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Partial Actions of Groups on Cell Complexes

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Abstract. In this paper, we study partial group actions on 2-complexes. Our results include a characterization, in terms of generating sets, of when a partial group action on a connected 2-complex has a connected globalization. Using this result, we give a short combinatorial proof that a group acting without fixed points on a connected 2-complex, with finite quotient, is finitely generated. This result is then generalized to characterize finitely generated groups as precisely those groups having a partial action, without fixed points, on a finite tree, with a connected globalization. Finally, using Bass-Serre theory, we determine when a partial group action on a graph has a globalization which is a tree.

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1. Introduction

This paper studies partial actions of groups on 2-complexes. Such partial actions basically arise via the restriction of a full action of a group G on a complex K to a subcomplex L of K. The notions and results of this paper apply equally well to simplicial complexes; what is important is that one is dealing with a combinatorial object rather than the geometric realization of such. Our main goal is to determine exactly what it means for a connected complex L to be a G-covering for an action of G on some (unspecified) connected complex K.

Partial group actions have appeared in the literature in various guises, but were first explicitly formalized in Exel's [3] and were then further developed by Lawson [5], who went on to provide many examples of partial actions; see [5, 4]. In [5], it is proved that every partial action of a group on a set has a globalization meaning, roughly, that every partial action can be obtained by restricting a full action. However, when a group acts partially on a set with extra structure the globalization does not necessarily inherit this structure; thus in [5], the globalization of a partial action of a group on a semilattice results in a partially ordered set which need not be a semilattice; while in [5] similar issues arise in the study of globalizations of partial actions on categories or topological spaces. Likewise, in this paper, we will see that the globalization of a partial action on a connected 2-complex may result

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in a complex which is not connected. The study of when connected globalizations (or globalizations with even more stringent conditions), arise leads to new insight into several classical results in the theory of group actions.

Like the classical theory of a group acting on a 2-complex (or a topological space), see for instance [8, 9, 11], we try to recover information about the group from the partial action. The relation of this theory to the theory of groups acting on a complex is analogous to the relation between graph immersions and coverings [12, 10]. An immersion of graphs is a sort of partial covering: what one obtains from restricting a covering map to a subgraph. Immersions retain many of the properties of a covering, but lose such properties as path lifting. The tradeoff is that one can use finite graphs to represent finitely generated subgroups of a free group (only finite index subgroups can be represented by finite sheeted covers). Now a group acting without fixed points on a finite complex must be a finite group. But the Bass-Serre Theorem [11] states that a group can act without fixed points on a tree if and only if the group is free; hence only the trivial group can act without fixed points on a finite tree. The situation for partial actions is quite different; any group can act partially without fixed points on a tree. However, after excluding some trivial sorts of partial actions, finitely generated groups can be characterized as exactly those groups acting partially without fixed points on a finite tree. Thus, as in the case of graph immersions, if we are willing to generalize the classical notions, we can "represent" finitely generated groups by finite complexes. Similarly, Bass-Serre theory states that any amalgamation or HNN-extension acts on a tree, but we can have such act partially on a segment.

Working with partial actions also gives a better understanding of some well-known results about groups acting on a complex. For instance, see [8, 9], it is well-known, that if a group G acts on a connected complex K and L is a subcomplex such that $GL = K$, then G is generated by the set of elements such that $gL \cap L \neq \emptyset$. This will turn out to have a natural interpretation in terms of the induced partial action of G on L and is, in fact, the exact condition needed for the partial action to have a connected globalization. Also, it is well-known that a group acting without fixed points on a connected topological space with compact quotient is finitely generated; see for instance [2]. We give a completely combinatorial proof, in the case of complexes, which is more enlightening in that it shows the result to be a consequence of the fact that there are only finitely many partial automorphisms of a finite complex. The Bass-Serre theorem can be interpreted as characterizing partial actions of groups on connected graphs with globalizations to actions on trees.

