Rock Mech Rock Eng (2017) 50:2119–2133
DOI 10.1007/s00603-017-1206-2

ORIGINAL PAPER

Reliability-Based Stability Analysis of Rock Slopes Using Numerical Analysis and Response Surface Method

N. Dadashzadeh[1] · H. S. B. Duzgun[2] · N. Yesiloglu-Gultekin[3]

Received: 15 January 2016 / Accepted: 20 March 2017 / Published online: 30 March 2017

Abstract While advanced numerical techniques in slope stability analysis are successfully used in deterministic studies, they have so far found limited use in probabilistic analyses due to their high computation cost. The first-order reliability method (FORM) is one of the most efficient probabilistic techniques to perform probabilistic stability analysis by considering the associated uncertainties in the analysis parameters. However, it is not possible to directly use FORM in numerical slope stability evaluations as it requires definition of a limit state performance function. In this study, an integrated methodology for probabilistic numerical modeling of rock slope stability is proposed. The methodology is based on response surface method, where FORM is used to develop an explicit performance function from the results of numerical simulations. The implementation of the proposed methodology is performed by considering a large potential rock wedge in Sumela Monastery, Turkey. The accuracy of the developed performance function to truly represent the limit state surface is evaluated by monitoring the slope behavior. The calculated probability of failure is compared with Monte Carlo simulation (MCS) method. The proposed methodology is found to be 72% more efficient than MCS, while the accuracy is decreased with an error of 24%.

Keywords Rock slope stability · Uncertainty · Numerical simulation · Response surface method · First-order reliability method

✉ H. S. B. Duzgun
duzgun@metu.edu.tr

N. Dadashzadeh
dadashzadeh.n@queensu.ca

N. Yesiloglu-Gultekin
nurgulyesiloglu@yahoo.com

[1] Department of Geological Sciences and Geological Engineering, Queen's University, Kingston, ON, Canada

[2] Department of Mining Engineering, Middle East Technical University, Ankara, Turkey

[3] Head of Research Department, General Directorate of Infrastructure Investment, Ministry of Transport, Maritime Affairs and Communications, Ankara, Turkey

1 Introduction

The assumptions and simplifications of conventional limit equilibrium slope stability methods may not be sufficient to represent the behavior of complex slope problems. With the recent advancements in computational approaches, it is possible to model the slope stability problems more realistically by adopting numerical simulation methods (Cundall 1971; Firpo et al. 2011; Griffiths and Lane 1999; Stead and Eberhardt 1997; Eberhardt et al. 2004). It is widely accepted that rock slope stability involves various parameters which have large degree of uncertainty with data deficiency related to them. Thus, incorporation of uncertainties in the analysis is required for realistic analysis and risk-based decisions. Although numerical methods are quite capable of simulating complex slope stability situations, they use deterministic values for the input variables. The uncertainty related to input values is implicitly considered in numerical methods by performing some sensitivity analysis, which require high computational performance. On the other hand, direct incorporation of the uncertainties in numerical methods serves for explicit consideration of uncertainties. Monte Carlo simulation (MCS) technique is one of the well-known methods which allow the systematic and quantitative treatment of the uncertainties. However, the main disadvantage of this technique is the extensive computational cost (requires hundreds of simulations) which increases substantially in the numerical analyses (Wong 1985). The first-order reliability method (FORM) is an efficient reliability based for considering the variability in the analysis of the parameters (Hasofer and Lind 1974). Defining a limit state function, which is a division surface between the failure and safe state of the slope, is mandatory in FORM. The limit state function of a slope can easily be obtained from limit equilibrium methods. There is considerable number of studies, which focus on the application of reliability-based methods in rock slope stability using limit equilibrium methods. Duzgun et al. (2003) introduced a methodology for reliability-based design of rock slopes. Jimenez-Rodriguez et al. (2006) proposed a system reliability approach for rock slope stability. Duzgun and Bhasin (2008) applied reliability techniques to investigate the stability of a rock slope in Norway. Li et al. (2009) presented a system reliability approach for rock wedges. However, since it is not possible to define an explicit mathematical function for the stability of a slope simulated by numerical methods, FORM cannot be directly used as a reliability assessment tool. On the other hand, the drawback with the high CPU cost of MCS technique makes the probabilistic analyses costly to be performed. Consequently, use of alternative efficient approaches for integrating reliability-based methods with numerical methods in engineering designs is becoming a topic of interest in the recent years. Liu and Cheng (2016) presents a system reliability analysis approach for layered soil slopes based on multivariate adaptive regression splines (MARS) and Monte Carlo simulation (MCS). The proposed approach is achieved in a two-phase process. First, MARS is constructed based on a group of training samples that are generated by Latin hypercube sampling (LHS). Second, the established MARS is integrated with MCS to estimate the system failure probability of slopes.

Response surface method (RSM) is a powerful technique, which allows developing explicit mathematical relationships between the parameters of a given output. The application of the RSM in engineering reliability analyses started at the end of 1980s and well developed in 1990s (Wong 1985; Faravelli 1989; Bucher and Bourgund 1990; Rajashekhar and Ellingwood 1993; Liu and Moses 1994; Kim and Na 1997). Efforts have been dedicated to the application of the RSM in geotechnical and soil slope stability problems. The first use of RSM in geotechnical application was performed by Wong (1985) in which a soil slope was modeled by finite element code. He repeated the slope model in MCS technique and received a reasonable match between the probability of failure (P_f) obtained from MCS and RSM. Zangeneh et al. (2002) employed the RSM to analyze the displacement of slopes in the earthquake studies. Ji and Low (2012) tried to improve the existing methods of slope reliability analysis by considering system reliability using a stratified RSM to define the performance functions of possible failure modes. Zhang et al. (2013) studied the system reliability of soil slopes with RSM. Li et al. (2015) proposed a stochastic RSM for reliability analysis involving correlated non-normal random variables (RV). Such studies profoundly facilitate the applications of reliability methods for complex problems. However, the application of reliability methods requires to be integrated with the state-of-the-art numerical simulations.

Li et al. (2016) reviewed previous studies on developments and applications of RSMs in different slope reliability problems. Then, the computational efficiency and accuracy of four commonly used RSMs (namely single quadratic polynomial-based response surface method (SQRSM), single stochastic response surface method (SSRSM) and multiple quadratic polynomial-based response surface method (MQRSM), and multiple stochastic response surface method (MSRSM) were systematically compared for cohesive and c–ϕ slopes, and their feasibility and validity in the four types of slope reliability problems were discussed.

Johari and Lari (2016) employed probabilistic rock wedge stability analyses. In their paper, a system reliability analysis of rock wedge stability was presented. To perform reliability analysis, a cut-set system was used for wedge analysis. The reliability indices of the individual components and the correlations between the components are determined by the first-order reliability method (FORM). The sequential compounding method (SCM) is used as an efficient numerical procedure to determine the reliability indices of failure modes and system reliability index, considering correlations between failure modes which are calculated by defining equivalent linear safety margin for each failure mode. The predicted system reliability indices and corresponding probabilities of failure from the proposed method were compared with those of the Monte Carlo simulation (MCS).

In this paper, an integrated methodology for probabilistic numerical modeling of rock slope stability is proposed. The proposed methodology integrates FORM and numerical simulations through the use of the RSM. In order to demonstrate the implementation of the proposed methodology, the three-dimensional distinct element code, 3DEC, is used to simulate a rock slope in Sumela Monastery, Turkey. The accuracy of the methodology is assessed by monitoring the slope behavior in each level to investigate whether the limit state function developed by RSM truly represents the slope behavior. The accuracy and efficiency of the approach is later studied by comparing the results with MCS method.

Although the use of probabilistic methods has been increased recently, as summarized above, the integration of numerical and reliability methods has not been well investigated in the rock slope stability assessments yet. Moreover, as RSM is a local approximation tool, studies have not been performed to investigate whether the developed functions by RSM truly represent the limit state surface of the slope stability problem. The numerical methods, specifically distinct element analysis, response surface analysis and reliability analysis for rock slopes are not new. However, integration of distinct element slope stability analysis by response surface model to obtain a reliability index for a given slope is novel. Majority of the works in the literature related to reliability-based rock slope stability analyses rely on limit equilibrium analysis. Although there are probabilistic numerical methods, they mainly do not incorporate response surface model approach, rather they use Monte Carlo simulation approach, which is not practical when a large number of blocks are modeled. Even the recent relevant works in the literature does not integrate three methods into a single one so that advantages of them are taken into account for rock slope stability analysis. Moreover, RSM applications in all the studies generate the limit state function iteratively until acceptable convergence of the reliability index is achieved. The advantage of the proposed methodology is that the design point in each iteration is examined in the numerical model and corresponding factor of safety (FOS) and slope behavior is recorded. Based on the FORM, the true response function is achieved when corresponding design point generates a FOS ≈ 1.0 in the numerical model. Furthermore, tracking the slope behavior in each iteration provides valuable information about critical values which can indicate failure behavior better. In all these respects, the proposed approach with its implementation to a case study captures and demonstrates advantages of the three integrated methods.

