electromagnetic vector field is of size $\varepsilon'\lambda \sim \varepsilon\lambda^2 \sim \varepsilon^{1-2\alpha}$. If $\alpha = 0$, we are in the same context as in [30], [32] and [14]. High values of the electromagnetic vector field corresponds to $\alpha \geqslant 1/2$. The reader can refer to [26] and [27] for results on Dirac equation in high electromagnetic fields. We prove, with Wigner measures approach, that the adiabatic decoupling becomes false as soon as $\alpha \geqslant 1/2$, i.e. for high electromagnetic vector fields. Roughly speaking, if $\alpha \geqslant 1/2$, the mass is not big enough so that the adiabatic Theorem applies and there happens some Landau-Zener type's transition specific to avoided crossings that we describe in terms of two-scale Wigner measures. In the following, we focus on the system

$$\begin{cases} i\varepsilon\partial_t\psi^\varepsilon = P_\varepsilon(t,x,\varepsilon D_x)\psi^\varepsilon, \\ \psi^\varepsilon|_{t=0} = \psi_0^\varepsilon \end{cases} \tag{6}$$

where P_ε is the matrix-valued symbol

$$P_\varepsilon(t,x,\xi) = \sum_{k=1}^{3} \gamma^0\gamma^k(\xi_k - A_k(t,x)) + V(t,x) + \varepsilon^\alpha m\gamma^0 \tag{7}$$

with $\alpha > 0$, $m > 0$, $A = (A_1, A_2, A_3)$ a $\mathscr{C}^\infty$ vector field (the magnetic potential vector) and V a $\mathscr{C}^\infty$ function (the electric potential).

The Wigner measures approach is of interest because the limit of physically meaningful quantities can be simply expressed in terms of Wigner measures. For example, in [14] (see also [30] and [2]), the authors calculate the weak limit of the relativistic current density J_k, $1 \leqslant k \leqslant 3$,

$$J_k^\varepsilon(t,x) = \gamma^0\gamma^k\psi^\varepsilon(t,x)\cdot\overline{\psi}^\varepsilon(t,x)$$

and of the relativistic position density n

$$n^\varepsilon(t,x) = |\psi^\varepsilon(t,x)|^2.$$

Indeed, a Wigner measure μ of the family (ψ^ε) is a positive matrix-valued measure on $T^*\mathbf{R}^4$ which satisfy – up to some subsequence – that, for all the observable $a \in \mathscr{C}_0^\infty(\mathbf{R}^8, \mathbf{C}^{4,4})$,

$$(\mathrm{op}_\varepsilon(a)\psi^\varepsilon|\psi^\varepsilon) \underset{\varepsilon\to 0}{\longrightarrow} \mathrm{tr}\langle a, \mu\rangle := \mathrm{tr}\left(\int a(x,\xi)\,\mathrm{d}\mu(x,\xi)\right),$$

where $\mathrm{op}_\varepsilon(a)$ denotes the semi-classical pseudo-differential operator of symbol a; the kernel of $\mathrm{op}_\varepsilon(a)$ is, with Weyl quantification,

$$k(x,y) = \int a\left(\frac{x+y}{2},\xi\right)\mathrm{e}^{\frac{i(x-y)\cdot\xi}{\varepsilon}}\frac{d\xi}{(2\pi\varepsilon)^4}.$$

Let us assume that (ψ_0^ε) satisfies some assumption called ε-oscillation which implies that, roughly speaking, the oscillations of (ψ_0^ε) are not greater than $\frac{1}{\varepsilon}$, (see [13]), namely

$$\limsup_{\varepsilon\to 0}\int_{|\xi|\geqslant R/\varepsilon}|\widehat{\psi}_0^\varepsilon(\xi)|^2\,\mathrm{d}\xi \underset{R\to+\infty}{\longrightarrow} 0.$$

Then, the weak limits of n^ε and J_k^ε can be expressed as

$$\mathrm{w}-\lim_{\varepsilon\to 0} n^\varepsilon(t,x)\,\mathrm{d}t\,\mathrm{d}x = \mathrm{tr}\left(\int_{\tau,\xi}\mu(t,x,\mathrm{d}\tau,\mathrm{d}\xi)\right),$$
$$\mathrm{w}-\lim_{\varepsilon\to 0} J_k^\varepsilon(t,x)\,\mathrm{d}t\,\mathrm{d}x = \mathrm{tr}\left(\int_{\tau,\xi}\gamma^0\gamma^k\mu(t,x,\mathrm{d}\tau,\mathrm{d}\xi)\right).$$

In [14], Gérard et al. describe the Wigner measures of (ψ_ε) in the case $\alpha = 0$. The fact that $\alpha \neq 0$ induces serious difficulties that our contribution aims to deal with. Actually, the analysis performed in [14] crucially uses the fact that as $\alpha = 0$, the eigenvalues of the matrix P_ε,

$$\lambda_\varepsilon^\pm(t,x,\xi) = \pm\sqrt{|\xi - A(t,x)|^2 + \varepsilon^{2\alpha}m^2} + V(t,x), \tag{8}$$

do not depend on ε, are distinct and of constant multiplicity 2.

For $\alpha \neq 0$,

$$\lim_{\varepsilon\to 0}\lambda_\varepsilon^+(t,x,A(t,x)) = \lim_{\varepsilon\to 0}\lambda_\varepsilon^-(t,x,A(t,x)).$$

The eigenvalues are still distinct for all (t,x) but they cross asymptotically as ε goes to 0. One said that the system (6) displays an avoided crossing, by contrast with a "real" crossing for which there should exist (t_0,x_0) such that $\lambda_\varepsilon^+(t_0,x_0) = \lambda_\varepsilon^-(t_0,x_0)$. Crossings are usually characterized by the codimension of the singular set. The crossing here is of codimension 3 since

$$(t,x,\tau,\xi) \mapsto \xi - A(t,x)$$

is of rank 3.

The works of Hagedorn and Joye ([16]–[19], [21]) and those of Colin de Verdière et al. [5] show that avoided crossings yield transfer of energy at leading order between the two modes and thus, that there is no adiabatic decoupling. This so-called *Landau-Zener* phenomenon has been first described independently and simultaneously by Landau and Zener in the 30's (see [22] and [33]). We shall discuss this transfer of energy in terms of Wigner measures in the same spirit than the works of Gérard and the author for "real" crossings (see [7]–[12]) in the sense explained above. Our purpose here is to apply to this avoided crossing the method introduced in [11] for eigenvalue of multiplicity one with the developments performed in [8] for higher multiplicity.

1.1. Wigner measures for Dirac equation. Let us first introduce some notations. We endow $T^*\mathbf{R}^4$ with the symplectic form

$$\sigma = \mathrm{d}\tau\wedge\mathrm{d}t + \mathrm{d}\xi\wedge\mathrm{d}x,$$

and we denote by $\{f,g\}$ the Poisson bracket of functions f and g,

$$\{f,g\} = \nabla_{\tau,\xi}f\cdot\nabla_{t,x}g - \nabla_{t,x}f\cdot\nabla_{\tau,\xi}g.$$

The vector field H_f is the Hamiltonian vector field associated with the function f, thus we have

$$\mathrm{d}g\,H_f = \{f,g\} = \sigma(H_f,H_g).$$

We set

$$p(\xi) = \sum_{k=1}^{3} \sigma_k \xi_k = \begin{pmatrix} \xi_3 & \xi_1 - i\xi_2 \\ \xi_1 + i\xi_2 & -\xi_3 \end{pmatrix}, \quad \mathbf{0} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \quad \mathbf{1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

so that we have

$$P_\varepsilon = \begin{pmatrix} \varepsilon^\alpha m \mathbf{1} & p(\xi - A(t,x)) \\ p(\xi - A(t,x)) & \varepsilon^\alpha m \mathbf{1} \end{pmatrix} + V(t,x).$$

We will abusively denote by $m\varepsilon^\alpha$ the matrix $m\varepsilon^\alpha \mathbf{1}$ when there will not be any ambiguity on the fact that $m\varepsilon^\alpha$ is a 2×2 matrix. With this convention, the spectral projectors $\Pi_\varepsilon^\pm$ associated with $\lambda_\varepsilon^\pm$ write

$$\Pi_\varepsilon^\pm(t,x,\xi) = \frac{1}{2}\left(\mathrm{Id} \pm \frac{1}{\sqrt{\varepsilon^{2\alpha} m^2 + |\xi - A|^2}} \begin{pmatrix} \varepsilon^\alpha m & p(\xi - A(t,x)) \\ p(\xi - A(t,x)) & \varepsilon^\alpha m \end{pmatrix} \right).$$

As ε goes to 0, $\Pi_\varepsilon^\pm$ tends to $\Pi_0^\pm$ which is smooth outside $\{\xi = A(t,x)\}$.

According to [14] and [15], any Wigner measure μ of (ψ^ε) satisfies outside $\{\xi = A(t,x)\}$

$$\mu = \mu^+ + \mu^-, \quad \text{with } \mu^\pm = \mu^\pm \Pi_0^\pm = \Pi_0^\pm \mu^\pm,$$

$$\mathrm{Supp}\ \mu^\pm \subset \{\tau + V \pm |\xi - A| = 0\}. \tag{9}$$

We set

$$S = \{\xi = A, \tau + V = 0\}.$$

This set S is the intersection of the set $\{\xi = A(t,x)\}$ above mentioned and of the characteristic set Σ of the system studied,

$$\Sigma = \{|\xi - A|^2 = (\tau + V)^2\}.$$

By [14] and [15], measures μ^+ and μ^- satisfy the transport equations,

$$\{\tau + V \pm |\xi - A|, \mu^\pm\} = [\mu^\pm, F^\pm] \quad \text{outside} \quad S,$$

with

$$\begin{aligned} F^\pm &= [\Pi_0^\pm, \{\tau + V \pm |\xi - A|, \Pi_0^\pm\}] + |\xi - A| \Pi_0^\pm \{\Pi_0^\mp, \Pi_0^\mp\} \Pi_0^\pm \\ &= \frac{i}{2|\xi - A|} \gamma_5 \sum_{k=1}^{3} \gamma^0 \gamma^k \left(\frac{\xi - A}{|\xi - A|} \times \left(E \mp B \times \frac{\xi - A}{|\xi - A|} \right) \right)_k \\ &\quad - \frac{i}{2|\xi - A|} \left(\frac{\xi - A}{|\xi - A|} \cdot B \right) \Pi_0^\pm. \end{aligned} \tag{10}$$

This latter formula comes from [30] (see also [15]) by tedious but straightforward computation.

Outside S, measures $\mu^{\pm}$ propagate along the classical trajectories associated with the Hamiltonian $\tau+\lambda_0^{\pm}=\tau+V\pm|\xi-A|$. Let us denote by $\rho_s^{\pm}=(t_s^{\pm},x_s^{\pm},\tau_s^{\pm},\xi_s^{\pm})$, these Hamiltonian curves passing at $(s=0,\rho=\rho_0)$,

$$\dot{\rho}_s^{\pm}=H_{\lambda_0^{\pm}}(\rho_s^{\pm}),\qquad \rho^{\pm}_{|s=0}=\rho_0.$$

The problems arise as such trajectories reach the singular set S. Consider trajectories which arrive in (or arise from) S in ρ_0 transversally to S; by this, we mean that in ρ_0 we have

$$\lim_{s\to 0^-} H_{\lambda^{\pm}}(\rho_s)\notin TS_{\rho_0}\quad (\text{or } \lim_{s\to 0^+} H_{\lambda^{\pm}}(\rho_s)\notin TS_{\rho_0}).$$

A simple calculation proves that, if this occurs, necessarily, there exist $r>0$, ω and ω' in $\mathbf{S}^2$ such that

$$r\omega=E(\rho_0)+B(\rho_0)\times\omega,\quad r\omega'=E(\rho_0)-B(\rho_0)\times\omega'. \tag{11}$$

These equations have a unique solution (r,ω,ω') if and only if in ρ_0,

$$E\cdot B\neq 0\quad\text{or}\quad (E\cdot B=0\quad\text{and}\quad |E|>|B|).$$

In particular, if $E\cdot B=0$,

$$\omega=\frac{r}{|E|^2}E-\frac{E\times B}{|E|^2},\qquad \omega'=\frac{r}{|E|^2}E+\frac{E\times B}{|E|^2},\qquad r=\sqrt{|E|^2-|B|^2}. \tag{12}$$

Consider

$$\Omega=\{E\cdot B\neq 0\quad\text{or}\quad (E\cdot B=0\quad\text{and}\quad |E|>|B|)\}.$$

According to Proposition 1 in [11], the existence of a unique (r,ω,ω') satisfying (11), is enough to prove that there exists a unique classical trajectory of $\tau+\lambda_0^{\pm}$ passing through the point $\rho_0\in S\cap\Omega$. Moreover, the curves $(\rho_s^{\pm})_{s<0}$ are smoothly continued by $(\rho_s^{\mp})_{s>0}$.

The existence of such trajectories yield that the transport equations outside S stated above are not enough to determine the Wigner measure μ. Besides, we prove in the Appendix that

$$\mu(S\cap\Omega)=0.$$

Therefore, μ does not concentrate on the crossing set S: the mass of μ carried out by the ingoing trajectories $(\rho_s^{\pm})_{s<0}$ parts between the outgoing trajectories $(\rho_s^{+})_{s>0}$ and $(\rho_s^{-})_{s<0}$. We want to describe the branching of μ above the crossing set. We prove that it depends on α and that it is determined by the way (ψ^{ε}) concentrates at the scale $\sqrt{\varepsilon}$ on the set consisting in all the classical trajectories entering in S. Consider $\rho_0\in\Omega\cap S$ and $\mathscr{V}$ some neighborhood of ρ_0 in Ω. Let us denote by $J^{\pm,\,\mathrm{in}}$ (resp. $J^{\pm,\,\mathrm{out}}$) the sets of all curves $\rho_s^{\pm}$ coming into (resp. out of) some point ρ in $S\cap\mathscr{V}$. The connection between $(\rho_s^{\pm})_{s<0}$ and $(\rho_s^{\mp})_{s>0}$ at $s=0$ yields that the sets

$$J=J^{+,\,\mathrm{in}}\cup J^{-,\,\mathrm{out}}\quad\text{and}\quad J'=J^{-,\,\mathrm{in}}\cup J^{+,\,\mathrm{out}}$$

are smooth codimension 3 submanifolds of $T^*\mathbf{R}^4$. Moreover, according to Proposition 3 in [11], there exist two smooth vector-valued functions u, $u'\in\mathscr{C}^{\infty}(\mathbf{R}^8,\mathbf{S}^2)$ such that

$$J=\{\xi=A+(\tau+V)u\},\qquad J'=\{\xi=A+(\tau+V)u'\},$$

with $u_{|S}=\omega$ and $u'_{|S}=-\omega'$.

If $E \cdot B = 0$ in $\mathscr{V}$, J and J' are involutive submanifolds of $T^*\mathbf{R}^4$ (we shortly recall the definition of involutive submanifolds in the next section). The situation is different in the case where $E \cdot B \neq 0$. These facts are proved in [10] in a general context and in [7] for Dirac equation but in the case of constant electromagnetic vector fields. In this paper, we focus on the case where $E \cdot B = 0$ near some $\rho_0 \in S$, so that we are in the geometric setting of [10]. Our aim is to study the concentration of (ψ^ε) above J and J' at the scale $\sqrt{\varepsilon}$ by means of two-scale Wigner measure for involutive submanifolds which have been introduced by Miller in [25] and developed in [9].

1.2. Two-scale Wigner measures. We consider for a while a more general setting and recall results of [9]. Let I be a codimension m submanifold of the cotangent space $T^*\mathbf{R}^D$. If $\rho \in I$, $\sigma(\rho)$ is a symplectic form on the vectorial space $T(T^*\mathbf{R}^D)_\rho$ and $TI_{|\rho}$ is a vectorial subspace of $T(T^*\mathbf{R}^D)_{|\rho}$. The vectorial subspace

$$TI_{|\rho}^{\perp} = \{\delta\rho \in T(T^*\mathbf{R}^D)_{|\rho}, \quad \forall \delta\rho' \in TI_{|\rho},\ \sigma(\rho)(\delta\rho, \delta\rho') = 0\}$$

is the orthogonal of $TI_{|\rho}$ for the symplectic structure induced by σ. The submanifold I is said to be involutive if and only if

$$\forall \rho \in I, \qquad TI_{|\rho}^{\perp} \subset TI_{|\rho}.$$

In coordinates, if I is an involutive submanifold and $\rho_0 \in I$, there exist local symplectic coordinates (x, ξ) near ρ_0 such that $I = \{x_1 = \cdots = x_m = 0\}$.