This paper is organized as follows: we begin with a combinatorial definition of a 2-complex; then the definition of a partial action is given followed by the definition of a globalization. The heart of the work begins with an investigation of when a partial action on a connected complex has a connected globalization; this will turn out to be equivalent to a natural hypothesis to place on a partial action. We then investigate what happens when one turns to partial actions without fixed points. We then dedicate a section to investigating the relationship between Bass-Serre theory and partial actions, characterizing when a partial action can be globalized to an action on a tree. Finally, we end with several questions and point out some interesting lines of further investigation.

We are indebted to M. Lawson for bringing to our attention the work of Macbeath [9]. He and Kellendonk [5] also examine [9] in relation to partial group actions, though from a topological point of view, with different goals than ours.

2. 2-Complexes

In this paper, complexes will be viewed as combinatorial objects as per [8]. First we define a *graph* or 1-*complex*. A graph K consists of the following data: a set $V(K)$ of *vertices*; a set $E(K)$ of *edges*; maps $\alpha, \omega : E(K) \to V(K)$ selecting the *initial*, respectively, *terminal* vertex of an edge; and an involution $(-)^{-1} : E(K) \to E(K)$ such that, for $e \in E(K)$, $e^{-1} \neq e$ and $\alpha(e) = \omega(e^{-1})$. The edge e^{-1} is called the *reverse* of e. An *orientation* of a graph consists of choosing a subset $E^{+}(K) \subseteq E(K)$ consisting of exactly one member of each pair $\{e, e^{-1}\}$. From now on, we will assume all graphs to be oriented.

A *path* $p = e_1 \cdots e_n$ (perhaps empty) is, as usual, a sequence of edges such that $\omega(e_j) = \alpha(e_{j+1})$; we give $\alpha(p)$ and $\omega(p)$ the obvious interpretations. A path is called *reduced* if it contains no subpath of the form ee^{-1}. A *loop* is a path p such that $\alpha(p) = \omega(p)$. If p and q are loops, one says that q is a *cyclic conjugate* of p if there are paths u, v such that $uv = p$, $vu = q$; that is, q is obtained from p by starting at a different point of the loop. The relation of being cyclic conjugates is clearly an equivalence relation. We call the equivalence class of a loop a *cycle*.

A 2-*complex* K consists of a graph K^1, called the 1-*skeleton* of K, a set $C(K)$ (possibly empty) of 2-*cells*, and an assignment to each 2-cell of a cycle in K^1 called its *boundary cycle*; the boundary cycle of a 2-cell is permitted to be empty and need not consist of reduced loops. The notation $V(K)$ and $E(K)$ will be used for the vertices and edges of K. Vertices, edges, will also be called 0-*cells*, 1-*cells*, respectively. We may sometimes use notation like $c \in K$ if the dimension of the n-cell c is clear from the context. A 2-complex is called *finite* if it has only finitely many n-cells for all n. One says that a 2-complex is *connected* if any two vertices can be joined by a path. We consider a graph to be a 2-complex without 2-cells. A connected graph whose only reduced loops are trivial is called a *tree*.

A *morphism* of 2-complexes $\varphi : K \to L$ consists of three functions, all denoted φ, sending n-cells to n-cells and respecting all the structure (that is, preserving orientation, incidence, and boundary cycles). The collection of 2-complexes forms a category with the obvious definitions of the identity morphism and composition. An automorphism will have its usual meaning. A *partial automorphism* of a 2-complex K is an isomorphism between sub-complexes of K. Note that we consider the "identity map" of the empty complex as a partial automorphism called the *empty partial automorphism*. We also make the observation that, since morphisms are assumed to preserve orientation, no edge is sent to its reverse by a partial automorphism.