2 First-Order Reliability Method (FORM)

According to the reliability theory, the probability of failure (P_f) of a system is evaluated by approximating a defined limit state function beyond which the determination of the supply capacity of an engineering system cannot meet certain demand requirements (Ang and Tang 1984). The limit state function is approximated either linearly or nonlinearly in higher orders (Fiessler and Rackwitz 1979; Hasofer and Lind 1974; Zhang and Du 2010). The FORM is introduced by Hasofer and Lind (1974) in which the P_f is estimated by approximating the limit state function with a hyper plane tangent to it at the most probable point (MPP) in the transformed space of independent standard normal variables.

In the practice of the FORM, a limit state function, $g(X_i)$, must be constructed. The $g(X_i)$ is defined in such a way that the operating or safe scenario is the availability of a resistance greater than the load:

$$g(X_i) = \mathrm{R}(X_i) - S(X_i) > 0 \tag{1}$$

and the non-operating or failure scenario is:

$$g(X_i) = R(X_i) - S(X_i) < 0 \tag{2}$$

where X_i is the vector of basic variables, $R(X_i)$ represents the resistance function and $S(X_i)$ represents the load function of the system. The variables must be transformed into a new space of statistically independent Gaussian variables, with zero mean (μ_{X_i}) and unit standard deviation (σ_{X_i}). The transformation from physical space (X_i) to standardized space or normalized space (U_i) is immediate in the case of independent Gaussian variables. When (random variables) RVs are not Gaussian or independent, a transformation must be applied to convert the variables into uncorrelated standard normal parameters. There are several transformation methods (Rosenblatt 1952; Nataf 1962; Fiessler and Rackwitz 1979) among which the Fiessler and Rackwitz (1979) has widely been used in reliability studies. Once the variables are transformed to the standard Gaussian U_i space, the reliability index of Hasofer–Lind, β_{HL}, is defined as the shortest distance from the origin to the limit state surface in the normalized U_i space. The point U_i^* corresponds to this shortest distance and is called the design point or the MPP. Once the MPP is estimated in the U_i space, the transformation methods can be used to find the corresponding point in the X_i space. Figure 1 illustrates the graphical representation of the FORM approximation for two variables.

The design point (U_i^*) is defined as:

$$U_i^* = \alpha_i \cdot \beta, \quad i = 1, 2, \ldots, n \tag{3}$$

where i is the number of RV, α_i is the direction cosine of the corresponding design point and β is the reliability index and can be iteratively obtained by:

$$\alpha_i = \frac{-\frac{\partial g}{\mathrm{d}u_i}(\beta \cdot \bar{\alpha})}{\left[\sum_{j=1}^{n} \frac{\partial g}{\mathrm{d}u_i}(\beta \cdot \bar{\alpha})^2\right]^{1/2}}, \quad i = 1, 2, \ldots, n \tag{4}$$

$$g(\alpha_1 \cdot \beta, \alpha_2 \cdot \beta, \ldots, \alpha_n \cdot \beta) = 0 \tag{5}$$

In order to implement the FORM in slope stability studies, the first step is to define a limit state or failure function, $g(X_i)$. The initial studies of reliability-based slope stability analysis were based on limit equilibrium methods. The limit equilibrium methods investigate the equilibrium of the driving forces of the slope mass with the resistance

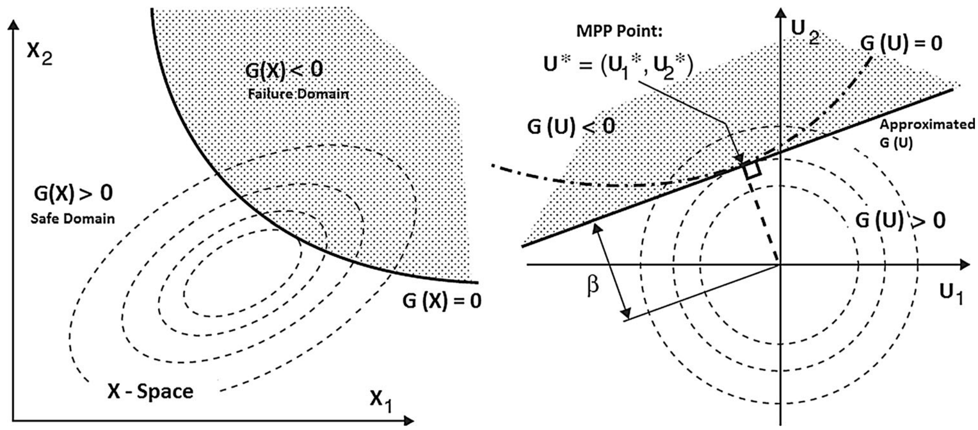


Fig. 1 Graphical representation of the FORM approximation

forces. Accordingly, this definition can be applied to generate the failure function of a slope due to limit equilibrium assumptions:

$$g(X_i) = R(X_i) - S(X_i) \quad (6)$$

where $R(X_i)$ denotes the resistance forces on the slope mass and $S(X_i)$ denotes all the driving forces on the slope. Alternatively, $g(X_i)$ can also be written as:

$$g(X_i) = \mathrm{FOS}(X_i) - 1 \quad (7)$$

where $\mathrm{FOS}(X_i)$ denotes the function of the factor of safety (FOS) for the slope with all RV affecting the performance of the slope (X_i). According to the different failure criteria for the rock mass of the slope as well as different failure mechanisms, the $g(X_i)$ may vary from case to case.

The limit state function is easy to define explicitly in limit equilibrium techniques. However, in case a reliability analysis is desired to be performed based on numerical simulations for the slope stability assessment, it is not possible to directly define an explicit $g(X_i)$ based on the RV. RSM can be implemented to generate an explicit function for $\mathrm{FOS}(X_i)$ when the slope is not simulated by limit equilibrium methods. Integrating the reliability analysis and numerical simulations can provide a better interpretation of the slope's performance.

3 Response Surface Method (RSM)

RSM was first introduced by Box and Wilson (1951), as a technique in empirical study of relationships between responses of parameters to a group of variables. The basic idea of the RSM is to develop an adequate functional relationship between a response of interest (output variable) influenced by several variables (input variables) based on a group of carefully designed mathematical and statistical experiments. An experiment is a series of tests or runs, in which changes are made in the input variables in order to identify the reasons for changes in the output response. In general, the structure of the relationship between the input and output (response) is unknown but can be truly approximated by the RSM in which the convergence to the real relation improves by a number of smooth functions (Khuri and Mukhopadhyay 2010). The RSM is performed by following two major steps, namely design and estimation (Wong 1985). The estimation step is the calculations of fitting an approximate response to the real surface based on a number of wisely selected sample points on the space. The design step deals with how to select the best sample points at which experiments will be run so that the fitting of the surface to the true one is satisfied. According to the estimation step, it is assumed that the true response, $G(X_i)$, of a system depends on i number of input variables, X_1, X_2, …, X_i, as:

$$G(X_i) = f(X_1, X_2, \ldots, X_i) + \varepsilon \quad (8)$$

where the function f is the true unknown and complicated response function, and ε is treated as a statistical error. There are several methods proposed to approximate $G(X_i)$ (Wong 1985; Bucher and Bourgund 1990; Rajashekhar and Ellingwood 1993; Kim and Na 1997; Zheng and Das 2000). The most common approach is the low-degree quadratic polynomial (Bucher and Bourgund 1990), due to their advantages of being simple and known properties.