We suppose that I is an involutive submanifold given by some system of equations $f = 0$ where $f = (f_1, \ldots, f_m) \in \mathscr{C}^\infty(\mathbf{R}^{2D}, \mathbf{R}^m)$, $\mathrm{Rank}(\mathrm{d}f) = m$ on $f = 0$ and $\{f_j, f_k\} = 0$. Let us denote by $\overline{\mathbf{R}^m}$, the ball obtained by adding a sphere at infinity to $\mathbf{R}^m$. We consider the set $\mathscr{A}$ of symbols $a = a(z, \zeta, \eta) \in \mathscr{C}^\infty(\mathbf{R}^D \times \mathbf{R}^D \times \mathbf{R}^m)$ which are uniformly compactly supported in the variables (z, ζ) with respect to η and which can be extended as a function of $\mathscr{C}^\infty(\mathbf{R}^D \times \mathbf{R}^D \times \overline{\mathbf{R}^m})$ by

$$a(z, \zeta, \infty\omega) = \lim_{R \to +\infty} a(z, \zeta, R\omega), \quad \text{in } e^\infty,\ \forall \omega \in \mathbf{S}^{m-1}.$$

We extend this definition to smooth matrix-valued function a by setting $a \in \mathscr{A}$ if and only if all the coefficients of the matrix a are symbols of $\mathscr{A}$. With any matrix-valued symbol $a \in \mathscr{A}$, we associate the two-scaled pseudo-differential operator,

$$\mathrm{op}_\varepsilon{}^I(a) := \mathrm{op}_h\left(a\left(z, \zeta, \frac{f(z, \zeta)}{\sqrt{\varepsilon}}\right)\right).$$

By Calderon-Vaillancourt's Theorem, the family of operators $\mathrm{op}_\varepsilon{}^I(a)$ is a bounded family of bounded operators in $L^2(\mathbf{R}^D)$. Let (ϕ^ε) be a bounded family in $L^2(\mathbf{R}^D, \mathbf{C}^N)$, $N \in \mathbf{N}$, we study the evolution as ε goes to 0 of

$$K_\varepsilon(a) := (\mathrm{op}_\varepsilon{}^I(a)\phi^\varepsilon | \phi^\varepsilon).$$

The limit of K_ε is described by a positive Radon measure on $\overline{N}(I)$, the compactified normal bundle to I.

Let us explain which is the bundle $\overline{N}(I)$. We associate with I its tangent bundle TI. Taking the quotient of the tangent space $T(T^*\mathbf{R}^D)_{|\rho}$ above some point ρ of I by $TI_{|\rho}$, we obtain the fiber above ρ of $N(I)$, the normal bundle to I. Then, $\overline{N}(I)_{|\rho}$ is the closed m-dimensional ball obtained by adding a sphere at infinity to $N(I)_{|\rho}$. The choice of the equation f yields local coordinates on $\overline{N}(I)_{|\rho}$ given by the continuation $\overline{\chi}$ of the isomorphism χ,

$$\chi : [\delta\rho] \in N(I)_{|\rho} \mapsto \eta = \mathrm{d}f(\rho)\delta\rho \in \mathbf{R}^m.$$

If ν is a measure on $\overline{N}(I)$, we denote by ν_f the measure on $\overline{\mathbf{R}^m}$ which is the image of ν by $\overline{\chi}$. Let us come back to the limit of $K_\varepsilon(a)$.

There exists a sequence $\varepsilon_k \underset{k\to+\infty}{\longrightarrow} 0$, *a matrix-valued positive measure* ν *on* $\overline{N}(I)$ *such that for all matrix-valued symbol* $a \in \mathscr{A}$,

$$K_{\varepsilon_k}(a) \underset{k\to+\infty}{\longrightarrow} \mathrm{tr}\left(\int_{\overline{\mathbf{R}^m}} a\, \mathrm{d}\nu_f\right) + \mathrm{tr}\left(\int_{f\neq 0} a\left(z,\zeta,\frac{f(z,\zeta)}{|f(z,\zeta)|}\infty\right)\mathrm{d}\mu\right),$$

where μ *is a Wigner measure of* (ϕ^ε).

We point out that ν determines μ above I by

$$\mu \mathbf{1}_I = \int_I \nu(z,\zeta,\mathrm{d}\eta).$$

These measures correspond to a second micro-localization in the spirit of [1], [23] (see also [6]): we add to the microlocal variables (z,ζ) a new variable η which belongs to $\overline{R^m}$. This additional coordinate is used for measuring the distance from points in $T^*\mathbf{R}^D$ to the submanifold I versus the scale $\sqrt{h}$. Some Wigner transform's approach of two-scale Wigner measures can be performed as in [12]. Consider W^ε, the usual Wigner transform of (ϕ^ε),

$$W^\varepsilon(z,\zeta) = \int \mathrm{e}^{iy\cdot\zeta}\phi^\varepsilon\left(z-\frac{\varepsilon}{2}y\right)\otimes\overline{\phi^\varepsilon}\left(z+\frac{\varepsilon}{2}y\right)\mathrm{d}y.$$

The two-scale Wigner transform of (ϕ^ε) is the distribution

$$W_2^\varepsilon\phi^\varepsilon(z,\zeta,\eta) = W^\varepsilon\phi^\varepsilon(z,\zeta)\otimes\delta\left(\eta-\frac{f(z,\zeta)}{\sqrt{h}}\right),$$

If one studies the action of the distribution W_2^ε on the class of test functions $\mathscr{A}$, two-scale Wigner measures appear as the limits points of W_2^ε.

Our aim in the following is to calculate two-scale Wigner measures associated with a family (ψ^ε) solution to (6), and with the involutive submanifolds J and J'.

1.3. The branching of the energy: matrix-valued Landau-Zener formula. We denote by $\overline{\nu}$ (resp. $\overline{\nu}'$) the measures associated with (ψ^ε) and J (resp. J'). We denote by Σ^+, Σ^-, $\dot{J}^{\pm,\mathrm{in}}$ and $\dot{J}^{\pm,\mathrm{out}}$ the sets

$$\Sigma^\pm = \{\lambda_0^\pm = 0\}, \qquad \dot{J}^{\pm,\mathrm{in}} = J^{\pm,\mathrm{in}}\setminus S, \qquad \dot{J}^{\pm,\mathrm{out}} = J^{\pm,\mathrm{out}}\setminus S.$$

We consider the bundles above $\Sigma^\pm$, $\overline{N}_{\Sigma^\pm}(\dot{J}^{\pm,\mathrm{in}})$ and $\overline{N}_{\Sigma^\pm}(\dot{J}^{\pm,\mathrm{out}})$ obtained respectively by adding a sphere at infinity to the fibers of $T\Sigma^\pm/T(\dot{J}^{\pm,\mathrm{in}})$ and of $T\Sigma^\pm/T(\dot{J}^{\pm,\mathrm{out}})$. Because of the properties of localization of Wigner measures, there exist scalar positive Radon measures $\nu^{\pm,\,\mathrm{in}}$ and $\nu^{\pm,\,\mathrm{out}}$ supported on $\overline{N}_{\Sigma^\pm}(\dot{J}^{\pm,\,\mathrm{in}})$ and $\overline{N}_{\Sigma^\pm}(\dot{J}^{\pm,\,\mathrm{out}})$ respectively such that

$$\overline{\nu} = \nu^{+,\,\mathrm{in}} + \nu^{-,\,\mathrm{out}}, \quad \overline{\nu}' = \nu^{-,\,\mathrm{in}} + \nu^{+,\,\mathrm{out}},$$

$$\Pi^\pm\ \nu^{\pm,\mathrm{in}}\Pi^\pm = \nu^{\pm,\,\mathrm{in}}, \quad \Pi^\pm\ \nu^{\pm,\,\mathrm{out}}\Pi^\pm = \nu^{\pm,\,\mathrm{out}}.$$

Moreover, if $\mathscr{L}^{\pm,\,\mathrm{in}}(H_{\lambda^\pm})$ (resp. $\mathscr{L}^{\pm,\,\mathrm{out}}(H_{\lambda^\pm})$) is the linearized Hamiltonian flow of $\tau + \lambda_0^\pm$ transversally to $\dot{J}^{\pm,\,\mathrm{in}}$ (resp. $\dot{J}^{\pm,\,\mathrm{out}}$), in $\Sigma^\pm$, the measures $\nu^{\pm,\,\mathrm{in}}$ (resp. $\nu^{\pm,\,\mathrm{out}}$) satisfy

$$\mathscr{L}^{\pm,\,\mathrm{in}}(H_{\lambda^\pm})\nu^{\pm,\,\mathrm{in}} = [\nu^{\pm,\,\mathrm{in}}, F^\pm] \quad \text{on} \quad \dot{J}^{\pm,\,\mathrm{in}},$$

$$\mathscr{L}^{\pm,\,\mathrm{out}}(H_{\lambda^\pm})\nu^{\pm,\,\mathrm{out}} = [\nu^{\pm,\,\mathrm{out}}, F^\pm] \quad \text{on} \quad \dot{J}^{\pm,\,\mathrm{out}}.$$

These propagation properties result from [14], [15] and [9]. Using (10), (11), the equations of J and J' and the fact that $|\xi - A| = \mp(\tau + V)$ on $J^{\pm,\,\mathrm{in}}$ and $J^{\pm,\,\mathrm{out}}$, we obtain that

$$F^\pm_{|\dot{J}^{\pm,\,\mathrm{in}}} = O(1), \qquad F^\pm_{|\dot{J}^{\pm,\,\mathrm{out}}} = O(1) \text{ near } S.$$

Therefore, the fact that the Hamiltonian flows are transverse to S yields that, in the set of distributions, measures $\nu^{\pm,\,\mathrm{in}}$ and $\nu^{\pm,\,\mathrm{out}}$ have traces on S that we denote by $\nu_S^{\pm,\,\mathrm{in}}$ and by $\nu_S^{\pm,\,\mathrm{out}}$. These four traces can be identified to measures on one set in which we can study the existing link between $\nu_S^{\pm,\,\mathrm{out}}$ and $\nu_S^{\pm,\,\mathrm{in}}$.

Lemma 1. *For $\rho_0 \in S$, the map*

$$\eta : T(T^*(\mathbf{R}^4))_{|\rho_0} \to \mathbf{R}^3$$

$$\delta\rho \mapsto \mathrm{d}(\tau + V)\,\delta\rho B + \mathrm{d}(\xi - A)\,\delta\rho \times E$$

induces some isomorphism between the limits of the fibres of $N_{\Sigma^\pm}(\dot{J}^{\pm,\,\mathrm{in}})$ and $N_{\Sigma^\pm}(\dot{J}^{\pm,\,\mathrm{out}})$ above some point ρ which goes to ρ_0 and the hyperplane normal to E for the Euclidian structure of $\mathbf{R}^3$.

This lemma is proved in the Appendix.

We extend the map η to the limits of the fibres of $\overline{N}_{\Sigma^\pm}(\dot{J}^{\pm,\,\mathrm{in}})$ and of $\overline{N}_{\Sigma^\pm}(\dot{J}^{\pm,\,\mathrm{out}})$. Provided this identification, the connection between $\nu_S^{\pm,\,\mathrm{in}}$ and $\nu_S^{\pm,\,\mathrm{out}}$ near the point ρ_0 is described by the following Theorem.

Theorem 1. *We assume that $\nu_S^{+,\,\mathrm{in}}$ and $\nu_S^{-,\,\mathrm{in}}$ are mutually singular.*

1) *The adiabatic regime*: $\alpha \in]0, \frac{1}{2}[$,

$$\nu_S^{+,\,\mathrm{out}} = \frac{|E|^2}{|E|^2 - |B|^2}\mathscr{R}_a \nu_S^{+,\,\mathrm{in}} \mathscr{R}_a^*, \qquad \nu_S^{-,\,\mathrm{out}} = \frac{|E|^2}{|E|^2 - |B|^2}\mathscr{R}_a \nu_S^{-,\,\mathrm{in}} \mathscr{R}_a^*,$$

with

$$\mathscr{R}_a = \begin{pmatrix} \mathbf{1} & -p\left(\frac{E\times B}{|E|^2}\right) \\ p\left(\frac{E\times B}{|E|^2}\right) & -\mathbf{1} \end{pmatrix}.$$

2) *The non-adiabatic regime*: $\alpha \geqslant \frac{1}{2}$,

$$\begin{aligned} \nu_S^{+,\mathrm{out}} &= T\nu_S^{-,\mathrm{in}} + (1-T)\mathscr{R}\nu_S^{+,\mathrm{in}}\mathscr{R}^* \\ \nu_S^{-,\mathrm{out}} &= T\nu_S^{+,\mathrm{in}} + (1-T)\mathscr{R}\nu_S^{-,\mathrm{in}}\mathscr{R}^*, \end{aligned} \tag{13}$$

where, *if* $\alpha = 1/2$,

$$T = \exp\left(-\frac{\pi}{(|E|^2 - |B|^2)^{3/2}}\Phi(\eta, m)^2\right),$$

$$\mathscr{R} = \frac{|E|}{\Phi(\eta, m)}(\mathscr{R}_{na}(\eta) + m\mathscr{R}_a),$$

and, *if* $\alpha > 1/2$,

$$T = \exp\left(-\frac{\pi}{(|E|^2 - |B|^2)^{3/2}}\Phi(\eta, 0)^2\right),$$

$$\mathscr{R} = \frac{|E|}{\Phi(\eta, 0)}\mathscr{R}_{na}(\eta),$$

and

$$\Phi(\eta, m) = \left[|\eta|^2 - \left(\frac{E \times B}{|E|^2}\cdot\eta\right)^2 + (|E|^2 - |B|^2)m^2\right]^{1/2},$$

$$\mathscr{R}_{na}(\eta) = \begin{pmatrix} \mathbf{0} & p\left(\frac{\eta\times E}{|E|^2}\right) \\ p\left(\frac{\eta\times\eta}{|E|^2}\right) & \mathbf{0} \end{pmatrix} + i\eta\cdot\frac{E\times B}{|E|^3}\begin{pmatrix} p\left(\frac{E}{|E|}\right) & \mathbf{0} \\ \mathbf{0} & p\left(\frac{E}{|E|}\right) \end{pmatrix}.$$

Remarks. 1) The matrix $\mathscr{R}$ describes the change of polarization at the crossing. Observe that if $B = 0$, the transfer coefficient T and the polarization matrix $\mathscr{R}$ are rather simple. In that case, we are reduce to the situation studied in [8].

2) The existence of different regimes in avoided crossings has already been noticed in [5] and in [18] where the same critical scale $\sqrt{\varepsilon}$ appears (the latter reference has been completed recently by [29]). If $\alpha \in]0, \frac{1}{2}[$, the mass $\varepsilon^\alpha m$ is big enough so that the measure is propagated as if there were no crossing ($\alpha = 0$). If $\alpha > \frac{1}{2}$, the mass is so small that we have the same result as if $m = 0$, i.e. as if we had a true crossing as studied in [11]–[8]. Finally, in the critical case $\alpha = \frac{1}{2}$, the Landau-Zener formula depend on m. Observe too that the part of the measure localized on $\{|\eta| = +\infty\}$ always propagates adiabatically.

3) It is likely that the adiabatic regime could be studied directly on the semi-classical scale and that the introduction of two-scale Wigner measure is not necessary to calculate the evolution of Wigner measures: the reflection of Wigner measure could be obtained by the same method as in Section 3 below.

4) In the case $E \cdot B \neq 0$, the two-scale Wigner measures which have to be considered are more complicated (see [7]) but similar results may be expected.

5) In the proofs below, we do not use the fact that $\xi - A$ is a linear function of ξ with $x, \xi \in \mathbf{R}^3$. Thus Theorem 1 can be generalized to systems of the same form that (6) where we turn $\xi - A$ into some function $l = l(t, x, \xi)$, $x, \xi \in \mathbf{R}^d$, $d \geqslant 2$ and where V may depend on ξ with the conditions of [11]: dl of rank 3 on $\{l = 0\}$ and $E \cdot B = 0$, with

$$E = \{\tau + V, l\}, \qquad B = (\{l_3, l_2\}, \{l_1, l_3\}, \{l_2, l_1\}).$$

We will proceed as follows. Section 2 is devoted to the reduction of system (6) to a model system, via a Fourier integral operator and a canonical transform. Then, in Section 3, we reduce Theorem 1 to some equivalent statement on the solution of the model system. In Section 4 and 5, we treat separately the "adiabatic" cases – namely $\alpha \in (0, 1/2)$ and ($\alpha \geqslant 1/2$, $|\eta| = +\infty$) – and the "non-adiabatic" ones: $\alpha \geqslant 1/2$ and $|\eta| < +\infty$. Finally, in the Appendix, we prove the restitution of the energy by the crossing and the geometric Lemma 1.