3. Partial Actions

Fix a 2-complex K. The set of all partial automorphisms of K forms a monoid $I(K)$ under composition of relations; that is, if φ and ρ are partial automorphisms, then the various component functions of φ and ρ are relations and can be composed as such, the result being a partial automorphism with $\mathrm{dom}(\varphi\rho) = \rho^{-1}(\mathrm{dom}(\varphi))$ and $\mathrm{ran}(\varphi\rho) = \varphi(\mathrm{ran}(\rho))$. Note that the empty partial automorphism is the zero of $I(K)$. The monoid $I(K)$ has the following additional property: for each $\varphi \in I(K)$, there exists a unique element φ^{-1} of $I(K)$ such that $\varphi\varphi^{-1}\varphi = \varphi$ and $\varphi^{-1}\varphi\varphi^{-1} = \varphi^{-1}$. A monoid with this property is called an *inverse monoid*; see [8] for an introduction to this interesting theory (which also comes up in the study of graph immersions [10]). There is a natural partial order on $I(K)$ given by $\varphi \leqslant \rho$ if φ is a restriction of ρ. This order is easily seen to be compatible with multiplication and taking inverses. More generally, if I is any inverse monoid, there is a natural partial order, compatible with multiplication and taking inverses, defined by $m \leqslant n$ if $m = en$ with e an idempotent; this order on $I(K)$ is exactly the restriction order.

We are now interested in motivating the notion of a partial action of a group on a 2-complex. The idea is to imagine that one has a group G acting on a 2-complex K by automorphisms and that L is a subcomplex. Then to each element of G, we can associate a partial automorphism of L by restriction. The map $f : G \to I(L)$ is then easily verified to have the following properties:

(1) $f(1) = 1$;
(2) $f(g^{-1}) = f(g)^{-1}$;
(3) $f(g_1)f(g_2) \leqslant f(g_1g_2)$.

This last property follows because, in general, it could happen that, for some vertex $v \in L$, one has $g_2v \notin L$, but $g_1g_2v \in L$. On the other hand, if $g_2v \in L$ and $g_1(g_2v) \in L$, then $(g_1g_2)v = g_1(g_2v)$.

In general, a map $f : G \to I$ from a group to an inverse monoid satisfying the above properties is called a *dual prehomomorphism*. Following Lawson [7], we define a *partial action* of a group on a 2-complex L to be a dual prehomomorphism $f : G \to I(L)$. One could explicitly write down axioms to describe partial actions as is done in [5] for the case of a partial action of a group on a set. We shall see shortly, as was shown to be the case by Kellendonk and Lawson [5] for partial actions of groups on sets (and certain other structures), that every partial action of a group on a 2-complex comes from restricting a total action and so this scenario should give the proper intuition. We will sometimes use the adjective *full* when talking about group actions in the usual sense.

Now suppose G is a group and H is a subgroup of G acting (fully) on a complex K. Then one can *extend* this partial action to G by having each element of $G\backslash H$ act via the empty partial automorphism (while letting H act as before). In a sense, this partial action does not detect the elements of $G\backslash H$ leading us to the following notion. Let $f : G \to I(L)$ give a partial action of G on L and let

$$D(G) = f^{-1}(I(L)\backslash 0) = \{g \in G \mid \mathrm{dom}(f(g)) \neq \emptyset\};$$

then we say that the partial action is *true* or G *truly* acts partially on L if G is generated by $D(G)$. Observe that the partial action of G on L is obtained by extension (in the above sense) of the induced partial action of $\langle D(G)\rangle$ on L. In this paper, we will be concerned primarily with true partial actions. To give the reader more of a feeling for this notion, let G act on a 2-complex K and let L be a subcomplex; then for the induced partial action of G on L,

$$D(G) = \{g \in G | gL \cap L \neq \emptyset\}.$$

In the theory of groups acting on complexes, this set plays an important role. We also observe that $D(G)$ is a *symmetric set* meaning that $g \in D(G)$ if and only if $g^{-1} \in D(G)$. Thus if a partial action is true, $D(G)$ is, in fact, a symmetric generating set for G. Partial actions and globalizations relative to symmetric generating sets were studied in [5] and the interested reader is referred there.