According to this approach, if $G(X_i)$ represents the real response surface of a system, the approximated surface based on quadratic polynomial is:

$$\hat{G}(X_i) = a + \sum_{i=1}^{n} b_i X_i + \sum_{i=1}^{n} c_i X_i^2 \quad (9)$$

where $\hat{G}(X_i)$ is the approximate response surface function; X_i is the ith RV ($i = 1, 2,\ldots, n$); n is the number of RV; and a, b_i, c_i are the polynomial coefficients which must be calculated. Based on Eq. (9), $2n + 1$ number of experiments is required to obtain the polynomial coefficients. According to Box and Draper (1987), the most important part of the RSM is the design of experiments (DoE). The objective of DoE is the selection of the points where the response should be evaluated. Among various sampling methods (Myers and Montgomery 1995; Montgomery 1997), a common approach is to evaluate $G(X)$ at $2n + 1$ combinations of central point, X_i, and along the line parallel to each coordinate axes at $X_i \pm f\sigma_{X_i}$. Parameters X_i and σ_{X_i} denote the mean and the standard deviation of the ith RV. f is usually set to be 1 for most of the approximations. However, there are several studies in which the importance of f is discussed (Youliang et al. 2008). It is to be noted that the main limitation of the RSM is that it is a local analysis method, where the developed response function is invalid for regions other than the ranges of designed experiments.

4 The Proposed Methodology

The limit state function of a slope can be defined as:

$$g(X_i) = \mathrm{FOS}(X_i) - 1 \tag{10}$$

FOS is a value that is used to examine the stability state of slopes. Generally, a slope fails when its material shear strength on the sliding surface is insufficient to resist the applied in situ shear stresses. According to Anon (2013), a "FOS" index can be defined for any relevant problem by taking the ratio of the calculated parameter value under given conditions to the critical value of the same parameter, at which the onset of an unacceptable outcome manifests itself. This requires to identify the actual and critical parameters. In recent numerical techniques, this goal is achieved based on parameter reduction techniques (Shen 2012). According to this method, the actual parameter value is achieved by direct resolution of the field and the constitutive equations governing the problem, while the critical parameter is calculated by solving inverse boundary value problem. In numerical simulations, this can be achieved using a trial-and-error technique for a range of parameter values until the critical value is found (Diederichs et al. 2007).

The calculation of the FOS in 3DEC is performed based on strength reduction method. The strength reduction method is increasingly popular numerical technique to evaluate FOS in geomechanics. According to this method, the FOS is calculated by progressively reducing the shear strength of the material to bring the slope to a state of limiting equilibrium. The method is commonly applied with the Mohr–Coulomb failure criterion. In this case, the FOS is defined according to the following equations:

$$\mathrm{FOS} = \frac{\tau}{\tau_{\mathrm{trial}}} \tag{11}$$

where τ is the actual strength being obtained from the material properties and corresponding constitutive models, and τ_{trial} is the critical strength of the problem. τ_{trial} is obtained from:

$$\tau_{\mathrm{trial}} = C_{\mathrm{trial}} + \sigma_n \tan\varphi_{\mathrm{trial}} \tag{12}$$

$$C_{\mathrm{trial}} = \frac{1}{\mathrm{SRF}} C \tag{13}$$

$$\emptyset_{\mathrm{trial}} = \arctan\left(\frac{1}{\mathrm{SRF}} \tan\emptyset\right) \tag{14}$$

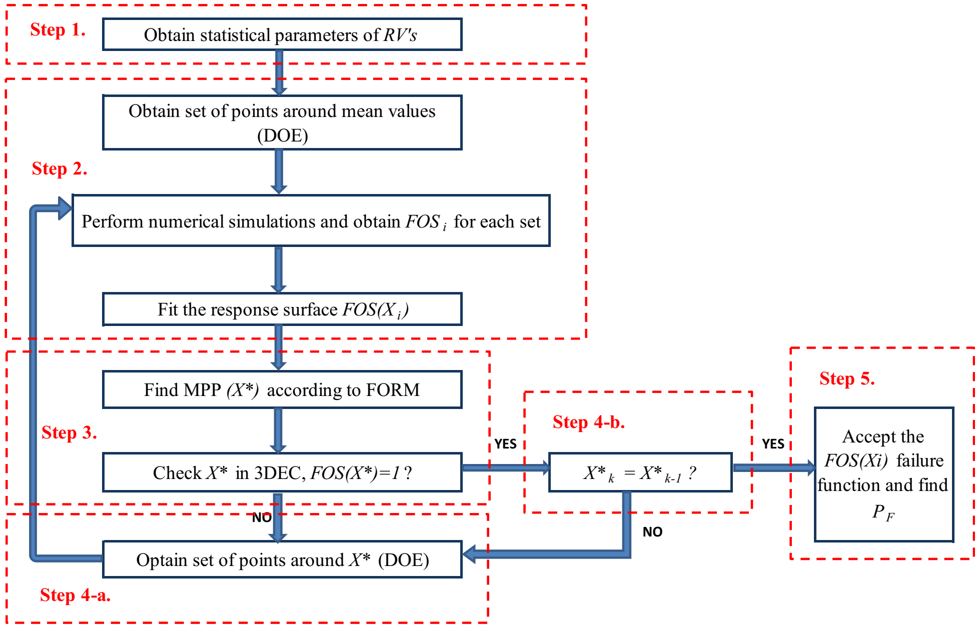
where SRF is strength reduction factor that is obtained by a series of simulations using trial values of the SRF to reduce the cohesion, C, and friction angle, φ, until slope failure occurs.

Once the function of $\mathrm{FOS}(X_i)$ is generated, FORM can easily be performed. However, unlike limit equilibrium methods, it is not possible to directly define an explicit function for FOS in slopes simulated by numerical methods. In such cases, RSM can be used to find the mathematical relationship between the parameters of slope stability and FOS. Considering the RSM as a local analysis technique, it is important to correctly find the surface where $\mathrm{FOS}(X_i) = 1$ is satisfied. Figure 2 illustrates the developed methodology in which the satisfactory convergence to the true limit state surface of FOS can be obtained. The proposed approach consists of five steps, where there are iterations through step 2 to step 5.

Step 1: This step is for obtaining statistical parameters of the RV. Any reliability analysis, in which the uncertainties in the involved parameters are quantified, requires the statistical parameters of the RV. The statistical parameters of each RV can be obtained from laboratory tests and field observations, as well as the literature studies. The statistical parameters of RV depend on the probability distribution function fitted to the RV. For example, if a normal distribution is the best fitting distribution then mean and standard deviation are required. In an ideal case, this is carried out by performing several number of laboratory tests or field observation depending on how the value of the parameter is measured. In most of the real cases, it may not be possible to conduct sufficient number of tests or collect data from the field. Such situations does not prohibit the use of probabilistic methods as given distribution functions and amount of variability in the literature can be used in a Bayesian statistical approach to predict the statistical parameters.

Step 2: It mainly involves integration of numerical analysis and FORM through RSM. The approximation of the surface where $\mathrm{FOS}(X_i) = 1$ and can be performed by combining RSM and FORM. The main idea is to gradually converge to the true surface by iteratively fitting a number of equations for the selected sample points. In each iteration, FORM is responsible to provide the center point where the other sets of random variables need to be

Fig. 2 Flowchart of the developed methodology



designed around it, while RSM provides the corresponding function fitted in the area of the center point.

The first iteration starts by designing the sample points around the mean values of the RVs. For a quadratic response function, $2n + 1$ number of sample points is necessary to calculate the coefficients (a_i, b_i, c_i). Once the design of the points around the mean is performed, each set is separately imported to the numerical simulation and the corresponding FOS is computed. Equation (9) can easily be generated by calculating the coefficients after the response of each set is obtained (FOS).

Step 3: This step is for checking whether the obtained FOS(X_i) satisfies the desired response, FOS(X_i) = 1. For this purpose, FORM is used to examine the validity of the generated function. Considering the definition of FORM, the MPP (U_i^*) always locates on the limit state surface $G(X_i)$ after transferring it to the physical space (X_i). In other words, if the generated function is the true limit state surface, the calculated X_i^* must provide the value of $FOS \cong 1$ in the numerical model. Hence, in order to check the validity of the FOS(X_i) in the first iteration, the corresponding X_i^* is calculated by FORM and then imported to the numerical simulation. When the obtained FOS is close to 1.0, the first condition of the methodology is fulfilled. Otherwise, the iteration must be continued up to the level in which the obtained X_i^* yields a value of $FOS\left(X_i^*\right) \cong 1.0$.

