Rock Mech Rock Eng (2016) 49:3817–3822
DOI 10.1007/s00603-016-0942-z

TECHNICAL NOTE

Deep Lined Circular Tunnels in Transversely Anisotropic Rock: Complementary Solutions

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Received: 30 November 2015 / Accepted: 29 February 2016 / Published online: 16 March 2016

Keywords Deep tunnel · Circular liner · Anisotropy · Transversely anisotropic rock · Water pressure · Undrained conditions

List of Symbols

A_s, I_s	Cross-section and moment of inertia of liner
E_x, E_y	Young's modulus of rock in *x*- and *y*-axis
E_s, ν_s	Young's modulus and Poisson's ratio of the liner
G_{xy}	Shear modulus of rock
M	Biot's modulus
r, θ	Polar coordinates
r_o	Radius of the tunnel
t	Liner thickness
T^s, M^s	Axial force and moment of liner
U_x, U_y	Displacement of the rock in cartesian coordinates
U_r^{IV}, U_θ^{IV}	Displacement of the liner in polar coordinates
u	Pore pressures
x, y	Cartesian coordinates of axes of elastic symmetry
z	Complex number, $z_k = x + \mu_k y$, $k = 1, 2$
α_x, α_y	Biot's constants in *x*- and *y*-axis
ε_x, ε_y, γ	Axial and shear strains in *x*- and *y*-axis
μ_1, μ_2	Roots of compatibility equation
ν_{xy}, ν_{xz}, ν_{yz}	Poisson's ratios of rock
σ_v, σ_h, τ_{vh}	Total normal and shear stresses at the far-field
σ_x, σ_y, τ_{xy}	Total normal and shear stresses in Cartesian coordinate system
$\Delta\sigma_r$, $\Delta\tau$	Radial and shear stresses at the liner-rock contact

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1 Introduction

An analytical solution for a supported deep tunnel in transversely anisotropic rock has been provided by Bobet (2011). Figure 1 shows the tunnel, with radius r_o, in a rock medium with x and y as the axes of elastic symmetry, and subjected to a general far-field stress that is defined by stresses σ_v, σ_h and τ_{vh}. The solution was found with the following assumptions: deep circular tunnel, transversely anisotropic elastic rock, elastic thin liner, tied rock-liner contact, plane strain conditions on any cross section perpendicular to the tunnel axis, simultaneous excavation and liner installation. Two formulations were presented, one for dry or saturated porous ground with no drainage at the rock-liner contact, and the other for undrained loading. With these assumptions, and using complex variable theory and conformal mapping techniques, the strain compatibility equation obtained was (in the following we use the same notation as in Bobet 2011):

$$\alpha_1 \frac{\partial^4 F}{\partial y^4} + \alpha_3 \frac{\partial^4 F}{\partial x^4} + \left(\frac{1}{G_{xy}} - 2\alpha_2\right) \frac{\partial^4 F}{\partial x^2 \partial y^2} = -\beta_1 \frac{\partial^2 u}{\partial y^2} - \beta_2 \frac{\partial^2 u}{\partial x^2}, \tag{1}$$

where u is the pore pressures, $F(x, y)$ is a stress function, which definition satisfies equilibrium:

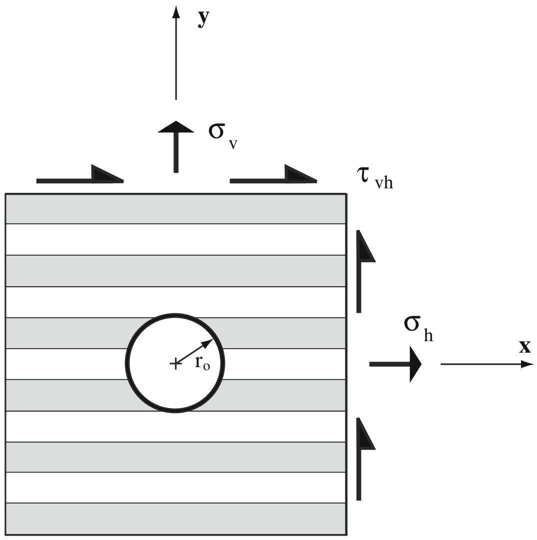

Fig. 1 Deep tunnel in transversely anisotropic rock

$$\begin{aligned}\sigma_x &= \frac{\partial^2 F}{\partial y^2}\\ \sigma_y &= \frac{\partial^2 F}{\partial x^2}\\ \tau_{xy} &= -\frac{\partial^2 F}{\partial x \partial y}\end{aligned} \tag{2}$$