We end this section by remarking that Exel [3] showed that, to each group G, one can associate an inverse monoid G^{Pr} with the property that partial actions of G correspond to full actions (in the sense of inverse semigroup theory) of G^{Pr}; this inverse semigroup was identified as the prefix expansion of Birget-Rhodes in [5] and was subsequently generalized in [7] to the theory of partial actions of inverse semigroups.

4. Globalization

If G acts partially on a 2-complex L, then a *globalization* (of the partial action) consists of a full action of G on a complex K and an embedding of L in K such that:

(1) $GL = K$;
(2) The partial action of G on L is the restriction of the action of G on K.

We denote this globalization by (G, L, K). One, of course, has an analogous definition for partial actions of a group on other structures. The notion of a globalization was introduced by Kellendonk and Lawson [5].

A globalization (G, L, K) is called *universal* if, given any other globalization (G, L, K'), there is a morphism $\varphi : K \to K'$ fixing L and preserving the action. In [5], Kellendonk and Lawson prove that if G acts partially on a set X, then there is a universal globalization. To construct this globalization, consider the equivalence relation on $G \times X$ given by $(g, x) \sim (h, y)$ if their exists $f \in G$ such that $gf^{-1} = h$ and $fx = y$ (this is like a tensor product of G and X). One can show that G acts on the quotient by $h[(g, x)] = [(hg, x)]$ where we use square brackets to denote equivalence classes. We call the quotient X^{G1}. It is shown in [5] that $x \mapsto [(1, x)]$ embeds X in X^{G1} and that, in fact, (G, X, X^{G1}) is the universal globalization of X.

Now if G acts partially on a 2-complex L, then G acts partially on $V(L)$, $E(L)$, and $C(L)$. It is not hard to see that if e_1, $e_2 \in E(L)$ and $(g, e_1) \sim (h, e_2)$, then $(g, \alpha(e_1)) \sim (h, \alpha(e_1))$. Indeed, if $gf^{-1} = h$ and $fe_1 = e_2$, then $f\alpha(e_1) = \alpha(e_2)$. A sequence of similar observations makes it clear that the collection $\{V(L)^{\mathrm{G1}}$, $E(L)^{\mathrm{G1}}, C(L)^{\mathrm{G1}}\}$ has a natural 2-complex structure and the resulting complex, call it L^{G1}, gives rise to the universal globalization of the partial action of G on L; note that in proving that $[(g, e)]^{-1} = [(g, e^{-1})] \neq [(g, e)]$, one needs to use the fact that partial automorphisms never take an edge to its reverse. We thus have the following theorem:

Theorem 4.1. *Every partial action of a group on a* 2*-complex has a universal globalization.*

The next section is concerned with when partial actions on connected complexes have connected globalizations. This turns out to be very related to some classical results in the theory of groups acting on complexes and topological spaces.

As an example, consider the extension to a group G of the trivial action of the identity on a single vertex. The universal globalization of this action is easily seen to be isomorphic to the left regular representation of G (where the set G is viewed as a collection of isolated vertices). In particular, the globalization is not connected, although the original partial action was on a connected graph.

5. Connected Globalizations

If G is a group acting partially on a connected 2-complex L, then a globalization (G, L, K) is called *connected* if K is connected. Our first main result is the following theorem, characterizing when connected globalizations exist.

Theorem 5.1. *Let G act partially on a non-empty connected 2-complex L. Then there is a connected globalization if and only if the partial action is true. In this case, all globalizations are connected.*

Proof. Suppose first that the partial action is true. Since the partial action has a universal globalization by Theorem 4.1, we may prove sufficiency by proving that, in fact, all globalizations are connected. So let (G, L, K) be a globalization; it suffices to show that if $v_0, v \in L$ and $g \in G$, then there is a path in K from v_0 to gv. By assumption, $g = g_1 \cdots g_n$ with $g_j \in D(G)$ all j. We induct on n (over all vertices v). If $n = 0$, the result follows since $v_0, v \in L$ and L is connected. Assume now that $v' \in L$ and $g_1, \ldots, g_{n-1} \in D(G)$ implies that there is a path in K from v_0 to $g_1 \cdots g_{n-1} v'$. Let $w \in L$ be such that $g_n w \in L$; such exists since $g_n \in D(G)$. Choose a path p in L from w to v. Now $gw = g_1 \cdots g_{n-1} v'$ where $v' = g_n w \in L$ so, by induction, there is a path q from v_0 to gw. Then the path q followed by the path gp goes from v_0 to gv as desired.