Step 4: It consists of two stages, namely 4-a and 4-b. 4-a is obtaining a new set of sample points in order to continue the iteration. The new set should define a new region in which FOS(X_i) is fitted. The region in each new level is designed around the X_i^* of the previous level. Hence, the mean values as the center point must be substituted by the X_i^* calculated from the previous iteration. Step 2 and 3 must be repeated until $FOS(X^*) \cong 1.0$ is satisfied. 4-b is for confirming the validity of the approximated limit state function as fulfilling the condition of $FOS(X^*) \cong 1.0$ is not sufficient. The reason is that any point on the real limit state surface takes the value of FOS = 1.0; while, there is only one valid MPP $\left(X_i^*\right)$ on the real limit state function according to FORM. On the other hand, the RSM is valid only for the local in which it is studied. Accordingly, the complete convergence only happens when the X_i^* determined in a subsequent level approaches to the X_i^* in the preceding level. Therefore, the two conditions of the proposed methodology to be fulfilled are given as:

$$\mathrm{FOS}\left(X_i^*\right) \cong 1.0 \tag{15}$$

$$X_k^* = X_{k-1}^*, k = \text{number of iteration} \tag{16}$$

Figure 3 shows a schematic illustration of the methodology for a two variable problem. The origin of the axes represents the mean point of the random variables.

Step 5: As the final step of the methodology, once both conditions in the previous steps are fulfilled, the function of FOS(X_i) in the last level can be accepted as a true approximation. Subsequently, the P_f of the slope under consideration is calculated according to FORM.

5 Implementation of the Methodology

5.1 Study Area

In order to implement the proposed methodology and investigate its computational efficiency, a rock slope from

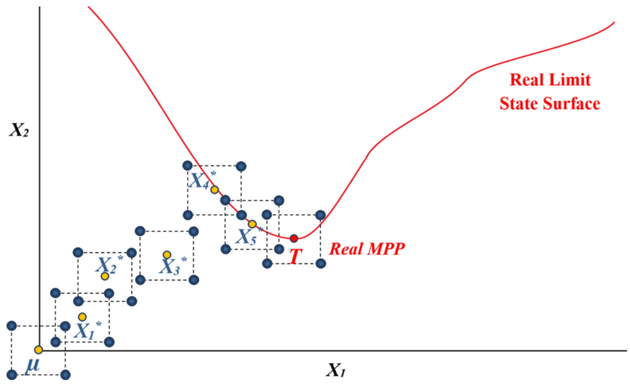


Fig. 3 Schematic illustration of the approximation of the real limit state surface due to RSM and FORM in a two random variable space

a historical site, the Sumela Monastery, is selected. The Sumela Monastery located in the Altindere National Park at Macka region of Trabzon in Turkey, is a Greek Orthodox monastery dedicated to the Virgin Mary. It was founded in AD 386 during the reign of the Emperor Theodosius I (375–395) by two priests from Athens, Barnabas and Sophronios (Miller 1968). The monastery is constructed in a steep rock cliff at a height of about 200 m from the toe of the cliff which is surrounded by the roads and settlements of the local citizens (Fig. 4). The Sumela Monastery is one of the major historical and touristic places of Turkey hosting around 180,000 local and foreign visitors every year (Gelisli et al. 2011). The rock formations are part of the Northern Zone of the Eastern Pontide volcanic province on the Black Sea coast, which is dominated by Late Cretaceous and Middle Eocene volcanics and volcaniclastic rocks (Gelisli et al. 2011). The formation of the Northern Zone consists of basaltic and andesitic lithic tuff, volcanogenic sandstone, shale, basaltic and andesitic lavas and conglomerate deposited in a rift basin setting. According to Gelisli et al. (2011), the region evolved into a carbonate platform after the deposition of the Hamurkesen Formation as a result of a decrease in tectonic activity and filling of the rift basins, giving rise to the Berdiga Formation during the Late Jurassic–Early Cretaceous. Alluvial deposits composed of clay, silt, sand and gravel are widely displayed adjacent to the rivers in this region.

The structure of the monastery has been subjected to several rockfalls from the cliff in the past couple of decades. In 2001, a hazardous rockfall event was reported causing damages to the monastery buildings and facilities. According to Gelisli et al. (2011), the fallen blocks were detached from the crest of the cliff on top of the Monastery structure following a path toward the settlements (Fig. 5). According to the past evidences and field observations, a wedge failure is detected in the vicinity of the monastery which threatens the structure as well as the downhill settlements. The potential wedge is shown in Fig. 5a. The rockfall event happened in 2001 had also been detached from the detected wedge. The wedge is highly fractured causing small block instabilities, potential to fall. More proofs of block detachments are obvious on the wedge. In this study, it is tried to establish the probability of the wedge failure in the area based on the proposed methodology. 3DEC is used to simulate the rock slope.

The laboratory and field studies are performed to characterize the geomechanical properties of the region. Based on the parameters of the rock mass classification system (Bieniawski 1989) listed in Table 1, the quality of the rock mass is classified as fair rock. On the basis of the data obtained from discontinuity surveys, it is found that the slope has two major joint sets (1 m:84°/182°–2 m:40°/46°). The kinematic analysis shown in Fig. 6 validates the formation of a wedge failure in the region. The statistical parameters related to the intact rock and discontinuity parameters are explained in Sect. 5.4.

In order to study the mechanical properties of the rock material, tests are conducted for six samples. The descriptive statistics related to the rock material properties obtained from these tests (uniaxial compressive strength, tensile strength (Brazilian), triaxial compressive strength) are given in Table 3.

In order to study the problem in 3DEC, the model is generated. A sensitivity analysis is performed to identify the random and deterministic parameters of the problem. Later, the P_f is calculated by defining the FOS(X_i) based on RSM and FORM. In order to verify that the generated limit state function for FOS truly represents the failure surface,

Fig. 4 Structure of the Sumela Monastery located inside a steep cliff

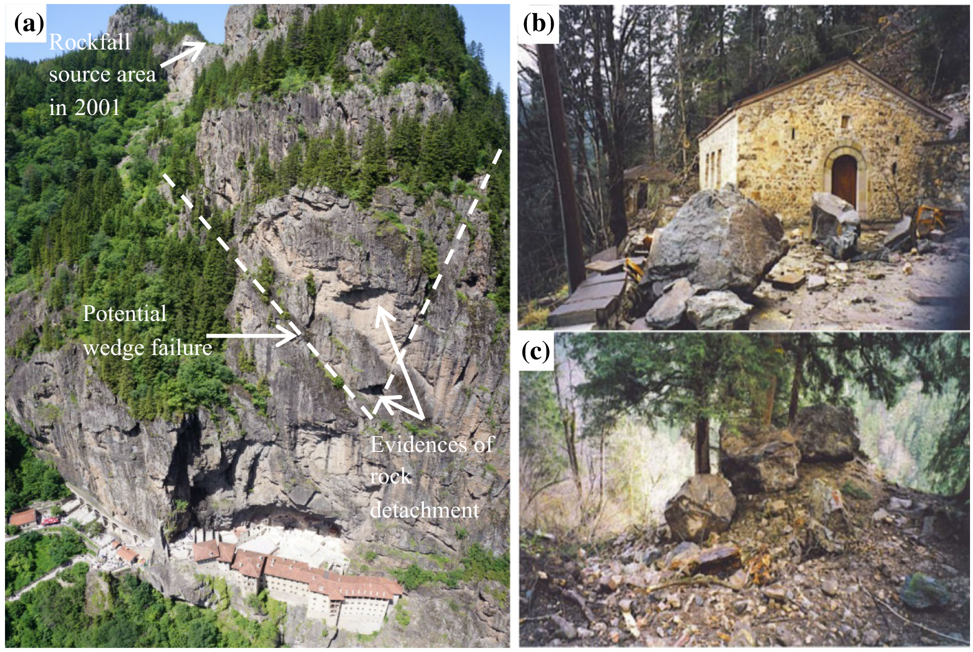


Fig. 5 **a** Potential wedge failure and rockfall source areas, **b**, **c** rockfall event in 2001 (Gelisli et al. 2011)

Table 1 Classification of the rock mass based on RMR system (Bieniawski 1989)

Parameter		Value	Rating
1	UCS (MPa)	230	12
2	Rock quality designation (RQD) (%)	40	8
3	Spacing of discontinuities (m)	>2	20
4.1	Discontinuity length (m)	5–10	2
4.2	Separation (mm)	20–30	0
4.3	Roughness	smooth	1
4.4	Infilling	>5 mm	0
4.5	Weathering	Highly	1
5	Groundwater	Damp	10
Total RMR rating			54
Class of the rock			Fair rock

the slope behavior is monitored and the results are discussed. Moreover, MCS is performed on the same model in 3DEC to investigate the accuracy and efficiency of the methodology.