2. Reduction to a Model Problem

Let us describe first the Fourier integral operators we use. If κ is a canonical transform of $T^*\mathbf{R}^4$, there exists some semi-classical Fourier integral operator K that we call *associated with* κ, which satisfies

$$\forall f \in L^2(\mathbf{R}^4), \forall a \in \mathscr{C}_0^\infty(\mathbf{R}^8), K^* \mathrm{op}_h(a) K f = \mathrm{op}_h(a \circ \kappa) f + O(h^2) \|f\|_{L^2}, \mathrm{in}\, L^2(\mathbf{R}^4). \tag{14}$$

The reader can refer to [28] for a complete study on Fourier Integral Operator or to [9] where this claim is proved.

This section is devoted to the proof of the following Proposition.

Proposition 1. *Consider $\rho_0 \in S$ such that $|E(\rho_0)| > |B(\rho_0)|$. There exist some local canonical transform κ from a neighborhood of ρ_0 into a neighborhood Ω of* 0,

$$\kappa : (t, x, \tau, \xi) \mapsto (s, z, \sigma, \zeta), \quad \kappa(\rho_0) = 0,$$

a matrix C and a Fourier integral operator K associated with κ such that $v^\varepsilon = K \mathrm{op}_\varepsilon(C) \psi^\varepsilon$ satisfies for all $a \in \mathscr{C}_0^\infty(\Omega)$,

$$\mathrm{op}_\varepsilon(a) \mathrm{op}_\varepsilon \begin{pmatrix} -\sigma + s & p(m_\varepsilon, \Gamma\tilde{\zeta}) \\ p(m_\varepsilon, \Gamma\tilde{\zeta}) & -\sigma - s \end{pmatrix} v^\varepsilon = O(\varepsilon) \ in\ L^2(\mathbf{R}^4), \tag{15}$$

where $\tilde{\zeta} = (\zeta_1, \zeta_2)$, $m_\varepsilon = k(s, z, \sigma, \zeta) m \varepsilon^\alpha$ with

$$k_{|S} = (|E|^2 - |B|^2)^{-1/4}, \tag{16}$$

and where the 2×2 matrix-valued function $\Gamma = \Gamma(s, z, \sigma, \zeta)$ is smooth with $\det(\Gamma) \neq 0$ *in Ω. Moreover in the coordinates (s, z, σ, ζ),*

$$J^{\pm, \mathrm{in}} = \{\sigma \mp s = 0,\ \tilde{\zeta} = 0, s < 0\}, \quad J^{\pm, \mathrm{out}} = \{\sigma \pm s = 0, \tilde{\zeta} = 0, s > 0\}, \tag{17}$$
$$J \cup J' = \{\tilde{\zeta} = 0\} \cap \Sigma = \{\sigma^2 = s^2\} \cap \{\tilde{\zeta} = 0\}.$$

The proof of Proposition 1 follows the same schedule than the proof of Theorem 2 in [11] through modifications required by the presence of $m\varepsilon^\alpha$. We proceed in two steps: first we transform the matrix P_ε by algebraic operations so that the equations of J and J' appear, then we define the canonical transform and conclude the proof.

2.1. An algebraic lemma. Let us first introduce some notations. We shall use the following matrices:

$$M_1 = \frac{1}{\sqrt{2}}\begin{pmatrix} \mathbf{1} & -\mathbf{1} \\ \mathbf{1} & \mathbf{1} \end{pmatrix}, \qquad M_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix},$$

which have the following properties:

$$M_1^* = M_1^{-1} \qquad M_2 = M_2^{-1} = M_2^* \tag{18}$$

$$M_1 \begin{pmatrix} m & p(\xi) \\ p(\xi) & -m \end{pmatrix} M_1^* = \begin{pmatrix} -p(\xi) & m \\ m & p(\xi) \end{pmatrix}, \tag{19}$$

$$M_2 \begin{pmatrix} -p(\xi) & m \\ m & p(\xi) \end{pmatrix} M_2 = -\begin{pmatrix} \xi_3 & p(\xi_1,\xi_2,m) \\ p(\xi_1,\xi_2,m) & -\xi_3 \end{pmatrix}. \tag{20}$$

We denote by L the submanifold of $T^*\mathbf{R}^4$ defined by

$$L = \{(\xi - A - (\tau + V)u) \times (u - u') = 0\}.$$

Then, by a simple computation, we have $J \cup J' = \Sigma \cap L$. Moreover, we set

$$e_1 = e_1(t,x,\tau,\xi) := \frac{u - u'}{|u - u'|},$$

so that

$$(e_1)_{|S} = \frac{E}{|E|}. \tag{21}$$

We choose some smooth functions (e_2, e_3) so that (e_1, e_2, e_3) is a direct orthonormal basis of $\mathbf{R}^3$ with moreover

$$(e_2)_{|S} = \frac{B \times E}{|E|\,|B|}, \qquad (e_3)_{|S} = \frac{B}{|B|}, \quad \text{if } B \neq 0. \tag{22}$$

Consider now the complex valued function $\theta = \theta(t,x,\tau,\xi)$ defined by

$$\theta = u \cdot e_2 + iu \cdot e_3.$$

Note that near ρ_0, $|\theta| < 1$. Actually, in view of $u_{|S} = \omega$ and of (12) we obtain

$$\theta_{|S} \in \mathbf{R}, \quad \text{and} \quad \theta_{|S} = \frac{|B|}{|E|} < 1. \tag{23}$$

Lemma 2. *There exists some smooth invertible matrix* $C_1 = C_1(t,x,\tau,\xi)$ *such that*

$$\tau + P_\varepsilon = C_1^* H_1 C_1$$

with

$$H_1 = \tau + V + Y\cdot g - \begin{pmatrix} \frac{(\xi-A)\cdot e_1}{\sqrt{1-|\theta|^2}} & p(g,\tilde{m}_\varepsilon) \\ p(g,\tilde{m}_\varepsilon) & -\frac{(\xi-A)\cdot e_1}{\sqrt{1-|\theta|^2}} \end{pmatrix}, \tag{24}$$

where $\tilde{m}_\varepsilon = \tilde{k}(s,z,\sigma,\zeta)m\varepsilon^\alpha$ *and where* $g=(g_1,g_2)$ *is an equation of L satisfying*

$$\begin{cases} \mathrm{d}g_{1|S} &= \frac{|E|^2}{r^2}\mathrm{d}[(\xi - A - (\tau+V)u)\cdot e_2], \\ \mathrm{d}g_{2|S} &= \frac{|E|}{r}\mathrm{d}[(\xi - A - (\tau+V)u)\cdot e_3]. \end{cases} \tag{25}$$

The smooth complex-valued vector $Y = Y(t,x,\tau,\xi) = (Y_1,Y_2)$ *satisfies*

$$\begin{cases} Y_{1|S} = -\frac{|B|}{|E|}, \\ Y_{2|S} = 0. \end{cases} \tag{26}$$

The smooth real-valued function $\tilde{k}$ *satisfies*

$$\tilde{k}_{|S} = \frac{|E|}{\sqrt{|E|^2 - |B|^2}}. \tag{27}$$

Proof of Lemma 2. We first use the quaternion structure of matrix $p(v)$. By Lemma 2 of [11], there exists a smooth unitary matrix $U = U(t,x,\tau,\xi)$ such that

$$Up(v)U^* = p(v\cdot e_2, v\cdot e_3, v\cdot e_1). \tag{28}$$

Therefore, if $U' = \begin{pmatrix} U & \mathbf{0} \\ \mathbf{0} & U \end{pmatrix} M_1$, we have

$$U'\begin{pmatrix} m\varepsilon^\alpha & p(\xi-A) \\ p(\xi-A) & -m\varepsilon^\alpha \end{pmatrix}U'^* = \begin{pmatrix} -p((\xi-A)\cdot e_2,(\xi-A)\cdot e_3,(\xi-A)\cdot e_1) & m\varepsilon^\alpha \\ m\varepsilon^\alpha & p((\xi-A)\cdot e_2,(\xi-A)\cdot e_3,(\xi-A)\cdot e_1) \end{pmatrix}.$$

We aim now to introduce the equations of L given by the vector-valued function f,

$$\begin{aligned} f &= ((\xi - A - (\tau+V)u)\cdot e_2, (\xi - A - (\tau+V)u)\cdot e_3) \\ &= ((\xi - A)\cdot e_2 - (\tau+V)\theta_1, (\xi-A)\cdot e_3 - (\tau+V)\theta_2). \end{aligned}$$

We set $S(\theta) = \begin{pmatrix} 1 & \theta_1 - i\theta_2 \\ \theta_1 + i\theta_2 & 1 \end{pmatrix}$ and we get:

$$\begin{aligned} U'(\tau+P_\varepsilon)U'^* = (\tau+V)&\begin{pmatrix} S(-\theta) & \mathbf{0} \\ \mathbf{0} & S(\theta) \end{pmatrix} \\ &+ \begin{pmatrix} -p(f,(\xi-A)\cdot e_1) & m\varepsilon^\alpha \\ m\varepsilon^\alpha & p(f,(\xi-A)\cdot e_1) \end{pmatrix}. \end{aligned} \tag{29}$$

Let us denote by $\mathcal{N}$ the set of 2×2 matrices

$$\mathcal{N} = \left\{ S(z) = \begin{pmatrix} 1 & \bar z \\ z & 1 \end{pmatrix}, \quad z\in\mathbf{C} \right\}.$$

Lemma 3. *Let $S = S(z)\in\mathcal{N}$. Then for all $v = (v_1, v_2, v_3) = (v', v_3)\in\mathbf{R}^3$,*

$$S(z)p(v', v_3)S(z) = z\overline{v'} + \bar z v' + p(v' + z^2\overline{v'}, (1-|z|^2)v_3),$$

The proof is straightforward.

We use matrices $\sqrt{S(\theta)}^{-1}$ and $\sqrt{S(-\theta)}^{-1}$ to get rid of the matrix-valued coefficient of $(\tau+k)$ in (29). Notice that

$$S(\theta)^{-1} = \frac{S(-\theta)}{1-|\theta|^2}, \quad \sqrt{S(\theta)}^{-1} = aS(-b\theta),$$

with

$$a = \frac{1}{2}\left(1-|\theta|^2\right)^{-1/2}\left(\sqrt{1+|\theta|} + \sqrt{1-|\theta|}\right),$$

$$b = \frac{1}{1+\sqrt{1-|\theta|^2}}.$$

Define

$$C_1 = M_2\begin{pmatrix} \sqrt{S(-\theta)} & \mathbf{0} \\ \mathbf{0} & \sqrt{S(\theta)} \end{pmatrix}U'. \tag{30}$$

Then, if $H_1 = (C_1^*)^{-1}(\tau + P_\varepsilon)C_1^{-1}$, Lemma 3 and (29) yield

$$H_1 = M_2\Bigg[\tau + k - a^2b(\theta\bar f + f\bar\theta) + m\varepsilon^\alpha a^2(1-b^2|\theta|^2)\begin{pmatrix} \mathbf{0} & \mathbf{1} \\ \mathbf{1} & \mathbf{0}\end{pmatrix} + a^2\cdots$$
$$\begin{pmatrix} -p(f+b^2\theta^2\bar f, (1-b^2|\theta|^2)(\xi - A)\cdot e_1) & \mathbf{0} \\ \mathbf{0} & p(f+b^2\theta^2\bar f, (1-b^2|\theta|^2)(\xi-A)\cdot e_1)\end{pmatrix}$$
$$\cdots\Bigg]M_2.$$

We set $\tilde k = a^2(1-b^2|\theta|^2)$ and we check that $\tilde k = \frac{1}{\sqrt{1-|\theta|^2}}$.

Let $g = (g_1, g_2)$ be such that $g_1 + ig_2 = a^2(f + b^2\theta^2\bar f)$. Then we get (24) through straightforward computations.

Moreover, we have $\mathrm{d}g_{|S} = \mathrm{d}f_{|S} + b^2\theta^2\mathrm{d}\bar f_{|S}$ and in view of $a^2(1+b^2\theta^2) = (1-|\theta|^2)^{-1}$, we obtain

$$\mathrm{d}g_{1|S} = a^2(1+b^2\theta^2)_{|S}\mathrm{d}f_{1|S} = \frac{1}{1-|\theta|^2{}_{|S}}\,\mathrm{d}f_{1|S},$$

$$\mathrm{d}g_{2|S} = a^2(1-b^2\theta^2)_{|S}\mathrm{d}f_{2|S} = \frac{1}{\sqrt{1-|\theta|^2{}_{|S}}}\,\mathrm{d}f_{2|S},$$

whence (25). Moreover, if $Y_1 g_1 + Y_2 g_2 = -a^2 b(\theta\bar{f} + \bar{\theta} f) = -2a^2 b \mathscr{R}e(\theta\bar{f})$, then

$$(Y_1 \mathrm{d}g_1 + Y_2 \mathrm{d}g_2)_{|S} = -2a^2 b\theta \mathrm{d}\bar{f}_{|S}.$$

Therefore, $(Y_2)_{|S} = 0$ and $(Y_1)_{|S} = -2(1 - |\theta|^2)a^2 b\theta$ above S, whence (26). □

2.2. The canonical transform. Observe that

$$\left\{\tau + V + Y \cdot g, \frac{(\xi - A) \cdot e_1}{\sqrt{1 - |\theta|^2}}\right\} = \frac{|E|^2}{\sqrt{|E|^2 - |B|^2}} > 0.$$

Therefore, arguing as for Proposition 4 in [11], we get the following Proposition.

Proposition 2. *There exist some function λ, $\lambda > 0$ near ρ_0, and some local canonical transform κ,*

$$\kappa : (t, x, \tau, \xi) \mapsto (s, z, \sigma, \zeta),$$
$$\kappa(\rho_0) = 0,$$

such that

$$\sigma = \lambda(\tau + V + Y_1 g_1 + Y_2 g_2),$$
$$s = \lambda \frac{(\xi - A) \cdot e_1}{\sqrt{1 - |\theta|^2}},$$
$$\lambda g(t, x, \tau, \xi) = (\check{\gamma}^1(s, z, \sigma, \zeta) \cdot \tilde{\zeta}, \check{\gamma}^2(s, z, \sigma, \zeta) \cdot \tilde{\zeta}) + (\sigma^2 - s^2)\beta,$$

where $\tilde{\zeta} = (\zeta_1, \zeta_2)$ and where $\beta = (\beta_1, \beta_2)$, $\check{\gamma}_1$ and $\check{\gamma}_2$ are smooth functions valued in $\mathbf{R}^2$ with $\det(\check{\gamma}_1, \check{\gamma}_2) \neq 0$. *Moreover,*

$$\lambda_{|S}^2 = |E|^{-2}\sqrt{|E|^2 - |B|^2}. \tag{31}$$

Let us identify J and J' in these new coordinates. Remark that, if $\rho \in L$, $g(\rho) = 0$, thus

$$\lambda^2 |\tau + V|^2 = \sigma^2, \quad \lambda^2 |\xi - A|^2 = s^2(1 - |\theta|^2) + \sigma^2 |\theta|^2.$$

Therefore, $L \cap \{s^2 = \sigma^2\} = L \cap \{|\xi - A|^2 = (\tau + V)^2\} = J \cup J'$. We study now the sign of σ which is $\mathrm{sgn}(\tau + V)$ i.e. $\mp$ on $J^{\pm,\mathrm{in}} \cup J^{\pm,\mathrm{out}}$, and the sign of s which is $\mathrm{sgn}((\xi - A) \cdot e_1)$ i.e. $+\mathrm{sgn}(\tau + V)$ on J and $-\mathrm{sgn}(\tau + V)$ on J' (since $u \cdot e_1 > 0$ and $u' \cdot e_1 < 0$). For example, we obtain $J = \{\sigma = s\} \cap \{g = 0\}$ and $J^{+,\mathrm{in}} \subset \{s < 0, \quad \sigma < 0\}$. Hence the equations of J, J', $J^{\pm,\mathrm{in}}$ and $J^{\pm,\mathrm{out}}$ stated in Proposition 1.