Suppose now that (G, L, K) is a connected globalization; we show that the partial action of G on L is true. Let $H = \langle D(G) \rangle$ be the subgroup generated by $D(G)$ and fix a vertex v_0 of L. Let $g \in G$ and consider a path $e_1 \cdots e_n$ in K from v_0 to gv_0. Suppose e_j goes from $g_{j-1} v_{j-1}$ to $g_j v_j$ with $g_0 = 1$, $g_n = g$, $v_n = v_0$, and $g_j \in G$, $v_j \in L$, $j = 1, \ldots, n$. Let $h_j = g_{j-1}^{-1} g_j$, $j = 1, \ldots, n$; note that $h_1 \cdots h_n = g_n$. We show that, for all j, $h_j \in H$; the result will then follow. Let $e_j = ke$ with $k \in G$ and $e \in L$. Then $g_{j-1} v_{j-1} = k\alpha(e)$ and $g_j v_j = k\omega(e)$. Thus $g_{j-1}^{-1} k, k^{-1} g_j \in D(G)$, so $h_j = g_{j-1}^{-1} g_j = g_{j-1}^{-1} k k^{-1} g_j \in H$. □

6. Partial Actions Without Fixed Points

If G acts partially on a 2-complex L, then there exists a natural quotient complex L/G defined by identifying two n-cells if they are in the same orbit under the G-action; once again, the fact that no edge of L/G is its own reverse follows from the fact that partial automorphisms cannot send an edge to its reverse. It is also easy to see that if (G, L, K) is a globalization of the action, then $L/G = K/G$. Finally, we note that if L is connected, respectively, finite then so is the quotient L/G.

A partial action of a group G on a 2-complex L is said to be *without fixed points* if, for any n-cell c, $gc = c$ implies that $g = 1$. Partial actions without fixed points were first considered in [4]. If c is an n-cell, we let the stabilizer of c, denoted G_c, be the set $\{g \in G | gc = c\}$. A partial action is then without fixed points if and only if the stabilizer of every n-cell is trivial. Note that if (G, L, K) is a globalization, then the stabilizer of an n-cell $c \in L$ is the same under both the partial action on L and the action of K. The following proposition is straightforward.

Proposition 6.1. *Let G act partially on a 2-complex L and (G, L, K) be a globalization. Suppose $g \in G$ and $c \in L$ in an n-cell; then $G_{gc} = gG_cg^{-1}$. In particular, the partial action of G on L is without fixed points if and only if the action of G on K is without fixed points.*

If K is a graph and $v \in K$ is a vertex, the *star of* v is the set $\alpha^{-1}(v)$. A morphism of graphs is called *star injective* or, alternatively, an *immersion* [12, 10] if it is injective when restricted to each star.

Proposition 6.2. *Let G act partially without fixed points on a 2-complex L. Then the natural quotient morphism $\varphi : L^1 \to (L/G)^{-1}$ is an immersion.*

Proof. Suppose e_1, e_2 are edges of L with common initial vertex v and that $ge_2 = e_1$ with $g \in G$. Then

$$gv = g\alpha(e_2) = \alpha(ge_2) = \alpha(e_1) = v$$

so $g = 1$. □

The above argument shows, more generally, that if v is a vertex and $G_v = 1$, then the projection from L to L/G is injective when restricted to the star of v.

Our next goal is to give a proof, using partial actions, that if a group G acts without fixed points on a connected 2-complex with compact quotient, then G is finitely generated. Our proof will be entirely combinatorial: we will use no compactness arguments. Since a 2-complex has compact geometric realization if and only if it is finite, we are really talking about the quotient being a finite 2-complex. We remind the reader that this result and its proof work equally well for simplicial complexes.