5.2 Model Generation in 3DEC

The geometry of the slope in 3DEC is created by cutting an original block in a way that the outcome represents boundaries of physical features in the problem. A polygon surface is created based on the topography of the study area. The surface is then imported to 3DEC to create the model boundaries. Once the discontinuity data are added to the model, the final geometry is generated (Fig. 7). The large potential wedge is located at height of approximately 200 m from the toe of the cliff with volume of about ten million m^3.

According to Anon (2013), when the problem is dealing with unconfined set of hard rock blocks at low stress level, such as shallow slopes in jointed rock where the movements consist mainly of sliding and rotation of blocks, it is reasonable to assume the infinite material rigidity in order to let the discontinuities dominate the problem. Accordingly, since the study area is comprised of unconfined basaltic rock blocks, the behavior of the intact material is assumed to be rigid rather than deformable. This let the model be mainly governed by the joints and discontinuities, which reflects the observed behavior in the field.

5.3 Identification of Random Variables (RV)

It is important to note that both numerical and probabilistic analyses demand a high computational cost. Hence, reducing the number of RV, which may have negligible effect on model response, can considerably increase the efficiency. Prior to start the main simulations to estimate the P_f of the wedge, in order to define the deterministic and random input variables, a sensitivity analysis is performed to investigate how the uncertainty in each parameter may affect the FOS. The mechanical parameters of intact rock and discontinuities are assumed to be random at the initial step. In order to have a better understanding of uncertainty influence imposed by each variable, a constant coefficient of variation (COV) of 0.5 is implemented to all random variables. A total of 25 models are run in 3DEC. In each set of models, the concerning random parameter is repeatedly

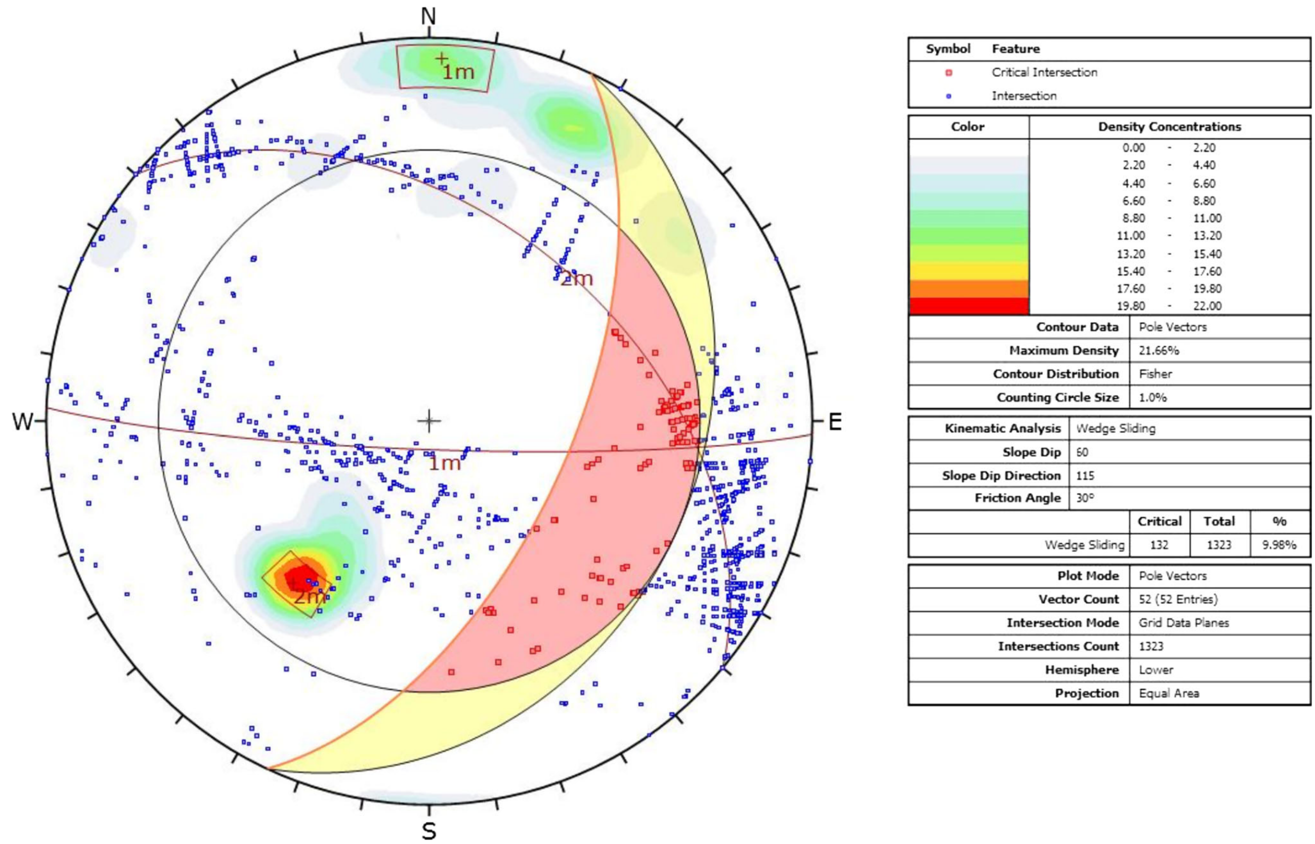


Fig. 6 Discontinuity distribution and kinematic analysis in the study region (RocScience 2015)

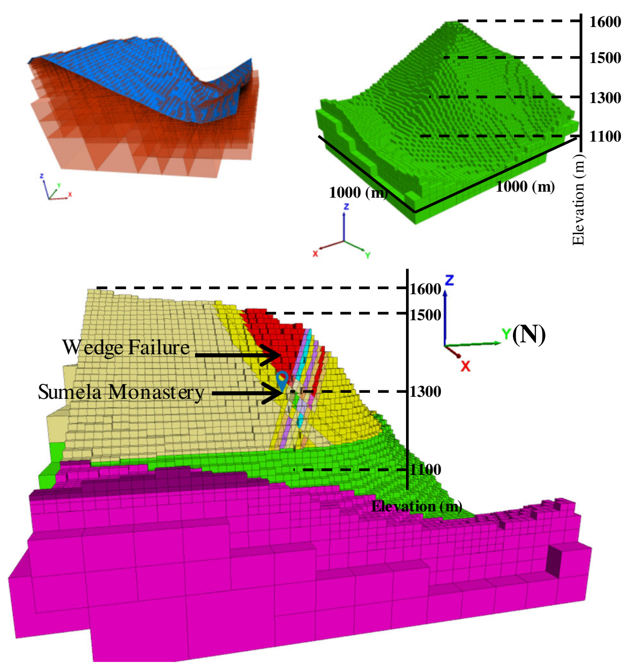


Fig. 7 Geometry of the study area developed in 3DEC (Itasca 2013)

changed while other parameters are kept constant in the mean values and the corresponding FOS is recorded. It is important to note that these simulations are only performed in order to decrease the number of basic variables by determining the response of the slope to the changes in the variability of the involved parameters. Figure 8 reveals that the variability in discontinuity cohesion (JC), normal stiffness (K_n) and shear stiffness (K_s) have the highest influence on FOS, respectively. The discontinuity friction's variability as it is also indicated in various probabilistic studies (e.g., Duzgun et al. 2003; Haderbache and Laouami 2013) is less influential than the variability in the cohesion and the stiffness parameters. This is mainly due to the linear nature of the Coulomb criterion. For this reason, variability of joint stiffness and cohesion is found to be the most effective in changing the values of FOS. As the model is generated in rigid form and the discontinuities are mainly governing the problem rather than the intact rock blocks. Therefore, as expected, the variability of intact parameters does not exhibit significant contribution to FOS. For this reason, they are considered to be deterministic.