Let us conclude now the proof of Proposition 1. We set

$$C = \lambda^{-1/2} C_1. \tag{32}$$

In the following calculations, we shall denote by $O(\varepsilon)$ either a family (R_ε) of operators or a family (r_ε) of functions such that, for every $\phi \in C_0^\infty$,

$$\frac{1}{\varepsilon}[\|\mathrm{op}_\varepsilon(\phi) R_\varepsilon\|_{\mathscr{L}(L^2)} + |\mathrm{op}_\varepsilon(\phi) r_\varepsilon|_{L^2}] \text{ is bounded.}$$

We have

$$\mathrm{op}_\varepsilon(\lambda H_1) = \mathrm{op}_\varepsilon(C^{*-1})\mathrm{op}_\varepsilon(\tau + P_\varepsilon)\mathrm{op}_\varepsilon(C^{-1}) + O(\varepsilon).$$

Consider now some Fourier integral operator K associated with κ and

$$v^\varepsilon = K\mathrm{op}_\varepsilon(C)\psi^\varepsilon.$$

Then we have

$$\begin{aligned}\mathrm{op}_\varepsilon\left(\sigma - \begin{pmatrix} s & p(\tilde f, m_\varepsilon) \\ p(\tilde f, m_\varepsilon) & -s\end{pmatrix}\right)\tilde v^\varepsilon &= K\mathrm{op}_\varepsilon(\lambda H_1)\tilde v^\varepsilon \\ &= K\mathrm{op}_\varepsilon(C^{*-1})\mathrm{op}_\varepsilon(\tau + P_\varepsilon)\psi^\varepsilon + O(\varepsilon) \\ &= O(\varepsilon),\end{aligned}$$

where we have used (14) and (6) and where $m_\varepsilon = \lambda\tilde m_\varepsilon$ and

$$\tilde f = (\check\gamma^1\cdot\tilde\zeta, \check\gamma^2\cdot\tilde\zeta) + (\sigma^2 - s^2)\beta.$$

We rewrite this equation as

$$\mathrm{op}_\varepsilon(D - DBD)v^\varepsilon = \mathrm{op}_\varepsilon(F)v^\varepsilon + O(\varepsilon), \tag{33}$$

with

$$D = \begin{pmatrix}\sigma - s & \mathbf{0} \\ \mathbf{0} & \sigma + s\end{pmatrix}, \quad B = \begin{pmatrix}\mathbf{0} & p(0,\beta) \\ p(0,\beta) & \mathbf{0}\end{pmatrix},$$
$$F = \begin{pmatrix}\mathbf{0} & p(\check\gamma^1\cdot\tilde\zeta, \check\gamma^2\cdot\zeta_2, m_\varepsilon) \\ p(\check\gamma^1\cdot\tilde\zeta, \check\gamma^2\cdot\zeta_2, m_\varepsilon) & \mathbf{0}\end{pmatrix}.$$

Since $DBF = BFD$ and $B_{|S} = 0$, $F_{|S} = 0$, we can argue as in [11] at the end of the proof of Theorem 2 (Section 3.3) and we get that there exists a matrix P such that $P_{|S} = \mathbf{1}$ and

$$\mathrm{op}_\varepsilon(D)v^\varepsilon = \mathrm{op}_\varepsilon(P)\mathrm{op}_\varepsilon(F)v^\varepsilon + O(\varepsilon).$$

Hence Proposition 1 with

$$\Gamma_{|S}\tilde\zeta = ((\check\gamma_1)_{|S}\cdot\tilde\zeta + (\check\gamma_2)_{|S}\cdot\tilde\zeta). \tag{34}$$

Therefore matrix Γ is invertible near S. □

3. Consequences of the Reduction to System (15)

Proposition 1 has two important consequences. The first one is that it states that there exists some involutive submanifold of $T^*\mathbf{R}^{d+1}$, $I = \{\tilde\zeta = 0\}$, such that

$$J \cup J' = \Sigma \cap I,$$

with transverse intersection. Thus, if ν is the two-scale Wigner measure of (ψ^ε) for I, we can identify measures $\overline{\nu}$, (resp. $\overline{\nu}'$) with ν above $\overline{N}(I)_{|J}$ (resp. $\overline{N}(I)_{|J'}$). Actually, if $\overline{N}_\Sigma(J)$ is the bundle above Σ obtained by adding a sphere at infinity of the fibers of $T\Sigma/TJ$, the canonical isomorphism from $T\Sigma/TJ$ onto $T(T^*\mathbf{R}^{d+1}_{|J})/TJ_J$ extends in some isomorphism

$$\theta_{\Sigma,I} : \overline{N}_\Sigma(J) \to \overline{N}(I),$$

that we use for identifying $\overline{\nu}$ and $\nu\,\mathbf{1}_{\overline{N}(I)_{|J}}$. The reader can refer to Lemma 4 in [9] for a proof of this fact. Because of this identification, we shall focus in calculating the two-scale Wigner measure of (ψ^{ε}) for I.

The second consequence is that because of the invariance of two-scale Wigner measure through canonical transform (see Lemma 2 in [9]), it is equivalent to study the two-scale Wigner measure of (v^{ε}) for I, or of (ψ^{ε}) for the same set. Indeed, the two-scale Wigner measures $\tilde{\nu}$ of (v^{ε}) and ν of (ψ^{ε}) are linked by

$$\forall a \in \mathscr{A}, \quad \langle a,\ \tilde{\nu}\rangle = \langle a \circ \overline{N}(\kappa),\ C\nu C^{*}\rangle, \tag{35}$$

where for $(\rho,\eta) \in \overline{N}(I)$, $\overline{N}(\kappa)(\rho,\eta) = (\kappa(\rho),\eta)$. We set

$$\tilde{\Pi}_0^{\pm} = \frac{1}{2}\left(1 \mp \frac{1}{\sqrt{s^2 + |\Gamma\tilde{\zeta}|^2}}\begin{pmatrix} s & p(\Gamma\tilde{\zeta},0) \\ p(\Gamma\tilde{\zeta},0) & -s \end{pmatrix}\right). \tag{36}$$

We decompose $\tilde{\nu}$ as $\tilde{\nu} = \tilde{\nu}^{+} + \tilde{\nu}^{-}$ with the commutation's relations:

$$\tilde{\Pi}_0^{\pm}\tilde{\nu}^{\pm}\tilde{\Pi}_0^{\pm} = \tilde{\nu}^{\pm}, \tag{37}$$

and the localization property: $\tilde{\nu}^{\pm}$ is supported on $\{\sigma \pm |s|\} = J^{\pm,\,\mathrm{in}} \cup J^{\pm,\,\mathrm{out}}$. Let us denote by $\tilde{\nu}_S^{\pm,\,\mathrm{in}}$ (resp. $\tilde{\nu}_S^{\pm,\,\mathrm{out}}$) the traces of $\tilde{\nu}^{\pm}$ on $s = 0^{-}$ (resp. $s = 0^{+}$). In view of (35), we have

$$\tilde{\nu}_S^{\pm,\,\mathrm{out}} \circ \overline{N}(\kappa)^{-1} = C_{|S}\,\nu_S^{\pm,\,\mathrm{out}}\,C_{|S}^{*}, \quad \tilde{\nu}_S^{\pm,\,\mathrm{in}} \circ \overline{N}(\kappa)^{-1} = C_{|S}\,\nu_S^{\pm,\,\mathrm{in}}\,C_{|S}^{*}. \tag{38}$$

In the coordinates (s,z,σ,ζ) we choose the equations of I, $\tilde{\zeta} = 0$. Let $\rho \in S$, this equation generates coordinates $\tilde{\eta} \in \mathbf{R}^4$ on $\overline{N}(I)_{|\rho}$. In these coordinates the branching of the energy is described by the following theorem.

Theorem 2. *Assume that $\tilde{\nu}^{+,\,\mathrm{in}}$ and $\tilde{\nu}^{-,\,\mathrm{in}}$ are mutually singular, then we have*

$$\begin{aligned} \tilde{\nu}_S^{+,\,\mathrm{out}} &= \tilde{T}\,\tilde{\nu}_S^{-,\,\mathrm{in}} + (1-\tilde{T})\tilde{\mathscr{R}}\,\tilde{\nu}_S^{+,\,\mathrm{in}}\tilde{\mathscr{R}} \\ \tilde{\nu}_S^{-,\,\mathrm{out}} &= \tilde{T}\,\tilde{\nu}_S^{+,\,\mathrm{in}} + (1-\tilde{T})\tilde{\mathscr{R}}\,\tilde{\nu}_S^{-,\,\mathrm{in}}\tilde{\mathscr{R}} \end{aligned} \tag{39}$$

with

1) *If* $\alpha \in \,]0,\frac{1}{2}[$,

$$\tilde{T} = 0, \quad \tilde{\mathscr{R}} = \begin{pmatrix} \mathbf{0} & p(0,0,1) \\ p(0,0,1) & \mathbf{0} \end{pmatrix}.$$

2) *If* $\alpha = \frac{1}{2}$,

$$\tilde{T} = \exp[-\pi(|\Gamma_{|S}\tilde{\eta}|^2 + k_{|S}^2 m^2)],$$

$$\tilde{\mathscr{R}} = (|\Gamma_{|S}\tilde{\eta}|^2 + k_{|S}^2 m^2)^{-1/2}\begin{pmatrix} \mathbf{0} & p(\Gamma_{|S}\tilde{\eta}, k_{|S}m) \\ p(\Gamma_{|S}\tilde{\eta}, k_{|S}m) & \mathbf{0} \end{pmatrix}.$$

3) *If* $\alpha > \frac{1}{2}$,

$$\tilde{T} = \exp[-\pi|\Gamma_{|S}\tilde{\eta}|^2], \quad \tilde{\mathscr{R}} = \frac{1}{|\Gamma_{|S}\tilde{\eta}|}\begin{pmatrix} \mathbf{0} & p(\Gamma_{|S}\tilde{\eta},0) \\ p(\Gamma_{|S}\tilde{\eta},0) & \mathbf{0} \end{pmatrix}.$$

Let us explain now why this theorem implies Theorem 1.

Proof of Theorem 1. We focus on the case $\alpha = 1/2$. Because of (38), we have

$$T = \tilde{T} \circ \overline{N}(\kappa)^{-1}, \quad \mathscr{R} = (C_{|S}^{-1} \tilde{\mathscr{R}} C_{|S}) \circ \overline{N}(\kappa)^{-1}.$$

The first observation consists in studying the link between $\tilde{\eta}$ and the function η defined in the introduction. In view of (34) and of Proposition 2, we have

$$\Gamma_{|S}\, \mathrm{d}\tilde{\zeta} = \lambda\, \mathrm{d}g_{|S}.$$

Moreover,

$$\eta = [\mathrm{d}(\xi - A)\delta\rho - \mathrm{d}(\tau + V)\delta\rho\, u] \times E.$$

Using (25), (27) and (11), we obtain

$$\Gamma_{|S}\, \mathrm{d}\tilde{\eta} = \left(-\frac{\lambda_{|S}|E|}{r^2}\eta \cdot e_3, \frac{\lambda_{|S}}{r}\eta \cdot e_2 \right). \tag{40}$$

Hence, by (27), (12), (31) and (22), we get

$$|\Gamma_{|S}\, \mathrm{d}\tilde{\eta}|^2 + k_{|S}^2 m^2 = (|E|^2 - |B|^2)^{-3/2} \Phi(\eta, m)^2.$$

This yields the value of T stated in Theorem 1.

It remains to calculate $C_{|S}^{-1}\tilde{\mathscr{R}} C_{|S}$. Notice that

$$\tilde{\mathscr{R}} = \Phi(\eta, m)^{-1}(|E|^2 - |B|^2)^{3/4} \begin{pmatrix} \mathbf{0} & p(\Gamma_{|S}\tilde{\eta}, mk_{|S}) \\ p(\Gamma_{|S}\tilde{\eta}, mk_{|S}) & \mathbf{0} \end{pmatrix}.$$

Moreover, because of (32), we have

$$C_{|S}^{-1}\tilde{\mathscr{R}} C_{|S} = C_1{}_{|S}^{-1}\tilde{\mathscr{R}} C_{1|S}.$$

We shall use the following lemma.

Lemma 4. *For any* $v = (v_1, v_2, v_3) \in \mathbf{R}^3$,

$$\begin{aligned} &(C_1)_{|S}^{-1} \begin{pmatrix} \mathbf{0} & p(v) \\ p(v) & \mathbf{0} \end{pmatrix} (C_1)_{|S} \\ &\quad = -\frac{v_3}{\sqrt{1-\theta^2}} \begin{pmatrix} \mathbf{1} & -p\left(\frac{E \times B}{|E|^2}\right) \\ p\left(\frac{E \times B}{|E|^2}\right) & -\mathbf{1} \end{pmatrix} + i\frac{\theta\, v_2}{\sqrt{1-\theta^2}} \begin{pmatrix} p\left(\frac{E}{|E|}\right) & \mathbf{0} \\ \mathbf{0} & p\left(\frac{E}{|E|}\right) \end{pmatrix} \\ &\qquad - \frac{1}{\sqrt{1-\theta^2}} \begin{pmatrix} \mathbf{0} & U^* p(\sqrt{1-|\theta|^2}v_1, v_2, 0)U \\ U^* p(\sqrt{1-|\theta|^2}v_1, v_2, 0)U & \mathbf{0} \end{pmatrix}. \end{aligned}$$

Proof. This lemma comes from simple computations and the use of

$$\sqrt{S(\theta_{|S})}p(v', 0)\sqrt{S(-\theta_{|S})} = p\left(v_1, \frac{v_2}{\sqrt{1-\theta^2}}, 0\right) + i\theta_{|S} p(0,0,1),$$

$$U^* p(\theta_{|S})U = -p\left(\frac{E \times B}{|E|^2}\right), \quad U^* p(0,0,1)U = p\left(\frac{E}{|E|}\right).$$

These latter equations are consequences of the definition of U (see (28)), of (22) and (12). □

By definition of $\Phi(\eta, m)$, we have

$$\mathscr{R} = \frac{|E|}{\Phi(\eta, m)} (|E|^2 - |B|^2)^{1/4} C_1^{-1} \begin{pmatrix} \mathbf{0} & p(v) \\ p(v) & \mathbf{0} \end{pmatrix} C_1,$$

with $v = (\lambda_{|S} \Gamma_{|S} \tilde{\eta}, m\, k_{|S})$. By (40), we obtain

$$v_3 = (|E|^2 - |B|^2)^{-1/4} m,$$

$$\theta v_2 = -(|E|^2 - |B|^2)^{-1/4} \eta \cdot \frac{E \times B}{|E|^3},$$

$$\left(\sqrt{1 - |\theta|^2} v_1, v_2\right) = (|E|^2 - |B|^2)^{-1/4} \left(\left(\frac{E}{|E|^2} \times \eta \right) \cdot e_2, \left(\frac{E}{|E|^2} \times \eta \right) \cdot e_3, \left(\frac{E}{|E|^2} \times \eta \right) \cdot e_1 \right).$$

In view of (28), we get the expression of $\mathscr{R}$ stated in Theorem 1. □

In the following sections, we drop the ˜ on $\tilde{\eta}$ and we focus on proving Theorem 2, for $\alpha \in (0, 1/2)$ or ($\alpha \geqslant 1/2$ and $|\eta| = \infty$) first, then for ($\alpha \geqslant 1/2$ and $\eta < +\infty$).

Before closing this section, let us state a last result concerning system (15). The dependance on the variable σ of the function γ_1 and γ_2 does not prevent from dealing with system (15) as with a system of evolution equations. Actually, arguing as in [11], Proposition 5, we can prove the following hyperbolic estimate.

Proposition 3. *Consider* (v^ε) *a bounded family in* $L^2(\mathbf{R}^4_{s,z}, \mathbf{C}^4)$ *satisfying* (15), *consider* $\rho \in \mathscr{C}_0^\infty(\mathbf{R}^3)$ *such that* $\rho(0) = 1$, *then there exists* $\delta > 0$ *and* $\varepsilon_0 > 0$ *such that the family* $\left(\rho\left(\frac{(s, \varepsilon D_{z_1}, \varepsilon D_{z_2})}{\delta}\right) v^\varepsilon\right)_{\varepsilon > \varepsilon_0}$ *is bounded in* $L^\infty(\mathbf{R}_s, L^2(\mathbf{R}^3_z))$.