The following result, although in some sense a trivial observation, is at the heart of what follows and so we designate it a theorem.

Theorem 6.3. *Suppose G has a true partial action without fixed points on a non-empty finite 2-complex L; then G is finitely generated.*

Proof. First we observe that since the action is without fixed points, if g_1, $g_2 \in D(G)$ act the same on some common n-cell of their respective domains, then these elements are the same. Suppose that the partial action is given by a dual prehomomorphism $f : G \to I(L)$. Then the above observation shows that $f|_{D(G)}$ is injective. But L finite implies that $I(L)$ is finite and so we can conclude that $D(G)$ is finite. Thus $G = \langle D(G) \rangle$ is finitely generated. □

We proceed with two lemmas.

Lemma 6.4. *Suppose that K is a 2-complex and L is a finite subcomplex; then L is contained in a finite connected subcomplex L' of K.*

Proof. For each connected component L_i of L, choose a vertex v_i as a base point. Then since K is connected, each of these base points can be connected to v_1 by a path. Choose one such path for each base point. The union of L with the edges and vertices used in these (finitely many) paths is then a finite subcomplex L' as desired. □

Lemma 6.5. *Suppose G has a true partial action on a non-empty 2-complex L and L' is a connected subcomplex with $GL' = L$. Then partial action of G on L' obtained by restriction is also true.*

Proof. It is straightforward to see that the restriction of the partial action to L' is a partial action. Now Theorem 5.1 tells us that there is a connected globalization (G, L, K). But then (G, L', K) is easily seen to be a connected globalization and so another application of Theorem 5.1 completes the proof. □

We now prove a more general version of the desired result.

Theorem 6.6. *Suppose G has a true partial action without fixed points on a non-empty connected 2-complex L with the quotient L/G finite; then G is finitely generated.*

Proof. Since L/G is finite, there exists a finite subcomplex L' of L containing at least one representative of each n-cell of L/G (one can do this by choosing a lift of each n-cell and then including, in the case of a 1 or 2-cell, its boundary). By Lemma 6.4, there exists a finite connected subcomplex L'' containing L'. Clearly $GL'' = L$ whence, by Lemma 6.5, the partial action of G on L'' is true. An application of Theorem 6.3 then shows that G is finitely generated. □

Since a full action of a group on a connected 2-complex is a true partial action, we obtain the desired result as a corollary.

Corollary 6.7. *Let G be a group acting without fixed points on a non-empty connected 2-complex with finite quotient; then G is finitely generated.*

The above results show that there are certain similarities and relationships between partial actions and full actions. We now wish to highlight some of the differences. First of all, we recall that a group acting (fully) without fixed points on a finite complex must be a finite group. The situation can be quite different for partial actions. Recall the theorem of Bass-Serre [11] that a group G acts without fixed points on a tree if and only if G is free. In particular, G can only so act on a finite tree if G is trivial. This should be contrasted with the following result.

Theorem 6.8. *Every group G has a true partial action without fixed points on a tree. The tree can be taken to be finite if and only if G is finitely generated.*

Proof. The trivial group acts without fixed points on a single vertex tree. Suppose G is non-trivial and let $X \subseteq G\setminus\{1\}$ be a collection of generators for G. Then the star of the identity in the Cayley graph of G with respect X is a tree T and the restriction of the natural left action on its Cayley graph to T is a true partial action without fixed points. In particular, if G is finitely generated, this tree can be take to be finite. Conversely, if G has a true partial action without fixed points on a finite tree, then G is finitely generated by Theorem 6.3. □

Of course, it is necessary to include true in the hypothesis of the above theorem since any group acts partially without fixed points on a one vertex tree by letting the identity fix the vertex and all other elements have empty domain.