5.4 Statistical Parameters of the Random Variables (RV)

Although discontinuity properties are identified to be the RV and the intact properties are considered to be deterministic variables, their statistical parameters are given for

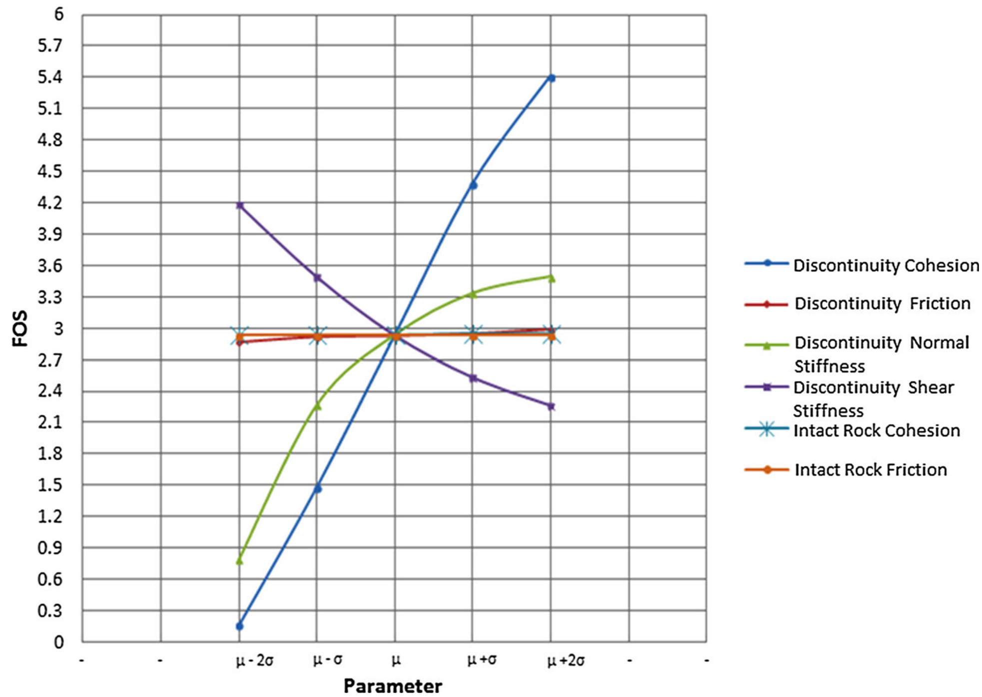


Fig. 8 Sensitivity analysis of FOS to the uncertainty in rock and discontinuity properties

the sake of completeness. Due to the fact that the study area, Sumela Monastry, is a cultural heritage site, by law, it is forbidden to collect suitable rock samples from the site. For this reason, rock samples for obtaining intact rock properties are collected from one of the neighboring fields of the site, which has the same geological and rock mass properties. It was not possible to collect reasonable number of rock discontinuity samples during the field investigations. On the other hand, tilt tests on the saw cut surfaces are performed in the laboratory. By using the tilt tests, RMR values and the literature review (e.g., Kainthola et al. 2013; Schultz 1993), the mechanical properties of discontinuities are described by assigning maximum and minimum values. The discontinuity shear strength parameters in the literature, which are similar to the case study, are searched by considering similar geological and intact rock properties. In this respect, although intact rock properties are not directly used, they are utilized for obtaining information for the characterization of the RV's. For this reason, their statistical parameters are also provided.

When a range of values are known for a random variable, simple statistical distributions, like uniform (UD), symmetric triangular (STD), upper triangular (UTD) and lower triangular (LTD) distributions, can be used for predicting the statistical parameters of the random variables (Ang and Tang 1984). Based on the predicted ranges for mechanical properties of intact rock and discontinuity properties and using the distributions given for ranges, the statistics of mean, standard deviation and COV are obtained. As the UD provides equal probability of obtaining values between the given ranges, it is adopted in this study. The equations of mean and COV for UD, STD, UTD and LTD are given in Table 2. The statistical parameters obtained for the intact properties of the rock are given in Table 3. Similarly, the statistical parameters of RV's used in the reliability analysis based on UD, STD, UTD and LTD are listed in Table 4. In the reliability analyses, UD values in Table 4 are used.

As it can be seen from Table 4, the friction angle has the least uncertainty, which is in line with the sensitivity analysis results. The constitutive model used in the 3DEC analysis is mainly built upon the values of the shear and normal stiffness values which indicates in fact model uncertainty. For this reason, their COV values are larger.

5.5 Reliability Analysis

In the proposed methodology, the response surface of FOS must be adjusted around the mean values of the random variables at the initial iteration. For this purpose, the set of input points around the mean must be designed. Since this study consists of three RVs, JC, K_n and K_s, seven sets of points are necessary in each iteration. Once the set of input parameters are defined, each set is simulated in 3DEC and the corresponding FOS is obtained. Table 5 indicates the results of simulations around the mean values of the RV.

According to Eq. (9), for the three RVs, seven coefficients must be calculated to generate the response surface function of FOS. The coefficients can easily be obtained by solving seven equations with seven unknowns based on

Table 2 Means and COV's corresponding to different distributions (Ang and Tang 1984)

PDF	Mean value, $\bar{X}$	COV
Uniform distribution (UD)	$\frac{1}{2}(X_l + X_u)$	$\frac{1}{\sqrt{3}}\left(\frac{X_u - X_l}{X_u + X_l}\right)$
Symmetric triangular distribution (STD)	$\frac{1}{2}(X_l + X_u)$	$\frac{1}{\sqrt{6}}\left(\frac{X_u - X_l}{X_u + X_l}\right)$
Upper triangular distribution (UTD)	$\frac{1}{3}(X_l + 2X_u)$	$\frac{1}{\sqrt{2}}\left(\frac{X_u - X_l}{2X_u + X_l}\right)$
Lower triangular distribution (LTD)	$\frac{1}{3}(2X_l + X_u)$	$\frac{1}{\sqrt{2}}\left(\frac{X_u - X_l}{X_u + 2X_l}\right)$

X denotes mean of the UD, X_l denotes lower range of the RV and X_u denotes upper range of the RV

Table 3 Basic descriptive statistics of the intact rocks parameters

Variables	Minimum	Maximum	Mean	SD	COV
UCS (MPa)	216.77	245.92	227.79	12.79	0.06
Tensile strength (MPa)	13.19	19.39	15.19	2.20	0.15
Young's modulus (GPa)	71.00	78.00	73.25	3.30	0.05
Poisson's ratio	0.22	0.29	0.26	0.03	0.11
Cohesion (c) (MPa)	60	71	65	3.85	0.06
Internal friction angle ϕ (°)	29.00	33.00	31.00	1.41	0.05

Table 4 Statistical parameters of RV

Random variables (RV)	Probability density function (PDF)	Mean	SD	COV
Discontinuity cohesion (JC) (MPa)	UD	3.90	0.35	0.09
	STD	3.90	0.24	0.06
	UTD	4.10	0.28	0.07
	LTD	3.70	0.28	0.08
Discontinuity friction angle (ϕ_d) (°)	UD	30.00	1.73	0.06
	STD	30.00	1.22	0.04
	UTD	31.00	1.41	0.05
	LTD	29.00	1.41	0.05
Normal stiffness (K_n) (GPa/m)	UD	30.00	2.31	0.08
	STD	30.00	1.63	0.05
	UTD	31.33	1.89	0.06
	LTD	28.67	1.89	0.07
Shear stiffness (K_s) (GPa/m)	UD	12.50	2.02	0.16
	STD	12.50	1.43	0.11
	UTD	13.67	1.65	0.12
	LTD	11.33	1.65	0.15

simulations listed in Table 5. Consequently, the response function of FOS for the first iteration is:

$$\mathrm{FOS}(X_1, X_2, X_3) = (0.649) + (1.953 * X_1) + (-0.445 * X_2) + (0.509 * X_3) + (-0.143 * X_1^2) + (0.008 * X_2^2) + (-0.023 * X_3^2) \quad (17)$$

Therefore, the failure limit state surface for the first iteration can be written as:

Table 5 Design of points around the mean and the corresponding FOS

Parameter	Set 1	Set 2	Set 3	Set 4	Set 5	Set 6	Set 7
X_1 (JC)	4.5	6.3	2.7	2.7	2.7	2.7	6.3
X_2 (K_n)	30	39	21	39	39	21	21
X_3 (K_s)	12	14.4	14.4	14.4	9.6	9.6	14.4
FOS	2.93	4.22	1.45	1.82	2.04	1.7	3.35

$$
\begin{aligned}
g(X_1, X_2, X_3) &= \mathrm{FOS}(X_1, X_2, X_3) - 1 \\
&= (0.649) + (1.953 * X_1) + (-0.445 * X_2) \\
&\quad + (0.509 * X_3) + (-0.143 * X_1^2) \\
&\quad + (0.008 * X_2^2) + (-0.023 * X_3^2) - 1
\end{aligned} \tag{18}
$$