Therefore, arguing as in [11] and [9], we can estimate $(\mathrm{op}_\varepsilon(a) v^\varepsilon \mid v^\varepsilon)$ for $a \in \mathscr{C}_0^\infty(K)$ where K is some compact subset of $\{|\tilde{\zeta}|^2 + s^2 \leqslant \delta\}$, for δ small enough; by Proposition 3 and Schur's lemma, we have

$$|(\mathrm{op}_\varepsilon(a) v^\varepsilon | v^\varepsilon)| \leqslant C \int_{-\infty}^{+\infty} \sup_{k + |\beta| \leqslant N} \sup_{(z, \sigma, \zeta) \in \mathbf{R}^7} |\partial_\sigma^k \partial_z^\beta a(s, z, \sigma, \zeta)| \, ds, \tag{41}$$

uniformly with respect to K.

4. The Adiabatic Cases: $(|\eta| = +\infty, \alpha \geqslant 1/2)$ or $\alpha \in (0, 1/2)$

In this section, we prove Equations (39) for $\alpha \in (0, 1/2)$ and for ($\alpha \geqslant 1/2$, $|\eta| = +\infty$), i.e.

$$\tilde{\nu}^{+,\mathrm{out}} = \tilde{\mathscr{R}}\, \tilde{\nu}^{+,\mathrm{in}}\, \tilde{\mathscr{R}}, \quad \tilde{\nu}^{-,\mathrm{out}} = \tilde{\mathscr{R}}\, \tilde{\nu}^{-,\mathrm{in}}\, \tilde{\mathscr{R}}. \tag{42}$$

In view of the definition of $\tilde{\Pi}^\pm$ (see (36)), we have

$$\tilde{\Pi}^+ = \begin{pmatrix} \mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix} \text{ above } J^{+,\mathrm{in}}, \qquad \tilde{\Pi}^+ = \begin{pmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{1} \end{pmatrix} \text{ above } J^{+,\mathrm{out}}, \tag{43}$$

$$\tilde{\Pi}^- = \begin{pmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{1} \end{pmatrix} \text{ above } J^{-,\,\mathrm{in}}, \quad \tilde{\Pi}^- = \begin{pmatrix} \mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix} \text{ above } J^{-,\,\mathrm{out}}. \tag{44}$$

Therefore, the polarization of $\tilde{\nu}^{\pm,\,\mathrm{in}}$ is not the same than the one of $\tilde{\nu}^{\pm,\,\mathrm{out}}$. This explains the presence of the matrix $\tilde{\mathscr{R}}$ in (42).

We follow the method initiated in [9]–[11] and developed in [8] for eigenvalues of multiplicity higher than 1. We shall focus on the plus mode, the proof for the minus mode is similar. We proceed in two steps. First, we reduce (42) to some statement on the family (v^ε) by use of suitably chosen symbols. Then, in a second step, we prove this statement by Weyl-Hörmander pseudo-differential calculus.

4.1. An equivalent statement to (42). Here again, we proceed in two steps. Equation (42) link the traces on S of $\tilde{\nu}^\pm$. The first step consists in translating (42) into some result on $\tilde{\nu}^\pm$ itself and not only on its traces. Then, it is easy to obtain an equivalent statement on (v^ε), simply by using the definition of two-scale Wigner measure. Let us introduce first some notations.

Consider the vector-valued function $X^\infty_\omega = X^\infty_\omega(s,z,\sigma,\zeta,\eta)$ defined by

for $\alpha > 1/2$,

$$X^\infty_\omega = \left(p\left(\frac{\Gamma\eta}{|\Gamma\eta|},0\right)\omega,(0,0)\right) \text{ if } s>0, \quad X^\infty_\omega = ((0,0),\omega) \text{ if } s<0,$$

for $\alpha = 1/2$,

$$X^\infty_\omega = \left(p\left(\frac{(\Gamma\eta,mk)}{\sqrt{|\Gamma\eta|^2+m^2k^2}}\right)\omega,(0,0)\right) \text{ if } s>0, \quad X^\infty_\omega = ((0,0),\omega) \text{ if } s<0,$$

for $\alpha \in (0,1/2)$,

$$X^\infty_\omega = (p(0,0,1)\omega,(0,0)) \text{ if } s>0, \quad X^\infty_\omega = ((0,0),\omega) \text{ if } s<0,$$

and the matrix-valued function $\Pi^\infty_{\omega,\omega'} = \Pi^\infty_{\omega,\omega'}(s,z,\sigma,\zeta,\eta)$ defined by

$$\Pi^\infty_{\omega,\omega'} := X^\infty_\omega \otimes \overline{X^\infty_{\omega'}} =: \begin{cases} \Pi^{\mathrm{in}}_{\omega,\omega'} & \text{if} \quad s<0 \\ \Pi^{\mathrm{out}}_{\omega,\omega'} & \text{if} \quad s>0. \end{cases}$$

Lemma 5. *Consider now* $\rho \in \mathscr{C}^\infty_0(\mathbf{R})$, $\rho(0) \equiv 1$.

1) *If* $\alpha \geqslant 1/2$, *Equation* (42) *for* $|\eta| = +\infty$ *is equivalent to*

$$\forall a_0 \in \mathscr{C}^\infty_0(\mathbf{R}^7 \times \mathbf{S}^1), \ \lim_{\varepsilon\to 0} \operatorname{tr}\left(\left\langle \frac{1}{\varepsilon}\rho'\left(\frac{s}{\varepsilon}\right) a_0\left(z,\sigma,\zeta,\frac{\eta}{|\eta|}\right)\Pi^\infty_{\omega,\omega'},\ \tilde{\nu}^+\mathbf{1}_{|\eta|=\infty}\right\rangle\right) = 0. \tag{45}$$

2) *If* $\alpha \in (0,1/2)$, *Equation* (42) *is equivalent to*

$$\forall a_0 \in \mathscr{A}, \quad \lim_{\varepsilon\to 0} \operatorname{tr}\left(\left\langle \frac{1}{\varepsilon}\rho'\left(\frac{s}{\varepsilon}\right) a_0(z,\sigma,\zeta,\eta)\Pi^\infty_{\omega,\omega'},\ \tilde{\nu}^+\right\rangle\right) = 0. \tag{46}$$

Proof. 1) We crucially use that the matrix $\tilde{\nu}^+$ satisfies the commutation's relations (37). We write the 4×4 matrix $\tilde{\nu}^+$ by blocks of 2×2 matrices

$$\tilde{\nu}^{\pm} = \begin{pmatrix} A^+ & B^+ \\ C^+ & D^+ \end{pmatrix}.$$

By (43) and (44), $\tilde{\nu}^+$ is of the form

$$\tilde{\nu}^+ = \begin{pmatrix} A^+ & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix} \text{ above } J^{+,\,\mathrm{in}},$$

$$\tilde{\nu}^+ = \begin{pmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & D^+ \end{pmatrix} \text{ above } J^{+,\,\mathrm{out}}.$$

Besides, the 2×2 matrix A^+ is utterly determined by the knowledge of $\mathrm{tr}(A^+\omega \otimes \overline{\omega}')$ for any ω, ω' in $\mathbf{S}^1$. The same fact holds for D^+. Moreover, above $J^{+,\,\mathrm{in}}$

$$\mathrm{tr}(A^+\omega \otimes \overline{\omega}') = \mathrm{tr}\left(\tilde{\nu}^+ \left(\begin{pmatrix} \omega \\ 0 \\ 0 \end{pmatrix} \right) \otimes \left(\begin{pmatrix} \overline{\omega}' \\ 0 \\ 0 \end{pmatrix} \right) \right),$$

and above $J^{+,\,\mathrm{out}}$,

$$\mathrm{tr}(D^+\omega \otimes \overline{\omega}') = \mathrm{tr}\left(\tilde{\nu}^+ \left(\begin{pmatrix} 0 \\ 0 \\ \omega \end{pmatrix} \right) \otimes \left(\begin{pmatrix} 0 \\ 0 \\ \overline{\omega}' \end{pmatrix} \right) \right).$$

Therefore, by the definition of $\tilde{\Pi}^{\mathrm{in}}_{\omega,\omega'}$, Equation (42) is equivalent to the fact that for any choice of ω and ω',

$$\mathrm{tr}(\tilde{\nu}^{+,\,\mathrm{out}}\, \Pi^{\mathrm{in}}_{\omega,\omega'}) = \mathrm{tr}(\tilde{\mathscr{R}}\, \tilde{\nu}^{+,\,\mathrm{in}}\, \tilde{\mathscr{R}}\, \Pi^{\mathrm{in}}_{\omega,\omega'}).$$

In view of

$$\mathrm{tr}(\tilde{\mathscr{R}}\, \tilde{\nu}^{+,\,\mathrm{in}}\, \tilde{\mathscr{R}}\, \Pi^{\mathrm{out}}_{\omega,\omega'}) = \mathrm{tr}(\tilde{\nu}^{+,\,\mathrm{in}}\, \tilde{\mathscr{R}}\, \Pi^{\mathrm{out}}_{\omega,\omega'} \tilde{\mathscr{R}}) \quad \text{and} \quad \tilde{\mathscr{R}}\, \Pi^{\mathrm{out}}_{\omega,\omega'} \tilde{\mathscr{R}} = \Pi^{\mathrm{in}}_{\omega,\omega'},$$

Equation (42) is equivalent to the fact that for any choice of ω and ω', on $|\eta| = +\infty$,

$$\mathrm{tr}(\tilde{\nu}^{+,\,\mathrm{out}}\, \Pi^{\mathrm{out}}_{\omega,\omega'}) = \mathrm{tr}(\tilde{\nu}^{+,\,\mathrm{in}}\, \Pi^{\mathrm{in}}_{\omega,\omega'}),$$

i.e. to (45).

The proof of 2) is similar. □

For the second step, we need more notations. We set

$$\lambda^\varepsilon = \sqrt{s^2 + m_\varepsilon^2 + |\Gamma\tilde{\zeta}|^2} = \sqrt{s^2 + m^2 k^2 \varepsilon^{2\alpha} + |\Gamma\tilde{\zeta}|^2},$$

$$A^\varepsilon = \begin{pmatrix} s & p(\Gamma\tilde{\zeta}, m_\varepsilon) \\ p(\Gamma\tilde{\zeta}, m_\varepsilon) & -s \end{pmatrix},$$

and we denote by (f^ε) the family such that, locally near 0,

$$\frac{\varepsilon}{i}\partial_s v^\varepsilon = \mathrm{op}_\varepsilon(A_\varepsilon)v^\varepsilon + \varepsilon f^\varepsilon,$$

with $(\mathrm{op}_\varepsilon(\phi)f^\varepsilon)$ bounded in $L^2_{s,z}$ for any ϕ compactly supported in a neighborhood of 0 small enough.

Then, for $\omega\in\mathbf{S}^1$, we denote by $X^\varepsilon_\omega = X^\varepsilon_\omega(s,\sigma,z,\zeta)$ the norm 1 eigenvector of A^ε for the eigenvalue λ^ε defined for $s\neq 0$ or $\tilde\zeta\neq 0$ by

$$X^\varepsilon_\omega = \frac{1}{\sqrt{2\lambda^\varepsilon(\lambda^\varepsilon - s)}}(p(\Gamma\tilde\zeta, m_\varepsilon)\omega, (\lambda^\varepsilon - s)\omega) = \frac{1}{\sqrt{2\lambda^\varepsilon(\lambda^\varepsilon - s)}}(\lambda^\varepsilon + A^\varepsilon)\begin{pmatrix} 0 \\ \omega \end{pmatrix}.$$

For ω, $\omega'\in\mathbf{S}^1$, we consider the matrix

$$\Pi^\varepsilon_{\omega,\omega'} = \Pi^\varepsilon_{\omega,\omega'}(s,z,\sigma,\zeta) := X^\varepsilon_\omega \otimes \overline{X}^\varepsilon_{\omega'}.$$

The function $\Pi^\varepsilon_{\omega,\omega'}$ depends on ε, thus, for $a\in\mathscr{A}$, the function

$$q : (s,z,\sigma,\zeta,\eta) \mapsto a(s,z,\sigma,\zeta,\eta)\Pi^\varepsilon_{\omega,\omega'}(s,z,\sigma,\zeta)$$

is not in $\mathscr{A}$. However, this function is smooth and one can consider the family of operators $\mathrm{op}_\varepsilon(q(s,z,\sigma,\zeta,\frac{\zeta}{\sqrt{\varepsilon}}))$ that we will denote by $\mathrm{op}_\varepsilon{}^I(a\Pi^\varepsilon_{\omega,\omega'})$.

Lemma 6. *Consider the scalar symbol $a\in\mathscr{A}$ compactly supported outside $s=0$ and in the ball $\{s^2+|\tilde\zeta|^2<\delta^2\}$, for $\delta>0$ as in Proposition 3. Then we have,*

$$(\mathrm{op}_\varepsilon{}^I(a\,\Pi^\varepsilon_{\omega,\omega'})v^\varepsilon|v^\varepsilon) \underset{\varepsilon\to 0}{\longrightarrow} \mathrm{tr}\langle a\,\Pi^\infty_{\omega,\omega'}, \tilde\nu^+\rangle.$$

Proof. Let us suppose that $s>0$ on $\mathrm{Supp}(a)$, the proof is similar in the other cases. We have

$$X^\varepsilon_\omega = \left(\frac{\sqrt{\lambda_\varepsilon + s}}{\sqrt{2\lambda_\varepsilon}}p\left(\frac{(\Gamma\tilde\zeta, m_\varepsilon)}{\sqrt{|\Gamma\tilde\zeta|^2 + m_\varepsilon^2}}\right)\omega, \frac{\sqrt{\lambda_\varepsilon - s}}{\sqrt{2\lambda_\varepsilon}}\,\omega\right).$$

The functions $(s,z,\sigma,\zeta)\mapsto \frac{\sqrt{\lambda_\varepsilon+s}}{\sqrt{2\lambda_\varepsilon}}$ and $(s,z,\sigma,\zeta)\mapsto\frac{\sqrt{\lambda_\varepsilon-s}}{\sqrt{2\lambda_\varepsilon}}$ are smooth functions on $\mathrm{Supp}(a)$. They go respectively to 1 and 0 as ε goes to 0. Therefore, we get

$$(\mathrm{op}_\varepsilon{}^I(a\Pi^\varepsilon_{\omega,\omega'})v^\varepsilon|v^\varepsilon) = (\mathrm{op}_\varepsilon{}^I(a\Pi^\infty_{\omega,\omega'})v^\varepsilon, v^\varepsilon) + o(1),$$

where we used (41). Hence the result. □

In order to prove (45) and (46), we use Lemma 6 as follows.

• In the non adiabatic case $\alpha\geqslant 1/2$, we consider a_0, δ and δ' as before, and for $R>0$, we define a as

$$a(s,z,\sigma,\zeta,\eta) = \rho\left(\frac{s}{\delta'}\right)a_0\left(z,\sigma,\zeta,\frac{\eta}{|\eta|}\right)\rho\left(\frac{|\tilde\zeta|^2}{\delta}\right)\left(1-\rho\left(\frac{|\eta|}{R}\right)\right).$$

The function a is in $\mathscr{A}$ and for $q^\varepsilon = a\Pi^\varepsilon_{\omega,\omega'}$, the operator $\mathrm{op}_\varepsilon(q^\varepsilon)$ satisfies

$$|\partial^\beta_{\sigma,z,\zeta_3}\partial^\gamma_{s,\tilde\zeta}q^\varepsilon| \leqslant C_{\beta,\gamma}(\delta',\delta)\left(\frac{1}{R\sqrt{\varepsilon}}\right)^{|\gamma|},$$

where we implicitly assumed $R\sqrt{\varepsilon}\leqslant 1$.