7. Bass-Serre Theory and Tree Globalizations

In this section, we use Bass-Serre theory to characterize when a group acting partially on a connected graph has a globalization to an action on a tree (called a *tree globalization*). Recall [11] that a *graph of groups* (G, Y) consists of a connected non-empty (oriented) graph Y, a group G_v for each vertex $v \in V(Y)$, a group G_e for each edge $e \in E(Y)$ (where we require $G_e = G_{e^{-1}}$), and, for each edge e, a monomorphism $G_e \to G_{\omega(e)}$ denoted by $a \mapsto a^e$. Let T be a maximal tree of Y. Then the *fundamental group* of (G, Y) at T, written $\pi_1(G, Y, T)$, is the quotient of the free product of the G_v and a free group on $E^+(Y)$ induced by the relations: $ea^e e^{-1} = a^{e^{-1}}$ and $e \in T \Longrightarrow e = 1$. It is shown in [11] that this group is independent of the choice of T.

If G is a group acting partially on a non-empty connected graph X, then, following Serre [11], we define a graph of groups as follows. Let $Y = X/G$ and choose a maximal tree T of Y. Then [11, 3.1 Proposition 14], adapted to partial actions, shows that there is a lift $j: T \to X$. We proceed, following [11, Section 5.4], to extend j to a section $j: E(Y) \to E(X)$. It suffices to define j on positively oriented edges. Since $j(T)$ contains a vertex from every orbit, given an edge $e \in E^+(Y)$, we can choose $j(e)$ with $\alpha(j(e)) \in j(T)$. Also, since $\omega(j(e))$ and $j(\omega(e))$ project to $\omega(e)$, we can find γ_e with $\omega(j(e)) = \gamma_e j(\omega(e))$. We let $\gamma_{e^{-1}} = \gamma_e^{-1}$. Define, for $e \in E$,

$$\sigma(e) = \begin{cases} 0 & e \in E^+(Y) \\ 1 & e \notin E^+(Y). \end{cases}$$

Then, for each edge $e \in E(Y)$, we have:

$$\begin{aligned} \alpha(j(e)) &= \gamma_e^{-\sigma(e)} j(\alpha(e)) \\ \omega(j(e)) &= \gamma_e^{1-\sigma(e)} j(\omega(e)). \end{aligned}$$

We continue to use G_v, G_e, to denote the stabilizer of a vertex v, edge e, respectively, of X under the partial action of G. Define a graph of groups (G, Y) by letting, for $v \in V(Y)$, $e \in E(Y)$, $G_v = G_{j(v)}$, $G_e = G_{j(e)}$. The monomorphism $G_e \to G_{\omega(e)}$ is given by

$$a \mapsto a^e = \gamma_e^{\sigma(e)-1} a \gamma_e^{1-\sigma(e)}.$$

One can verify easily that there is a homomorphism $\varphi: \pi_1(G, Y, T) \to G$ defined by the inclusion of G_v into G and $e \mapsto \gamma_e$.

Theorem 7.1. *There is a globalization* (G, X, X') *such that* X' *is a tree if and only if* $\varphi: \pi_1(G, Y, T) \to G$ *is an isomorphism* (*where we keep the above notation*).

Proof. Suppose first that there is such a globalization; then, since $X \subseteq X'$ and $X'/G = X/G = Y$, we see that T, j, and the γ_e form an appropriate version of the above construction for the action of G on X' and so the result follows from [11, 5.4 Theorem 13].

Suppose now that φ is an isomorphism. We first claim that the partial action is true. Indeed, it follows, since φ is surjective that G is generated by the G_v and γ_e (where e is, say, a positively oriented edge). But for $g \in G_v, gv = v$ and hence $g \in D(G)$. On the other hand, if e is a positively oriented edge, then $\gamma_e j(\omega(e)) = \omega(j(e))$ and so $\gamma_e \in D(G)$. It follows, by Theorem 5.1, that X has a connected globalization (G, X, X'). We show that X' is a tree. Again, since $X \subseteq X'$ and $X'/G = X/G = Y$, we see that T, j, and the γ_e form an appropriate version of the above construction for the action of G on X'. Another application of [11, 5.4 Theorem 13] then shows that X' is a tree. □

We remark here that although we in some sense have cheated by using the results of Bass-Serre theory to prove the above theorem, one could give a proof from first principles. In fact, almost all of the proofs in [11] implicitly consider globalizations of partial actions and this is in fact the approach taken by Dicks and Dunwoody in their book [1].