Once the limit state failure function is generated, FORM can be performed to find the MPP (X_i^*) in this region. The response function of FOS and corresponding failure limit state $g(X_i)$ is accepted when obtained MPP illustrates a FOS close to 1.0 in 3DEC. Table 6 lists the results of MPP in the first iteration and corresponding FOS in 3DEC. It is obvious that the iteration must be continued until the first condition of the methodology is satisfied. The similar procedure must be continued by designing the input sets around the MPP $\left(X_i^*\right)$ of the previous iteration instead of mean values until the first condition is satisfied. Additionally, any point on the failure function takes a value of FOS $\cong$ 1.0. Since the RSM is locally valid, the true region must also be verified. This is obtained by satisfying the second condition of the methodology $(X_k^* = X_{k-1}^*)$ in which the convergence to the MPP is also obtained. Consequently, the final failure function is obtained as:

$$
\begin{aligned}
g(X_i) &= (5.124) + (-0.073 * X_1) + (0.682 * X_2) \\
&\quad + (-1.758 * X_3) + \left(0.147 * X_1^2\right) \\
&\quad + \left(-0.016 * X_2^2\right) + \left(0.066 * X_3^2\right) - 1
\end{aligned} \tag{19}
$$

Once the limit state surface is generated, any reliability method can easily be utilized to calculate the P_f of the slope.

6 Results and Discussion

In order to confirm the validation of the converged function of FOS, the response of the model is monitored in the center point of each iteration by plotting the vertical velocity and unbalanced force histories. The slope shows a steady state when the iterations are in the safe region. It is also observed that the slope starts to fail in 3DEC while convergence to the limit state surface according to the RSM is obtained. Figure 9 shows the history plots for iteration one and eleven, respectively. It is shown that, at the center point of the iteration one, where the FOS is calculated to be 2.93, the velocity of the slope follows almost constant rate of increase due to the execution of the model (Fig. 9a); At the center point of the iteration eleven, a sudden increase in the vertical velocity after 12,000 steps occurs (Fig. 9b). Any sudden increase in displacement or velocity of a model indicates a joint slip or block failure or plastic flow within the model (Anon 2013). Moreover, according to the maximum unbalanced force plots, not any instability condition is observed in iteration one (Fig. 9c). The unbalanced force follows a constant value which indicates a constant movement within the model. However, iteration eleven shows a large force imbalance in the model after 14,000 steps (Fig. 9d).

Table 6 Results of the MPP and corresponding FOS

Iteration	X_1^*	X_2^*	X_3^*	FOS (X^*)
1	3.85	28.3	12.6	2.72
2	3.11	26.4	12.9	2.42
3	2.88	25	12.6	2.2
4	2.75	24.1	13.2	1.92
5	2.62	21.9	13	1.64
6	2.53	21.2	13	1.47
7	2.46	20.8	13.3	1.25
8	2.39	20.5	13.2	1.2
9	2.27	19.9	13.4	1.13
10	2.18	19.1	13.3	1.09
11	2.17	19.4	13.4	1.05
12	2.06	18.9	13.5	1.07

The maximum vertical velocity in the center point of each iteration is plotted in Fig. 10. It is clear that vertical velocity increases by converging to the limit state surface. Slope instability is quite obvious by the sudden increase in the vicinity of the limit state surface where FOS is converging to 1.0. As it can be seen from Fig. 10, the first jump on velocity occurs after iteration 5 which is a valuable indicator for estimating the critical failure stage of the slope. In fact this stage is the beginning of failure, which can be used for monitoring purposes. The later stages indicate sudden onset failure.

In order to investigate the accuracy and efficiency of the proposed methodology, P_f of the slope is obtained by MCS method after 300 simulations in 3DEC. According to Table 7, in case of 300 attempts, the efficiency is increased by 72%, while the accuracy is decreased with error of 24%. In this case, since the slope is not dealing with low values of P_f, the error can be ignored by considering the high efficiency of the proposed methodology.

7 Conclusion

In this paper, a method to analyze reliability of rock slopes using the RSM and numerical simulation is developed. The numerical simulation which is usually expensive is repeated only a limited number of times to give point estimates of the response of FOS corresponding to the uncertainties in the model parameters. A function is then fit to these point estimates so that the response of FOS can be reasonably approximated within the region of interest. The approximation function, called the response surface, is replaced by subsequent repetitive computations required in

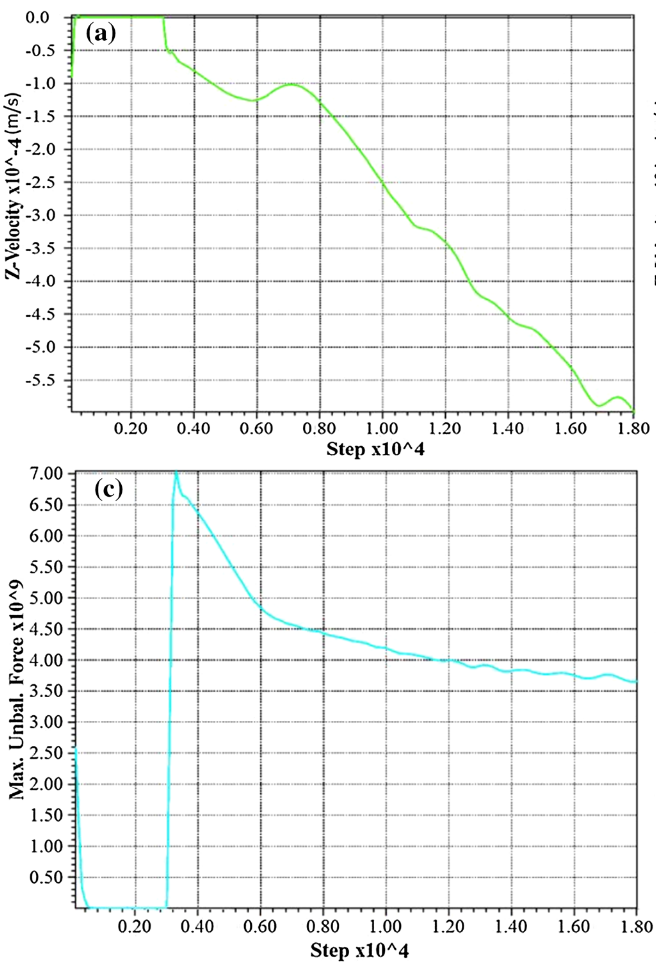


Fig. 9 **a** History of vertical velocity in the center point of iteration one, **b** history of vertical velocity in the center point of iteration eleven, **c** history of maximum unbalanced force in the center point of iteration one and **d** history of maximum unbalanced force in the center point of iteration eleven

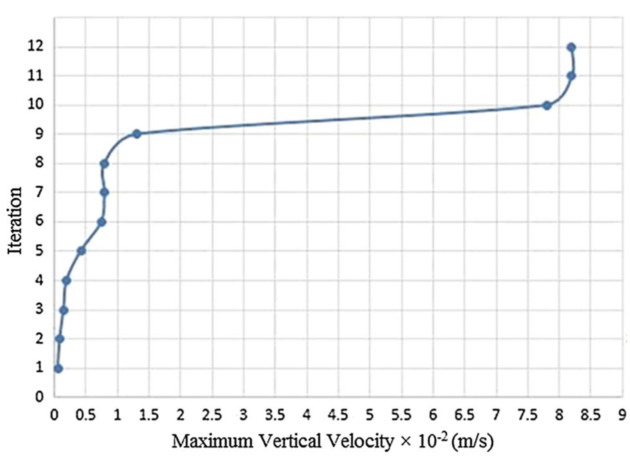


Fig. 10 Maximum vertical velocity in center points versus iteration number

the reliability analysis. The procedure is applied to a large potential wedge in a rock slope in the Sumela Monastery, Turkey. It is concluded that:

Table 7 P_f based on MCS and proposed methodology

Method	P_f (%)	Error (%)	Number of simulations
MCS	13.1	–	300
Proposed approach	16.3	24	84

- The challenging volume of simulations in MCS method can be decreased by using the proposed methodology in this paper.
- FORM can be used in cases with inexistence of predefined explicit limit state functions.
- Instead of conventional limit equilibrium methods, it is possible to define a function to FOS utilizing the state-of-the-art numerical techniques. FOS function based on effective random parameters of the slope is valuable in engineering design and practices.
- By studying the slope behavior during the development of the limit state function, it is possible to indicate the

range of critical velocity or displacement zones. Instrumentation and monitoring of slope stability in the field along with generated information can provide a good understanding of the slope stability hazards and effective implementation of the protection measures.