If we consider the Weyl-Hörmander metric

$$g_\varepsilon = \mathrm{d}z^2 + \frac{\mathrm{d}s^2}{R^2\varepsilon} + \varepsilon^2(\mathrm{d}\sigma^2 + \mathrm{d}\zeta_3^2) + \varepsilon\frac{\mathrm{d}\tilde{\zeta}^2}{R^2},$$

we have, with Hörmander's notations, $\frac{g_\varepsilon}{g_\varepsilon^\sigma} \leqslant \left(\frac{\sqrt{\varepsilon}}{R}\right)^2$; thus the gain of this symbolic calculus is $\frac{\sqrt{\varepsilon}}{R}$. Equation (45) is equivalent to

$$\lim_{\delta'\to 0}\lim_{R\to\infty}\lim_{\delta\to 0}\lim_{\varepsilon\to 0}(\mathrm{op}_\varepsilon{}^I(\partial_s a\Pi^\varepsilon_{\omega,\omega'})v^\varepsilon|v^\varepsilon) = 0. \tag{47}$$

• In the adiabatic case: $\alpha\in(0,1/2)$, for a_0 as before, $\delta'>0$, $\delta<\delta_0$ (where δ_0 is defined in Proposition 3), we define a as

$$a(s,z,\sigma,\zeta,\eta) = \rho\left(\frac{s}{\delta'}\right)a_0(z,\sigma,\zeta,\eta)\rho\left(\frac{|\tilde{\zeta}|^2}{\delta}\right).$$

We get that if $q_\varepsilon = a\Pi^\varepsilon_{\omega,\omega'}$,

$$|\partial^\beta_{\sigma,z,\zeta_3}\partial^\gamma_{s,\tilde{\zeta}}q^\varepsilon| \leqslant C_{\beta,\gamma}(\delta',\delta)\left(\frac{1}{\varepsilon^\alpha}\right)^{|\gamma|}.$$

This estimate allows us to use Weyl-Hörmandes symbolic calculus for which we refer to Sections 18.4, 18.5 and 18.6 in [20]. The symbol q^ε belongs to the class $S(1,g_\varepsilon)$ where g_ε is the metric

$$g_\varepsilon = \mathrm{d}z^2 + \frac{\mathrm{d}s^2}{\varepsilon^{2\alpha}} + \varepsilon^2(\mathrm{d}\sigma^2 + \mathrm{d}\zeta_3^2) + \varepsilon^{2(1-\alpha)}\mathrm{d}\tilde{\zeta}^2.$$

Since $\frac{g_\varepsilon}{g_\varepsilon^\sigma} \leqslant \varepsilon^{2(1-2\alpha)}$, the gain of this symbolic calculus is $\varepsilon^{1-2\alpha}$ which goes to 0 as ε goes to 0. Then (46) is equivalent to

$$\lim_{\delta'\to 0}\lim_{\delta\to 0}\lim_{\varepsilon\to 0}(\mathrm{op}_\varepsilon{}^I(\partial_s a\Pi^\varepsilon_{\omega,\omega'})v^\varepsilon|v^\varepsilon) = 0, \tag{48}$$

and the proof of (47) will apply too to (48).

4.2. Proof of (47) and (48). For short, we focus on the case $\alpha\geqslant 1/2$. We crucially use Weyl-Hörmander metric g_ε defined above and Corollary 41. We set

$$L = (\mathrm{op}_\varepsilon{}^I(\partial_s a\Pi^\varepsilon_{\omega,\omega'})v^\varepsilon|v^\varepsilon),$$

and we decompose L into $L = L^1 + L^2 + L^3$ with

$$\begin{aligned}
L^1 &= \frac{i}{\varepsilon}\left(\left[\mathrm{op}_\varepsilon{}^I(a\Pi^\varepsilon_{\omega,\omega'}), \mathrm{op}_\varepsilon\begin{pmatrix} s & p(\Gamma\tilde{\zeta},m_\varepsilon) \\ p(\Gamma\tilde{\zeta},m_\varepsilon)^* & -s\end{pmatrix}\right]v^\varepsilon|v^\varepsilon\right),\\
L^2 &= (\mathrm{op}_\varepsilon{}^I(a\partial_s\Pi^\varepsilon_{\omega,\omega'})v^\varepsilon|v^\varepsilon),\\
L^3 &= i(\mathrm{op}_\varepsilon{}^I(a\Pi^\varepsilon_{\omega,\omega'})f^\varepsilon|v^\varepsilon) - i(\mathrm{op}_\varepsilon{}^I(a\Pi^\varepsilon_{\omega,\omega'})v^\varepsilon|f^\varepsilon).
\end{aligned}$$

Then, we prove the convergence to 0 of L^1, L^2 and L^3. For $b = b(s,z,\sigma,\zeta,\eta)\in\mathscr{C}^\infty$, we set

$$b^\flat_\varepsilon(s,z,\sigma,\zeta) = b\left(s,z,\sigma,\zeta,\frac{\tilde{\zeta}}{\sqrt{\varepsilon}}\right),$$

$$b^\sharp_\varepsilon(s,z,\sigma,\zeta) = b(s,z,\varepsilon\sigma,\varepsilon\zeta,\sqrt{\varepsilon}\tilde{\zeta}),$$

so that we have for $b \in \mathscr{A}$,

$$\mathrm{op}_\varepsilon{}^I(b) = \mathrm{op}_\varepsilon(b^\flat_\varepsilon) = \mathrm{op}_1(b^\sharp_\varepsilon).$$

• We begin with L^3. By the Weyl-Hörmander symbolic calculus, if $\chi \in \mathscr{C}_0^\infty(\Omega)$ with $\chi = 1$ near the support of a, we have

$$\mathrm{op}_\varepsilon{}^I(a\Pi^\varepsilon_{\omega,\omega'}) = \mathrm{op}_\varepsilon{}^I(a\chi\Pi^\varepsilon_{\omega,\omega'}) = \mathrm{op}_\varepsilon{}^I(a\Pi^\varepsilon_{\omega,\omega'})\mathrm{op}_\varepsilon(\chi) + \mathrm{op}_1\left(S\left(\frac{\sqrt{\varepsilon}}{R}\right)^N, g_\varepsilon\right),$$

for all $N \in \mathbf{N}$. On the other hand, we can apply Corollary 41 to the symbol $b_\varepsilon = a^\flat_\varepsilon \Pi^\varepsilon_{\omega,\omega'}$. Using the explicit expressions of a and of $\Pi^\varepsilon_{\omega,\omega'}$, we get

$$|L^3| \leqslant C\int_{-\infty}^{+\infty}\left|\rho\left(\frac{s}{\delta'}\right)\right| ds + O\left(\left(\frac{\sqrt{\varepsilon}}{R}\right)^N\right).$$

• Let us consider now L^1. Note that $A^\sharp_\varepsilon \in S(\lambda^\sharp_\varepsilon, g_\varepsilon)$, therefore by symbolic calculus, we obtain

$$L^1 = \frac{1}{2}(\mathrm{op}_\varepsilon(\{a^\flat_\varepsilon\Pi^\varepsilon_{\omega,\omega'}, A\} - \{A, a^\flat_\varepsilon\Pi^\varepsilon_{\omega,\omega'}\})v^\varepsilon | v^\varepsilon) + O\left(\frac{1}{R^2}\right).$$

We set

$$b_\varepsilon = \{a^\flat_\varepsilon\Pi^\varepsilon_{\omega,\omega'}, A\} - \{A, a^\flat_\varepsilon\Pi^\varepsilon_{\omega,\omega'}\}. \tag{49}$$

Observe that $\Pi^\varepsilon_{\omega,\omega'} = U(s, \lambda^\varepsilon, \Gamma\tilde{\zeta}, m_\varepsilon)$ where $U = U(r,t,X,m)$ is homogeneous of degree 0. Therefore, $\partial_s\Pi^\varepsilon_{\omega,\omega'}$ and $\nabla_{\tilde{\zeta}}\Pi^\varepsilon_{\omega,\omega'}$ are homogeneous of degree -1 in the variables s, λ^ε, $\tilde{\zeta}$ and m_ε. However, $\partial_\sigma\Pi^\varepsilon_{\omega,\omega'}$ and $\nabla_{\tilde{z}}\Pi^\varepsilon_{\omega,\omega'}$ have a better degree of homogeneity, they are homogeneous function of degree 0 in λ^ε, s and $\tilde{\zeta}$. Note that, in (49), the derivatives $\partial_s\Pi^\varepsilon_{\omega,\omega'}$ and $\nabla_{\tilde{\zeta}}\Pi^\varepsilon_{\omega,\omega'}$ appear with some factor $\tilde{\zeta}$, which compensates the -1 degree of homogeneity of these functions. Therefore, applying Corollary 41, we obtain the estimate

$$|(\mathrm{op}_\varepsilon(b_\varepsilon)v^\varepsilon, v^\varepsilon)| \leqslant C\int_{-\infty}^{+\infty}\rho\left(\frac{s}{\delta'}\right)\mathrm{d}s = O(\delta').$$

Thus

$$\limsup_{(\delta',\delta)\to(0,0)}\ \limsup_{R\to+\infty}\limsup_{\varepsilon\to 0}|L^1| = 0.$$

• Finally, let us deal with the remainder term L^2. We shall use the following properties of X_ω and $\Pi^\varepsilon_{\omega,\omega'}$.

Lemma 7. 1) *There exists some matrix-valued function* ϕ*, homogeneous of degree* 0 *in* s, ε^α *and* $\tilde{\zeta}$*, such that if* (ω,ω') *is some orthonormal basis of* $\mathbf{R}^2$,

$$\partial_s X^\varepsilon_\omega = \frac{1}{2\lambda^\varepsilon}(\partial_s A^\varepsilon - \partial_s\lambda^\varepsilon)X^\varepsilon_\omega + (\phi\,\omega\cdot\omega')X^\varepsilon_{\omega'}.$$

2) *If* (ω,ω') *is some orthonormal basis of* $\mathbf{R}^2$,

$$\partial_s\Pi^\varepsilon_{\omega,\omega} = \left[A^\varepsilon, \frac{A^\varepsilon}{2(\lambda^\varepsilon)^2}\partial_s\Pi^\varepsilon_{\omega,\omega}\right] - (\phi\,\omega\cdot\omega)(\Pi^\varepsilon_{\omega',\omega}+\Pi^\varepsilon_{\omega,\omega'}),$$

$$\partial_s\Pi^\varepsilon_{\omega,\omega'} = \left[A^\varepsilon, \frac{A^\varepsilon}{2(\lambda^\varepsilon)^2}\partial_s\Pi^\varepsilon_{\omega,\omega'}\right] - (\phi\,\omega\cdot\omega')(\Pi^\varepsilon_{\omega,\omega}+\Pi^\varepsilon_{\omega',\omega'}). \tag{50}$$

We postpone the proof of this Lemma at the end of the section.

Let us conclude for L^2. The homogeneity of ϕ and $\Pi^\varepsilon_{\omega,\omega'}$ in s, λ^ε and $\tilde{\zeta}$ yields that

$$\limsup_{(\delta',\delta)\to(0,0)}\ \limsup_{R\to+\infty}\limsup_{\varepsilon\to 0}|(\mathrm{op}_\varepsilon{}^I(a(\phi\,\omega\cdot\omega')\Pi^\varepsilon_{\omega,\omega'})v^\varepsilon|v^\varepsilon)|=0.$$

Moreover, we transform the bracket part of $\partial_s\Pi^\varepsilon_{\omega,\omega'}$ as in [11] so that we can reuse the equation. In the metric g_ε, since $(\lambda^\varepsilon)^\sharp_\varepsilon \geqslant C\sqrt{\varepsilon}R$ on $\mathrm{Supp}(a^\sharp_\varepsilon)$

$$(a(\lambda^\varepsilon)^{-2}A^\varepsilon)^\sharp_\varepsilon\partial_s\Pi^\varepsilon_{\omega,\omega'}\in S\left(\frac{1}{\varepsilon R^2},g_\varepsilon\right).$$

We obtain

$$\begin{aligned}\mathrm{op}_\varepsilon{}^I\left(\left[A^\varepsilon, a\frac{A^\varepsilon}{2(\lambda^\varepsilon)^2}\partial_s\Pi^\varepsilon_{\omega,\omega}\right]\right) \in\ & \frac{1}{2}\mathrm{op}_\varepsilon(A^\varepsilon)\mathrm{op}_\varepsilon{}^I(a(\lambda^\varepsilon)^{-2}A^\varepsilon\partial_s\Pi^\varepsilon_{\omega,\omega'})\\ &-\frac{1}{2}\mathrm{op}_\varepsilon{}^I(a(\lambda^\varepsilon)^{-2}A^\varepsilon\partial_s\Pi^\varepsilon_{\omega,\omega'})\mathrm{op}_\varepsilon(A^\varepsilon)\\ &+\mathrm{op}_1\left(S\left(\frac{1}{R^2},g_\varepsilon\right)\right).\end{aligned}$$

Therefore, we can use again the equation and we get

$$\begin{aligned}&\left(\mathrm{op}_\varepsilon{}^I\left(\left[A^\varepsilon, a\frac{A^\varepsilon}{2(\lambda^\varepsilon)^2}\partial_s\Pi^\varepsilon_{\omega,\omega}\right]\right)v^\varepsilon|v^\varepsilon\right)\\ &\quad= O\left(\frac{1}{R^2}\right)+O(\varepsilon)-\frac{\varepsilon}{2i}(\mathrm{op}_\varepsilon{}^I(\partial_s(a(\lambda^\varepsilon)^{-2}A^\varepsilon\partial_s\Pi^\varepsilon_{\omega,\omega'}))v^\varepsilon|v^\varepsilon).\end{aligned}$$

Since $|\partial^\beta_{\sigma,z}\partial_s(a(\lambda^\varepsilon)^{-2}A^\varepsilon\partial_s\Pi^\varepsilon_{\omega,\omega'})^\flat_\varepsilon|\leqslant\frac{C}{(s^2+\varepsilon R^2)^{3/2}}$, as ε goes to 0,

$$\begin{aligned}&\left(\mathrm{op}_\varepsilon{}^I\left(\left[A^\varepsilon, a\frac{A^\varepsilon}{2(\lambda^\varepsilon)^2}\partial_s\Pi^\varepsilon_{\omega,\omega}\right]\right)v^\varepsilon|v^\varepsilon\right)\\ &\quad= O\left(\frac{1}{R^2}\right)+o(1)+C\varepsilon\int_{-\infty}^{+\infty}\frac{ds}{(s^2+\varepsilon R^2)^{3/2}}=o(1)+O\left(\frac{1}{R^2}\right).\end{aligned}$$

Hence, $\limsup_{R\to+\infty}\limsup_{\varepsilon\to 0}L^2=0$. It remains to prove Lemma 7 to complete the proof of (47) for $\alpha\geqslant 1/2$. Similar proof applies in the case $\alpha\in(0,1/2)$ where, roughly speaking $\varepsilon^{1-\alpha/2}$ plays the rule of $1/R^2$. □

Proof of Lemma 7. 1) Consider ω, $\omega'\in\mathbf{S}^1$ such that $\omega\cdot\omega'=0$. The vectors X_ω and $X_{\omega'}$ form a orthonormal basis of the subspace of all the eigenvectors of A^ε for the eigenvalue λ^ε. Moreover, using that $X_\omega\cdot X_{\omega'}=0$, we obtain that

$$\partial_s X_\omega\cdot X_{\omega'}=\frac{1}{2\lambda^\varepsilon(\lambda^\varepsilon-s)}\left[(\lambda^\varepsilon+A^\varepsilon)(\partial_s\lambda^\varepsilon+\partial_s A^\varepsilon)\begin{pmatrix}0\\ \omega\end{pmatrix}\right]\cdot\begin{pmatrix}0\\ \omega'\end{pmatrix}.$$

We get, through straightforward computation,

$$\partial_s X_\omega \cdot X_{\omega'} = \frac{1}{2\lambda^\varepsilon(\lambda^\varepsilon - s)} (p(\Gamma\tilde\zeta, m_\varepsilon)^* p(\partial_s\Gamma\tilde\zeta, m_\varepsilon)\omega\cdot\omega').$$

We set

$$\phi(s,z,\sigma,\zeta) = \frac{1}{2\lambda^\varepsilon(\lambda^\varepsilon - s)} p(\Gamma\tilde\zeta, m_\varepsilon)^* p(\partial_s\Gamma\tilde\zeta, m_\varepsilon). \tag{51}$$

Since $|X_\omega| = 1$, we also have $\partial_s X_\omega \cdot X_\omega = 0$. Thus, the vector $\partial_s X_\omega$ is of the form

$$\partial_s X_\omega = (\phi\,\omega\cdot\omega')X_{\omega'} + Y,$$

where Y is an eigenvector of A^ε for the eigenvalue $-\lambda^\varepsilon$. The derivation in the variable s of the relation $A^\varepsilon X_\omega = \lambda^\varepsilon X_\omega$ yields

$$(\partial_s\lambda^\varepsilon - \partial_s A^\varepsilon)X_\omega = (A^\varepsilon - \lambda^\varepsilon)\partial_s X_\omega = -2\lambda^\varepsilon Y = -2\lambda^\varepsilon(\partial_s X_\omega - (\phi\omega\cdot\omega')X_{\omega'}),$$

whence 1).