More explicitly Dicks and Dunwoody speak of a presentation of a G-set as consisting of a set X and a partial product $G \times X \to X$. There is of course a free G set on X, namely $G \times X$, and one then takes the minimal G-set quotient on $G \times X$ such that $(g,x) = (1,y)$ if $gx = y$ according to the partial product. One can easily show that X need not inject in general into the G-set so presented; in fact, a partial action is precisely a presentation in which X injects into the G-set given by this presentation (which in this case turns out to be the globalization).

Starting with a graph of groups, Dicks and Dunwoody construct a true partial action of the fundamental group of this graphs on a tree (using the language of presentations). They then construct an action of the group on a tree via globalization (again using the language of presentations).

With the tool of partial actions in hand, we quickly obtain the following refinement of the usual Bass-Serre Theorem.

Theorem 7.2. *Let* (G, Y) *be a graph of groups and* T *a maximal tree in* Y*. Then* $\pi_1(G, Y, T)$ *acts partially on a tree* T' *with* $Y = T'/G$ *and with a tree globalization. Furthermore, the tree* T' *can be chosen so that the projection to* Y *is bijective on edges (and hence an immersion). Conversely, any group acting partially on a tree with tree globalization is isomorphic to the fundamental group of an appropriate group of graphs.*

Proof. The last statement has already been proven. As to the rest, it is shown in [11] that $\pi_1(G, Y, T)$ acts on a tree T'' with quotient Y. By performing the above construction with $X = T''$ and choosing T' to be the smallest subgraph of T'' containing $j(Y)$, one obtains a tree so that the projection to $T'/G = Y$ is bijective on edges (and (G, T, T'') is a tree globalization). □

Corollary 7.3. *Let* (G, Y) *be a graph of groups with* Y *finite and let* T *be a maximal tree. Then* $\pi_1(G, Y, T)$ *acts partially on a finite tree* T' *with* $T'/G = Y$ *and with a tree globalization.*

So, for example, an amalgamated free product or HNN-extension acts partially on a segment with tree globalization.

8. Questions

This paper is only an introduction to the topic. As such, we end the paper with several questions and problems. First, it would be useful to find necessary and sufficient conditions for a true partial action of a group on a connected 2-complex to have a simply connected globalization. A theorem of Macbeath [9] could then be used to read off a presentation of the group.

Another interesting question is when a true partial action without fixed points on a 2-complex has globalization which is planar. This could have applications to the study of Fuchsian groups [8]. Similarly, it would be interesting to characterize when a partial action of a group on a graph has a globalization which is connected and hyperbolic (in the graph metric). An analogous question can be asked for 2-complexes. Such an investigation would be useful in the study of word hyperbolic groups.

A different, but, nonetheless, interesting line of investigation would be to see what information about a group can be obtained from the inverse monoid G^{Pr} mentioned in Section 3. Partial actions of G on a 2-complex correspond to actions of G^{Pr} on a 2-complex, fixed point free translates to only idempotents can fix an n-cell. Thus G^{Pr} can detect if G is finitely generated. What other information about the group can be gleaned from this monoid? Exel [3] shows that this inverse monoid has several interesting C^*-algebra invariants which give information about the group.

As a final observation, for which I am indebted to S. Margolis, we mention that partial group actions can be used to approximate group actions. For example, the natural left action of a group on its Cayley graph (or complex) can be approximated by or, more technically, is a direct limit of the partial actions obtained by restricting the action to larger and larger neighborhoods of the identity. This type of approximation, which can be viewed as a Todd-Coxeter type process, merits further investigation. Indeed, a computer, with its finite memory space, can only store such partial actions and so partial actions can be viewed as the study of actions through a computer's eye.

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