- The P_f of a large wedge in Sumela Monastery, Turkey, is calculated and compared with MCS method. Using the proposed methodology, the efficiency is increased by 72%, while the accuracy is decreased with error of 24%.
- The accuracy of the calculated P_f in this method is lower considering the MCS technique. However, such an accuracy difference can be tolerated for the given order of magnitudes of P_f.
- It is to be noted that, unlike deterministic analyses where acceptable values are established for the FOS, probabilistic analyses lack acceptable limits of P_f, so that the stability/instability assessment can be made by the comparison of the computed P_f of the given slope with acceptable values. Hence, the proposed approach provides a rigorous basis for evaluating the calculated P_f value in terms of safety.

Acknowledgements The data related to the case study were obtained from the project (No: 114M015) funded by The Scientific and Technological Research Council of Turkey. The authors acknowledge the funding as well as the project researchers for their invaluable help during data collection. The authors also thank to Cagdas Caglayan from Directorate of Surveying and Monuments of Trabzon, Republic of Turkey, Ministry of Culture and Tourism, for his help during the field studies.

References

Ang AHS, Tang WH (1984) Probability concepts in engineering planning and design. Decision, risk, and reliability, vol 2. Wiley, New York

Anon (2013) 3DEC Three-Dimensional Distinct Element Code V.5.0, User's Manual. Itasca Consulting Group Inc

Bieniawski ZT (1989) Engineering rock mass classifications. Wiley, New York

Box G, Draper NR (1987) Empirical model building and response surface. Willey, New York

Box GEP, Wilson KB (1951) On the experimental attainment of optimum conditions. J R Stat Soc 13:1–45

Bucher CG, Bourgund YM (1990) A fast and efficient response surface approach for structural reliability problems. Struct Saf 7:57–66

Cundall P (1971) A computer model for simulating progressive, large scale movements in blocky rock systems. In: Proceedings of the international symposium rock fracture, ISRM. Paper No. II-8

Diederichs MS, Lato M, Quinn P, Hammah R (2007) Shear strength reduction approach for slope stability analyses. In: 1st Canada–US rock mechanics symposium

Duzgun HSB, Bhasin RK (2008) Probabilistic stability evaluation of Oppstadhornet rock slope, Norway. Rock Mech Rock Eng 42:729–749. doi:10.1007/s00603-008-0011-3

Duzgun HSB, Yucemen MS, Karpuz C (2003) A methodology for reliability-based design of rock slopes. Rock Mech Rock Eng 36:95–120. doi:10.1007/s00603-002-0034-0

Eberhardt E, Stead D, Coggan JS (2004) Numerical analysis of initiation and progressive failure in natural rock slopes—the 1991 Randa rockslide. Int J Rock Mech Min Sci 41:69–87. doi:10.1016/S1365-1609(03)00076-5

Faravelli L (1989) Response surface approach for reliability analysis. Eng Mech 115:2763–2781

Fiessler B, Rackwitz R (1979) Quadratic limit states in structural reliability. J Eng Mech Div 105:661–676

Firpo G, Salvini R, Francioni M, Ranjith PG (2011) Use of digital terrestrial photogrammetry in rocky slope stability analysis by distinct elements numerical methods. Int J Rock Mech Min Sci 48:1045–1054. doi:10.1016/j.ijrmms.2011.07.007

Gelisli K, Seren A, Babacan AE et al (2011) The Sumela Monastery slope in Macka, Trabzon, Northeast Turkey: rock mass properties and stability assessment. Bull Eng Geol Environ 70:577–583. doi:10.1007/s10064-010-0343-6

Griffiths DV, Lane PA (1999) Slope stability analysis by finite element. J Geotech 49:387–403

Haderbache L, Laouami N (2013) A new rapid approach in assessing slope stability beneath a random field. Indian J Geo-Mar Sci 42(6):734–744

Hasofer AM, Lind NC (1974) Exact and invariant second-moment code format. J Eng Mech Div 100:111–121

Itasca (2013) 3DEC Three-Dimensional Distinct Element Code V.5.0. Itasca Consulting Group Inc

Ji J, Low BK (2012) Stratified response surfaces for system probabilistic evaluation of slopes. J Geotech Geoenviron Eng 138:1398–1406. doi:10.1061/(ASCE)GT.1943-5606.0000711

Jimenez-Rodriguez R, Sitar N, Chacón J (2006) System reliability approach to rock slope stability. Int J Rock Mech Min Sci 43:847–859. doi:10.1016/j.ijrmms.2005.11.011

Johari A, Lari AM (2016) System reliability analysis of rock wedge stability considering correlated failure modes using sequential compounding method. Int J Rock Mech Min Sci 82:61–70

Kainthola A, Verma D, Thareja R, Singh TN (2013) A review on numerical slope stability analysis. Int J Sci Eng Technol Res 2(6):1315–1320

Khuri AI, Mukhopadhyay S (2010) Response surface methodology. WIREs. Comput Stat 2:128–149. doi:10.1002/wics.73

Kim SH, Na SW (1997) Response surface method using vector projected sampling points. Struct Saf 19:3–19

Li D, Zhou C, Lu W, Jiang Q (2009) A system reliability approach for evaluating stability of rock wedges with correlated failure modes. Comput Geotech 36:1298–1307. doi:10.1016/j.compgeo.2009.05.013

Li D-Q, Jiang S-H, Cao Z-J et al (2015) A multiple response-surface method for slope reliability analysis considering spatial variability of soil properties. Eng Geol 187:60–72. doi:10.1016/j.enggeo.2014.12.003

Li D-Q, Zheng D, Cao Z-J, Tang XS, Phoon KK (2016) Response surface methods for slope reliability analysis: review and comparison. Eng Geol 203:3–14

Liu LL, Cheng YM (2016) Efficient system reliability analysis of soil slopes using multivariate adaptive regression splines-based Monte Carlo simulation. Comput Geotech 79:41–54

Liu YW, Moses F (1994) A sequential response surface method and its application in the reliability analysis of aircraft structural systems. Struct Saf 16:39–46

Miller W (1968) Trebizond: the last Greek empire. The Macmillan co, London

Montgomery DC (1997) Design and analysis of experiments. Wiley, New York

Myers RH, Montgomery DC (1995) Response surface methodology: process and product optimization using designed experiments. Wiley, New York

Nataf A (1962) Determination des distribution dont les marges sont donnees. C R lacademie des Sci 225:42–43

Rajashekhar M, Ellingwood B (1993) A new look at the response surface approach for reliability analysis. Struct Saf 12:205–220. doi:10.1016/0167-4730(93)90003

Rocscience (2015) Dips, V.6.016. Toronto, Canada

Rosenblatt M (1952) Remarks on a multivariate transformation. Ann Math Stat 23:470–472

Schultz RA (1993) Brittle strength of basaltic rock masses with applications to Venus. J Geophys Res 98(E6):10883

Shen H (2012) Non-deterministic analysis of slope stability based on numerical simulation. Technische Universität Bergakademie, Freiberg

Stead D, Eberhardt E (1997) Developments in the analysis of footwall slopes in surface coal mining. Eng Geol 46:41–61. doi:10.1016/S0013-7952(96)00084-1

Wong F (1985) Slope reliability and response surface method. Geotech Eng 111:32–53

Youliang D, Aiqun L, Yang D (2008) A new look at the response surface method for reliability analysis using chaos theory. Earthq Eng Eng Vib 7:329–335. doi:10.1007/s11803-008-0852-5

Zangeneh N, Azizian A, Lye L, Popescu R (2002) Application of response surface methodology in numerical geotechnical analysis. In: 55th Canadian Society for geotechnical conference, Hamilton, ON

Zhang J, Du X (2010) A second-order reliability method with first-order efficiency. J Mech Des 132:101006

Zhang J, Huang HW, Phoon KK (2013) Application of the Kriging-based response surface method to the system reliability of soil slopes. J Geotech Geoenviron Eng 139:651–655. doi:10.1061/(ASCE)GT.1943-5606.0000801

Zheng Y, Das P (2000) Improved response surface method and its application to stiffened plate reliability analysis. Eng Struct 22:544–551. doi:10.1016/S0141-0296(98)00136-9