2) Because of 1), if $M = \frac{1}{2\lambda^\varepsilon}((\partial_s A^\varepsilon - \partial_s\lambda^\varepsilon))$, we have for $\omega\cdot\omega' = 0$,

$$\partial_s\Pi^\varepsilon_{\omega,\omega'} = M\Pi^\varepsilon_{\omega,\omega'} + \Pi^\varepsilon_{\omega,\omega'}M + (\phi\,\omega\cdot\omega')(\Pi^\varepsilon_{\omega,\omega} + \Pi^\varepsilon_{\omega',\omega'})$$
$$\partial_s\Pi^\varepsilon_{\omega,\omega} = M\Pi^\varepsilon_{\omega,\omega} + \Pi^\varepsilon_{\omega,\omega}M + (\phi\,\omega\cdot\omega)(\Pi^\varepsilon_{\omega',\omega} + \Pi^\varepsilon_{\omega,\omega'}).$$

Observe that since

$$\partial_s((A^\varepsilon)^2) = \partial_s((\lambda^\varepsilon)^2) = A^\varepsilon\partial_s A^\varepsilon + A^\varepsilon\partial_s A^\varepsilon = 2\lambda^\varepsilon\partial_s\lambda^\varepsilon,$$

we have $A^\varepsilon M + MA^\varepsilon = 2\partial_s\lambda^\varepsilon\tilde\Pi^-_\varepsilon$. Therefore, using

$$A^\varepsilon\Pi^\varepsilon_{\omega,\omega'} = \Pi^\varepsilon_{\omega,\omega'}A^\varepsilon = \lambda^\varepsilon\Pi^\varepsilon_{\omega,\omega'}, \quad A^\varepsilon\Pi^\varepsilon_{\omega,\omega} = \Pi^\varepsilon_{\omega,\omega}A^\varepsilon = \lambda^\varepsilon\Pi^\varepsilon_{\omega,\omega},$$

we obtain

$$A^\varepsilon\partial_s\Pi^\varepsilon_{\omega,\omega'}A^\varepsilon = -(\lambda^\varepsilon)^2\partial_s\Pi^\varepsilon_{\omega,\omega'} + 2(\lambda^\varepsilon)^2(\phi\,\omega\cdot\omega')(\Pi^\varepsilon_{\omega,\omega} + \Pi^\varepsilon_{\omega',\omega'}),$$
$$A^\varepsilon\partial_s\Pi^\varepsilon_{\omega,\omega}A^\varepsilon = -(\lambda^\varepsilon)^2\partial_s\Pi^\varepsilon_{\omega,\omega} + 2(\lambda^\varepsilon)^2(\phi\,\omega\cdot\omega)(\Pi^\varepsilon_{\omega,\omega'} + \Pi^\varepsilon_{\omega',\omega}).$$

Hence 2). □

5. The Non-adiabatic Case: $\alpha \geqslant 1/2$ and $|\eta| < +\infty$

In this section, we aim at proving Landau-Zener formula (39) in $\{|\eta| < +\infty\}$ for $\alpha \geqslant 1/2$. We proceed in two steps. We begin by stating a normal form which holds in any ball $B \subset \mathbf{R}^2_\eta$. Then, we are reduced to deal with some abstract scattering problem which can be solved explicitly. This allows to obtain Landau-Zener formula for measure $\tilde\nu$, thus for ν.

5.1. A normal form at finite distance.

Proposition 4. *For any ball $B \subset \mathbf{R}^2_\eta$, there exists a matrix $C \in \mathscr{C}^\infty_0(\mathbf{R}^{10}_{s,z,\sigma,\zeta,\eta})$ such that if $u^\varepsilon = (1 + \sqrt{\varepsilon}\mathrm{op}_\varepsilon{}^I(C))v^\varepsilon$, then, in $L^2(\mathbf{R}^4)$,*

$$\forall a \in \mathscr{C}^\infty_0(\mathbf{R}^8 \times B), \quad \mathrm{op}_\varepsilon{}^I(a)\mathrm{op}_\varepsilon\begin{pmatrix} -\sigma + s & p(\Gamma_{|S}\tilde\zeta, m_{\varepsilon|S}) \\ p(\Gamma_{|S}\tilde\zeta, m_{\varepsilon|S}) & -\sigma - s \end{pmatrix}u^\varepsilon = O(\varepsilon).$$

Remark 1. Notice that (u^ε) and (v^ε) have the same two-scale Wigner measure for I.

Proof. We set $J = \begin{pmatrix} \mathbf{1} & \mathbf{0} \\ \mathbf{0} & -\mathbf{1} \end{pmatrix} \in \mathbf{R}^{4,4}$, $\tilde{H}_2 = \begin{pmatrix} -\sigma + s & p(\Gamma\tilde{\zeta}, m_\varepsilon) \\ p(\Gamma\tilde{\zeta}, m_\varepsilon) & -\sigma - s \end{pmatrix}$,

$$\Gamma^0 = \Gamma(s,z,0,\zeta), \quad \Gamma^{00} = \Gamma(0,z,0,\zeta),$$

$$m_\varepsilon^0 = \varepsilon^\alpha m\, k(s,z,0,\zeta), \quad m_\varepsilon^{00} = \varepsilon^\alpha m\, k(0,z,0,\zeta),$$

$$\tilde{H}_2^0 = \begin{pmatrix} -\sigma + s & p(\Gamma^0\tilde{\zeta}, m_\varepsilon^0) \\ p(\Gamma^0\tilde{\zeta}, m_\varepsilon^0) & -\sigma - s \end{pmatrix}, \quad \tilde{H}_2^{00} = \begin{pmatrix} -\sigma + s & p(\Gamma^{00}\tilde{\zeta}, m_\varepsilon^{00}) \\ p(\Gamma^{00}\tilde{\zeta}, m_\varepsilon^{00}) & -\sigma - s \end{pmatrix}.$$

We prove the following lemma.

Lemma 8. *There exist four smooth matrices $C_j = C_j(s,z,\sigma,\zeta,\eta)$, $\tilde{C}_j = \tilde{C}_j(s,z,\sigma,\zeta,\eta)$, $j \in \{1,2\}$, C_j, $\tilde{C}_j \in \mathscr{C}_0^\infty(\mathbf{R}^{10})$ such that for all $a \in \mathscr{C}_0^\infty(\mathbf{R}^{10} \times B)$,*

$$||\mathrm{op}_\varepsilon{}^I(a)[(1 + \sqrt{\varepsilon}\,\mathrm{op}_\varepsilon{}^I(J(C_1 + \varepsilon^{\alpha-1/2}\tilde{C}_1)J))\mathrm{op}_\varepsilon(\tilde{H}_2) - \mathrm{op}_\varepsilon(\tilde{H}_2^0)(1 + \sqrt{\varepsilon}\,\mathrm{op}_\varepsilon{}^I(C_1 + \varepsilon^{\alpha-1/2}\tilde{C}_1))]||_{\mathscr{L}(L^2)} = O(\varepsilon), \quad (52)$$

$$||\mathrm{op}_\varepsilon{}^I(a)[(1 + \sqrt{\varepsilon}\,\mathrm{op}_\varepsilon{}^I(C_2 + \varepsilon^{\alpha-1/2}\tilde{C}_2))\mathrm{op}_\varepsilon(\tilde{H}_2^0) - \mathrm{op}_\varepsilon(\tilde{H}_2^{00})(1 + \sqrt{\varepsilon}\,\mathrm{op}_\varepsilon{}^I(C_2 + \varepsilon^{\alpha-1/2}\tilde{C}_2))]||_{\mathscr{L}(L^2)} = O(\varepsilon). \quad (53)$$

This lemma yields Proposition 4. Actually, if a is compactly supported in all the variables s, z, σ, ζ, η, then, we have in $\mathscr{L}(L^2)$,

$$\mathrm{op}_\varepsilon{}^I(a)\mathrm{op}_\varepsilon(\zeta_j b) = \sqrt{\varepsilon}\,\mathrm{op}_\varepsilon{}^I(a\eta_j b) + O(\sqrt{\varepsilon}).$$

Hence

$$||\mathrm{op}_\varepsilon{}^I(a)\mathrm{op}_\varepsilon(\zeta_j b)||_{\mathscr{L}(L^2)} = O(\sqrt{\varepsilon}), \quad (54)$$

for $j \in \{0,1,2,3\}$ and $b \in \mathscr{C}_0^\infty(\mathbf{R}^{10})$. Therefore, writing

$$\Gamma\tilde{\zeta} = \Gamma_{|S}\tilde{\zeta} + O(|\zeta|^2), \quad m_\varepsilon^{00} = m_{\varepsilon|S} + \varepsilon^\alpha\, O(|\tilde{\zeta}|),$$

and using that $\alpha \geqslant 1/2$, we obtain that, for any a compactly supported in all the variables,

$$||\mathrm{op}_\varepsilon{}^I(a)\mathrm{op}_\varepsilon(p(\Gamma^{00}\tilde{\zeta}, m_\varepsilon^{00})) - \mathrm{op}_\varepsilon{}^I(a)\mathrm{op}_\varepsilon(p(\Gamma_{|S}\tilde{\zeta}, m_{\varepsilon|S}))||_{\mathscr{L}(L^2)} = O(\varepsilon).$$

Therefore, Equations (52) and (53) yield Proposition 4. □

Proof of Lemma 8. Set $C_1^\varepsilon = C_1 + \varepsilon^{\alpha-1/2}\tilde{C}_1$, Equation (52) is equivalent to

$$||\mathrm{op}_\varepsilon{}^I(a)[(1+\sqrt{\varepsilon}\,\mathrm{op}_\varepsilon{}^I(C_1^\varepsilon))\mathrm{op}_\varepsilon(J\tilde{H}_2) - \mathrm{op}_\varepsilon(J\tilde{H}_2^0)(1+\sqrt{\varepsilon}\,\mathrm{op}_\varepsilon{}^I(C_1^\varepsilon))]||_{\mathscr{L}(L^2)} = O(\varepsilon).$$

Note that $J\tilde{H}_2 = \begin{pmatrix} -\sigma + s & p(\Gamma\tilde{\zeta}, m_\varepsilon) \\ -p(\Gamma\tilde{\zeta}, m_\varepsilon) & \sigma + s \end{pmatrix}.$ We use symbolic calculus to expand in power of ε the left hand side. Because of (54), we obtain that C_1^ε must satisfy,

$$\sigma[J, C_1^\varepsilon] = \begin{pmatrix} \mathbf{0} & p((\Gamma^0 - \Gamma)\eta, \varepsilon^{\alpha-1/2}m(k^0 - k)) \\ -p((\Gamma^0 - \Gamma)\eta, \varepsilon^{\alpha-1/2}m(k^0 - k)) & \mathbf{0} \end{pmatrix}.$$

Therefore, we get

$$C_1 = \frac{1}{2\sigma}\begin{pmatrix} \mathbf{0} & p((\Gamma^0-\Gamma)\eta,0) \\ p((\Gamma^0-\Gamma)\eta,0) & \mathbf{0}\end{pmatrix}\chi(\eta),$$
$$\tilde{C}_1 = \frac{1}{2\sigma}\begin{pmatrix} \mathbf{0} & p(0,m(k^0-k)) \\ p(0,m(k^0-k)) & \mathbf{0}\end{pmatrix}\chi(\eta),$$

for some function $\chi\in\mathscr{C}_0^\infty(\mathbf{R}^4_\eta)$ identically equal to 1 on B.

A similar proof determines C_2 and $\tilde{C}_2$, whence Lemma 8. □

5.2. Landau-Zener formula. Because of Proposition 4, once given some ball $B\subset\mathbf{R}^2_\eta$, we are reduced to the study of the traces on $s=0^+$ and $s=0^-$ of the two-scale Wigner measure of a family (u^ε) satisfying

$$\forall a\in\mathscr{C}_0^\infty(\mathbf{R}^8\times B),\quad \mathrm{op}_\varepsilon{}^I(a)\mathrm{op}_\varepsilon\begin{pmatrix} -\sigma+s & p(\Gamma_{|S}\tilde{\zeta},m_{\varepsilon|S}) \\ p(\Gamma_{|S}\tilde{\zeta},m_{\varepsilon|S}) & -\sigma-s\end{pmatrix}u^\varepsilon$$
$$=O(\varepsilon)\quad\text{in } L^2(\mathbf{R}^4).$$

Moreover, by applying a cut-off function, we may suppose that Γ and k are compactly supported and turn $\Gamma_{|S}$, $k_{|S}$ into $\phi(\eta)\Gamma_{|S}$ and $\phi(\eta)k_{|S}$ with ϕ compactly supported and identically equal to 1 on B. This way, our system is micro-localized in the ball B; which is enough to calculate the two-scale Wigner measures in B. We are left with a system of the form

$$\frac{\varepsilon}{i}\partial_s u^\varepsilon = \mathrm{op}_\varepsilon{}^I\begin{pmatrix} s & \phi(\eta)p(\Gamma_{|S}\tilde{\zeta},m_{\varepsilon|S}) \\ \phi(\eta)p(\Gamma_{|S}\tilde{\zeta},m_{\varepsilon|S}) & -s\end{pmatrix}u^\varepsilon+\varepsilon f^\varepsilon,$$

where $(\mathrm{op}_\varepsilon{}^I(a)f^\varepsilon)$ is uniformly bounded in $L^2_{s,z}$ for symbols a compactly supported in $\mathbf{R}^8\times B$. However, f^ε does not contribute to the description of the traces on $s=0^+$, $s=0^-$ of the two-scale Wigner measure of (u^ε). Actually, if $S_\varepsilon(s,s')$ denotes the evolution operator associated with the free system

$$\frac{\varepsilon}{i}\partial_s \underline{u}^\varepsilon = \mathrm{op}_\varepsilon{}^I\begin{pmatrix} s & \phi(\eta)p(\Gamma_{|S}\tilde{\zeta},m_{\varepsilon|S}) \\ \phi(\eta)p(\Gamma_{|S}\tilde{\zeta},m_{\varepsilon|S}) & -s\end{pmatrix}\underline{u}^\varepsilon,\quad \underline{u}^\varepsilon_{|s=0}=u^\varepsilon_{|s=0},$$

then we have

$$u^\varepsilon=\underline{u}^\varepsilon+i\int_0^s S_\varepsilon(0,t)f^\varepsilon(t)\mathrm{d}t.$$

Hence, $u^\varepsilon=\underline{u}^\varepsilon+O(\sqrt{|s|})$ in $L^2(\mathbf{R}^3_z)$. Therefore, the traces of the two-scale Wigner measures of (u^ε) and $(\underline{u}^\varepsilon)$ on $s=0^\pm$ are the same. Let us denote by G the compact operator

$$G=\mathrm{op}_\varepsilon{}^I(\phi(\eta)p(\Gamma_{|S}\eta,\varepsilon^{\alpha-1/2}m\,k_{|S})).$$

The family $(\underline{u}^\varepsilon)$ satisfies

$$\frac{\varepsilon}{i}\partial_s\underline{u}^\varepsilon=\begin{pmatrix} s & \sqrt{\varepsilon}\,G \\ \sqrt{\varepsilon}\,G^* & -s\end{pmatrix}.$$

As a consequence of Proposition 7 in [11], we have the following Lemma.

Lemma 9. *There exist* $\alpha_j^\varepsilon = \alpha_j(z)$, $\omega_j^\varepsilon = \omega_j^\varepsilon(z)$, $j \in \{1,2\}$, *such that, as* ε *goes to* 0, *for any* $\chi \in \mathscr{C}_0^\infty(\mathbf{R})$, $\chi(GG^*)\alpha_1^\varepsilon$, $\chi(G^*G)\alpha_2^\varepsilon$, $\chi(GG^*)\omega_1^\varepsilon$ *and* $\chi(G^*G)\omega_2^\varepsilon$ *are bounded families in* $L^2(\mathbf{R}_z^3, \mathbf{R}^2)$ *and as* ε *goes to* 0

for $s < 0$,

$$\chi(GG^*)\begin{pmatrix} u_1^\varepsilon(s,z) \\ u_2^\varepsilon(s,z) \end{pmatrix} = \chi(GG^*)\mathrm{e}^{i\frac{s^2}{2\varepsilon}}\left|\frac{s}{\sqrt{\varepsilon}}\right|^{i\frac{GG^*}{2}} \alpha_1^\varepsilon + o(1),$$

$$\chi(G^*G)\begin{pmatrix} u_3^\varepsilon(s,z) \\ u_4^\varepsilon(s,z) \end{pmatrix} = \chi(G^*G)\mathrm{e}^{-i\frac{s^2}{2\varepsilon}}\left|\frac{s}{\sqrt{\varepsilon}}\right|^{-i\frac{G^*G}{2}} \alpha_2^\varepsilon + o(1),$$

for $s > 0$,

$$\chi(GG^*)\begin{pmatrix} u_1^\varepsilon(s,z) \\ u_2^\varepsilon(s,z) \end{pmatrix} = \chi(GG^*)\mathrm{e}^{i\frac{s^2}{2\varepsilon}}\left|\frac{s}{\sqrt{\varepsilon}}\right|^{i\frac{GG^*}{2}} \omega_1^\varepsilon + o(1),$$

$$\chi(G^*G)\begin{pmatrix} u_3^\varepsilon(s,z) \\ u_4^\varepsilon(s,z) \end{pmatrix} = \chi(G^*G)\mathrm{e}^{-i\frac{s^2}{2\varepsilon}}\left|\frac{s}{\sqrt{\varepsilon}}\right|^{-i\frac{G^*G}{2}} \omega_2^\varepsilon + o(1),$$

Moreover

$$\begin{pmatrix} \omega_1^\varepsilon \\ \omega_2^\varepsilon \end{pmatrix} = S^\varepsilon \begin{pmatrix} \alpha_1^\varepsilon \\ \alpha_2^\varepsilon \end{pmatrix} \tag{55}$$

with

$$S^\varepsilon = \begin{pmatrix} a(GG^*) & -\overline{b}(GG^*)G \\ b(G^*G)G^* & a(G^*G) \end{pmatrix},$$

with $a(\lambda) = \mathrm{e}^{-\pi\frac{\lambda}{2}}$, $b(\lambda) = \frac{2i\mathrm{e}^{i\frac{\pi}{4}}}{\lambda\sqrt{\pi}} 2^{-i\lambda/2}\mathrm{e}^{-\pi\frac{\lambda}{4}}\Gamma(1 + i\frac{\lambda}{2})\,\mathrm{sh}(\frac{\pi\lambda}{2})$, $a(\lambda)^2 = 1 - \lambda|b(\lambda)|^2$.

These formula allow to calculate two-scale Wigner measures thanks to Lemma 8 in [11] which states that for $\chi \in \mathscr{C}_0^\infty(\mathbf{R})$ and for $\alpha = 1/2$,

$$\begin{aligned} \mathrm{op}_\varepsilon{}^I\left(\chi\left(\left|\phi\left(\frac{\eta}{R}\right)\right|^2 (|\Gamma_{|S}\eta|^2 + m^2|k_{|S}|^2)\right)\right) &= \chi(GG^*) + o(1) \\ &= \chi(G^*G) + o(1), \end{aligned} \tag{56}$$

for $\alpha > 1/2$,

$$\mathrm{op}_\varepsilon{}^I\left(\chi\left(\left|\phi\left(\frac{\eta}{R}\right)\Gamma_{|S}\eta\right|^2\right)\right) = \chi(GG^*) + o(1) = \chi(G^*G) + o(1). \tag{57}$$

In view of

$$\tilde{\Pi}_0^+ = \begin{pmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{1} \end{pmatrix} \text{ on } J^{+,\,\mathrm{out}}, \quad \tilde{\Pi}_0^- = \begin{pmatrix} \mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix} \text{ on } J^{-,\,\mathrm{out}},$$

$$\tilde{\Pi}_0^+ = \begin{pmatrix} \mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix} \text{ on } J^{+,\,\mathrm{in}}, \quad \tilde{\Pi}_0^- = \begin{pmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{1} \end{pmatrix} \text{ on } J^{-,\,\mathrm{in}},$$

we obtain that $\tilde{\nu}^+ \mathbf{1}_{s<0}$ (resp. $\tilde{\nu}^- \mathbf{1}_{s<0}$) is the two-scale Wigner measure of the family $e^{i\frac{s^2}{2\varepsilon}} \left| \frac{s}{\sqrt{\varepsilon}} \right|^{i\frac{GG^*}{2}} (\alpha_1^\varepsilon, 0)$ (resp. $e^{-i\frac{s^2}{2\varepsilon}} \left| \frac{s}{\sqrt{\varepsilon}} \right|^{-i\frac{G^*G}{2}} (0, \alpha_2^\varepsilon)$). Similarly, $\tilde{\nu}^+ \mathbf{1}_{s>0}$ (resp. $\tilde{\nu}^- \mathbf{1}_{s>0}$) is the two-scale Wigner measure of $e^{-i\frac{s^2}{2\varepsilon}} \left| \frac{s}{\sqrt{\varepsilon}} \right|^{-i\frac{G^*G}{2}} (0, \omega_2^\varepsilon)$ (resp. $e^{i\frac{s^2}{2\varepsilon}} \left| \frac{s}{\sqrt{\varepsilon}} \right|^{i\frac{GG^*}{2}} (\omega_1^\varepsilon, 0)$). We use Lemma 9 in [11]: for every $\chi \in C_0^\infty(\mathbf{R}^{10})$), we have in $\mathscr{L}(L^2)$,

$$\left| \frac{s}{\sqrt{\varepsilon}} \right|^{-i\frac{GG^*}{2}} \mathrm{op}_\varepsilon{}^I(\chi) \left| \frac{s}{\sqrt{\varepsilon}} \right|^{i\frac{GG^*}{2}} = \left| \frac{s}{\sqrt{\varepsilon}} \right|^{i\frac{G^*G}{2}} \mathrm{op}_\varepsilon{}^I(\chi) \left| \frac{s}{\sqrt{\varepsilon}} \right|^{-i\frac{G^*G}{2}} + o(1) = \mathrm{op}_\varepsilon{}^I(\chi) + o(1).$$

Therefore, $\tilde{\nu}^{+,\mathrm{out}}$ (resp. $\tilde{\nu}^{-,\mathrm{out}}$) is the two-scale Wigner measure of the family $(\alpha_1^\varepsilon, 0)$ (resp. $(0, \alpha_2^\varepsilon)$). Similarly, $\tilde{\nu}^{+,\mathrm{in}}$ (resp. $\tilde{\nu}^{-,\mathrm{in}}$) is the two-scale Wigner measure of the family $(0, \omega_2^\varepsilon)$ (resp. $(\omega_1^\varepsilon, 0)$). Observe that Equation (55) yields,

$$\begin{pmatrix} \omega_1^\varepsilon(z) \\ \begin{pmatrix} 0 \\ 0 \end{pmatrix} \end{pmatrix} = a(GG^*) \begin{pmatrix} \alpha_1^\varepsilon(z) \\ \begin{pmatrix} 0 \\ 0 \end{pmatrix} \end{pmatrix} - \overline{b}(GG^*) \begin{pmatrix} \mathbf{0} & G \\ G^* & \mathbf{0} \end{pmatrix} \begin{pmatrix} \begin{pmatrix} 0 \\ 0 \end{pmatrix} \\ \alpha_2^\varepsilon(z) \end{pmatrix},$$

$$\begin{pmatrix} \begin{pmatrix} 0 \\ 0 \end{pmatrix} \\ \omega_2^\varepsilon(z) \end{pmatrix} = a(G^*G) \begin{pmatrix} \begin{pmatrix} 0 \\ 0 \end{pmatrix} \\ \alpha_2^\varepsilon(z) \end{pmatrix} + b(G^*G) \begin{pmatrix} \mathbf{0} & G \\ G^* & \mathbf{0} \end{pmatrix} \begin{pmatrix} \alpha_1^\varepsilon(z) \\ 0 \end{pmatrix}.$$

These relations yield (39) via (56) and (57), which completes the proof of Theorem 1. □

6. Appendix

6.1. Restitution of the energy by the crossing. We denote by Ω the set

$$\Omega = \{E \cdot B \neq 0 \quad \text{or} \quad (E \cdot B = 0 \quad \text{and} \quad |E| > |B|)\}.$$

Proposition 5. *If* (ψ^ε) *is a family of solutions to* (6) *and* μ *a Wigner measure of* (ψ^ε), *then we have*

$$\mu(S \cap \Omega) = 0.$$

Proof of Proposition 5. We set

$$Q_0 = \begin{pmatrix} \mathbf{0} & p(\xi - A) \\ p(\xi - A) & \mathbf{0} \end{pmatrix}.$$

We have

$$P_\varepsilon(x, \xi) = Q_0 + \varepsilon^\alpha m \gamma^0 + V(t, x).$$

Observe that $(\tau + V - \varepsilon^\alpha m - Q_0)(\tau + V + \varepsilon^\alpha m + Q_0) = (\tau + V)^2 - |\xi - A|^2 - \varepsilon^{2\alpha} m^2$. Therefore, if

$$Q := \mathrm{op}_\varepsilon(\tau + V - \varepsilon^\alpha m - Q_0) \mathrm{op}_\varepsilon(\tau + V + \varepsilon^\alpha m + Q_0),$$

we have

$$Q = \mathrm{op}_\varepsilon((\tau + V)^2 - |\xi - A|^2 - \varepsilon^{2\alpha} m^2) + \frac{\varepsilon}{2i} \mathrm{op}_\varepsilon(\{\tau + V - Q_0,\ \tau + V + Q_0\}) + o(\varepsilon) \tag{58}$$

in $\mathscr{L}(L^2)$. Let us denote by R the matrix

$$R := \frac{1}{2}\{\tau + V - Q_0,\ \tau + V + Q_0\} = \begin{pmatrix} ip(B) & p(E) \\ p(E) & ip(B) \end{pmatrix}.$$

Since $i\partial_t \psi^\varepsilon = \mathrm{op}_\varepsilon(P_\varepsilon)\psi^\varepsilon$, for any matrix-valued symbol $a \in \mathscr{C}_0^\infty(\mathbf{R}^8_{t,x,\tau,\xi})$ we have

$$\frac{1}{\varepsilon}(\mathrm{op}_\varepsilon(a)\, Q\psi^\varepsilon|\psi^\varepsilon) - \frac{1}{\varepsilon}(\mathrm{op}_\varepsilon(a)\psi^\varepsilon|Q\psi_\varepsilon) = 0.$$

Using (58), we obtain

$$\begin{aligned}&\frac{1}{\varepsilon}[\mathrm{op}_\varepsilon(a),\ \mathrm{op}_\varepsilon((\tau+V)^2 - |\xi - A|^2)]\\ &\qquad + \frac{1}{i}((\mathrm{op}_\varepsilon(a)\mathrm{op}_\varepsilon(R) + \mathrm{op}_\varepsilon(R^*)\mathrm{op}_\varepsilon(a))\psi^\varepsilon|\psi^\varepsilon) = o(1),\end{aligned}$$

whence $\{\mu, (\tau+V)^2 - |\xi - A|^2\} = \mu R^* + R\mu$. Consider $\Phi \in \mathscr{C}_0^\infty(\mathbf{R}^3)$ such that $0 \leqslant \Phi \leqslant 1$ with $\Phi(0) = 1$. Using test functions of the form $\Phi\left(\frac{\xi - A(t,x)}{\delta}\right)$ for some positive δ, then letting δ which go to 0, we get

$$R\mathbf{1}_{\xi = A(t,x)}\mu + \mathbf{1}_{\xi = A(t,x)}\mu R^* = 0.$$

Consider $\chi = \chi(t,x,\tau,\xi)$ a scalar test function and $M = \langle \chi,\ \mu\mathbf{1}_{\{\{\xi = A\}\cap\Omega\}}\rangle$. The matrix M is a positive hermitian matrix satisfying $MR^* + RM = 0$. Let us prove that $M = 0$. Observe that

$$R^2 = \begin{pmatrix} |E|^2 - |B|^2 & 2iB\cdot E \\ 2iB\cdot E & |E|^2 - |B|^2 \end{pmatrix}.$$

Since we have $RMR^* = -R^2M$, the matrix $-R^2M$ is hermitian and positive. Thus, in the case $E\cdot B = 0$, it is obvious that the fact that $|E| > |B|$ yields $M = 0$. We focus now on the case $E\cdot B \neq 0$. Matrix M can be decomposed in 2×2 blocks

$$M = \begin{pmatrix} A & C \\ C^* & D \end{pmatrix},$$

where A and D are 2×2 hermitian positive matrices. Moreover, if $-R^2M$ is hermitian positive, we obtain that we must have $(E\cdot B)(A + D) = 0$. Therefore, necessarily, $A = D = 0$, which yields $C = 0$ and $M = 0$. This comes from the fact that the non-diagonal coefficient $\mu_{i,j}$ of measure-valued matrix μ is a measure absolutely continuous with respect to the diagonal measure-valued coefficients $\mu_{i,i}$ and $\mu_{j,j}$. □

6.2. Proof of Lemma 1. Let us study the limits of the fibers of $\overline{N}_{\Sigma^\pm}(\dot{J}^{\pm,\mathrm{in}})$ and $\overline{N}_{\Sigma^\pm}(\dot{J}^{\pm,\mathrm{out}})$ above some point ρ which tends to S. Consider H and H' the limits of the two Hamiltonian vector fields along J and J' above S, we have

$$\begin{aligned}\lim_{s\to 0^-} H_{\lambda^+}(\rho_s^+) &= \lim_{s\to 0^+} H_{\lambda^-}(\rho_s^-) = H_{\tau+V}(\rho_0) - \omega\cdot H_{\xi - A}(\rho_0) := H,\\ \lim_{s\to 0^+} H_{\lambda^+}(\rho_s^+) &= \lim_{s\to 0^-} H_{\lambda^-}(\rho_s^-) = H_{\tau+V}(\rho_0) + \omega'\cdot H_{\xi - A}(\rho_0) := H'.\end{aligned}$$

Let $H^\perp$ and $(H')^\perp$ be the orthogonal of H and H' respectively for the symplectic form on $T(T^*\mathbf{R}^4)$. The measures $\nu_S^{+,\mathrm{in}}$ and $\nu_S^{-,\mathrm{out}}$ are measures on the compactification of $H^{\perp}/TJ$, and similarly, the measures $\nu_S^{-,\mathrm{in}}$ and $\nu_S^{+,\mathrm{out}}$ live on the compactification of $(H')^{\perp}/TJ'$. Note that $TJ = TS \oplus \mathbf{R}H$ and $TJ' = TS \oplus \mathbf{R}H'$, therefore if we set

$$F = TJ + TJ' = TS \oplus \mathbf{R}H \oplus \mathbf{R}H',$$

the planes $H^{\perp}/TJ$ and $(H')^{\perp}/TJ'$ can be identified to $T(T^*\mathbf{R}^4)/F$.

Consider $\delta\rho \in T(T^*\mathbf{R}^d)_{|\rho}$, $\rho \in S$. We decompose $\delta\rho$ as

$$\delta\rho = lH + l'H' + \delta s + \delta\tilde{\rho},$$

with l, $l' \in \mathbf{R}$ and $\delta s \in TS_{|\rho}$. The class of $\delta\rho$ modulo F is characterized by the class of $\delta\tilde{\rho}$ modulo J, i.e. by $\mathrm{d}(\xi - A)\,\delta\tilde{\rho} - \mathrm{d}(\tau + V)\,\delta\tilde{\rho}u$ (because $\xi - A = (\tau + V)u$ is an equation of J). Observe that

$$\mathrm{d}(\xi - A)H - \mathrm{d}(\tau + V)Hu = 0, \quad \mathrm{d}(\xi - A)\,\delta s - \mathrm{d}(\tau + V)\,\delta su = 0,$$

$$\mathrm{d}(\xi - A)H' - \mathrm{d}(\tau + V)H'u = -2r\frac{E}{|E|^2}.$$

Therefore, the class of $\delta\rho$ modulo F is utterly determined by the knowledge of

$$\begin{aligned}\eta &= (\mathrm{d}(\xi - A)\,\delta\rho - \mathrm{d}(\tau + V)\,\delta\rho u) \wedge E \\ &= \mathrm{d}(\xi - A)\,\delta\rho \wedge E + \mathrm{d}(\tau + V)\,\delta\rho B,\end{aligned}$$

where we have used (11). This concludes the proof of Lemma 1. □

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