σ_x, σ_y and τ_{xy} are the stresses in the x–y coordinate system, with x and y representing the axes of elastic symmetry. α_1, α_2, α_3, G_{xy}, β_1 and β_2 are elastic constants defined as

$$\begin{aligned}\varepsilon_x &= \alpha_1\,\sigma_x - \alpha_2\,\sigma_y + \beta_1\,u\\ \varepsilon_y &= -\alpha_2\,\sigma_x + \alpha_3\,\sigma_y + \beta_2\,u\\ \gamma &= \frac{\tau_{xy}}{G_{xy}}\\ \alpha_1 &= \frac{1-\nu_{xz}^2}{E_x}\\ \alpha_2 &= \frac{(1+\nu_{xz})\,\nu_{yx}}{E_y}\\ \alpha_3 &= \left(1-\frac{E_x}{E_y}\nu_{yx}^2\right)\frac{1}{E_y}\\ \beta_1 &= \alpha_1\,\alpha_x - \alpha_2\,\alpha_y\\ \beta_2 &= -\alpha_2\,\alpha_x + \alpha_3\,\alpha_y\end{aligned} \tag{3}$$

where ε_x, ε_y and γ are the strains in the x–y coordinate system, E_x and E_y are the Young's modulus in the x and y directions, ν_{xz} and ν_{yx} are the Poisson's ratios in the xz and yx directions, respectively, G_{xy} is the shear modulus, and α_x and α_y are the Biot's constants. Note that $\nu_{xy} = -\nu_{yx} \cdot E_x/E_y$.

Equation (1) can be rewritten in terms of the complex variable $z_k = x + \mu_k y$, $k = 1, 2$, where μ_k is a complex number. The result is

$$\left[\alpha_1\,\mu_k^4 + \left(\frac{1}{G_{xy}} - 2\alpha_2\right)\mu_k^2 + \alpha_3\right]\frac{\partial^4 F}{\partial z^4} = -\beta_1\frac{\partial^2 u}{\partial y^2} - \beta_2\frac{\partial^2 u}{\partial x^2} \tag{4}$$

Stresses and strains, given (3), can be found introducing the functions $\phi(z_k) = F'(z_k) = \partial F/\partial z_k$, $k = 1, 2$, such that

$$\begin{aligned}\sigma_x &= 2\,\mathrm{Re}\left[\mu_1^2\,\phi_1'(z_1) + \mu_2^2\,\phi_2'(z_2)\right] + \frac{\partial^2 F_o}{\partial y^2}\\ \sigma_y &= 2\,\mathrm{Re}\left[\phi_1'(z_1) + \phi_2'(z_2)\right] + \frac{\partial^2 F_o}{\partial x^2}\\ \tau &= -2\,\mathrm{Re}\left[\mu_1\,\phi_1'(z_1) + \mu_2\,\phi_2'(z_2)\right] - \frac{\partial^2 F_o}{\partial x \partial y},\end{aligned} \tag{5}$$

where F_o is a particular solution of (1) and μ_1 and μ_2 are the roots of the equation:

$$\alpha_1\,\mu_k^4 + \left(\frac{1}{G_{xy}} - 2\alpha_2\right)\mu_k^2 + \alpha_3 = 0. \tag{6}$$

Displacements are obtained by integration of strains, Eq. (3), given the stresses in (5). They are expressed as

$$\begin{aligned}U_x &= 2\,\mathrm{Re}\left[(\alpha_1\mu_1^2 - \alpha_2)\,\phi_1(z_1) + (\alpha_1\mu_2^2 - \alpha_2)\,\phi_2(z_2)\right]\\ U_y &= 2\,\mathrm{Re}\left[\left(-\alpha_2\mu_1 + \frac{\alpha_3}{\mu_1}\right)\phi_1(z_1) + \left(-\alpha_2\mu_2 + \frac{\alpha_3}{\mu_2}\right)\phi_2(z_2)\right].\end{aligned} \tag{7}$$

Bobet (2011) provided closed-form analytical formulations for the stresses and displacements of the rock and the liner when the roots of Eq. (6) are pure imaginary numbers, i.e. of the form $\mu_k = i\,|\mu_k|$, where i is the imaginary unit, $i^2 = -1$, and $|\mu_k|$ is a real number equal to the modulus of the complex number μ_k. However, these are not the only solutions of (6). A solution of the form $\mu_1 = a + ib$, $\mu_2 = -a + ib$ is also possible, where a and b are real numbers.

This paper complements the results provided by Bobet (2011), obtained for pure imaginary roots, with roots of the following form: $\mu_1 = a + ib$, $\mu_2 = -a + ib$. The same approach and the same notation given by the previous publication are used.

2 Tunnel in Dry Rock or in Rock Below the Water Table with Impermeable Liner

The problem is decomposed into four different problems taking advantage of the principle of superposition in elasticity. See Fig. 2. Problem I is that of the rock without the tunnel and subjected to the far-field stresses, Fig. 1b; Problem II consists of the tunnel opening where stresses are applied to the tunnel wall such that they are those of the far-field but with opposite sign, Fig. 1c; Problem III describes the rock-liner interaction, Fig. 1d; and Problem

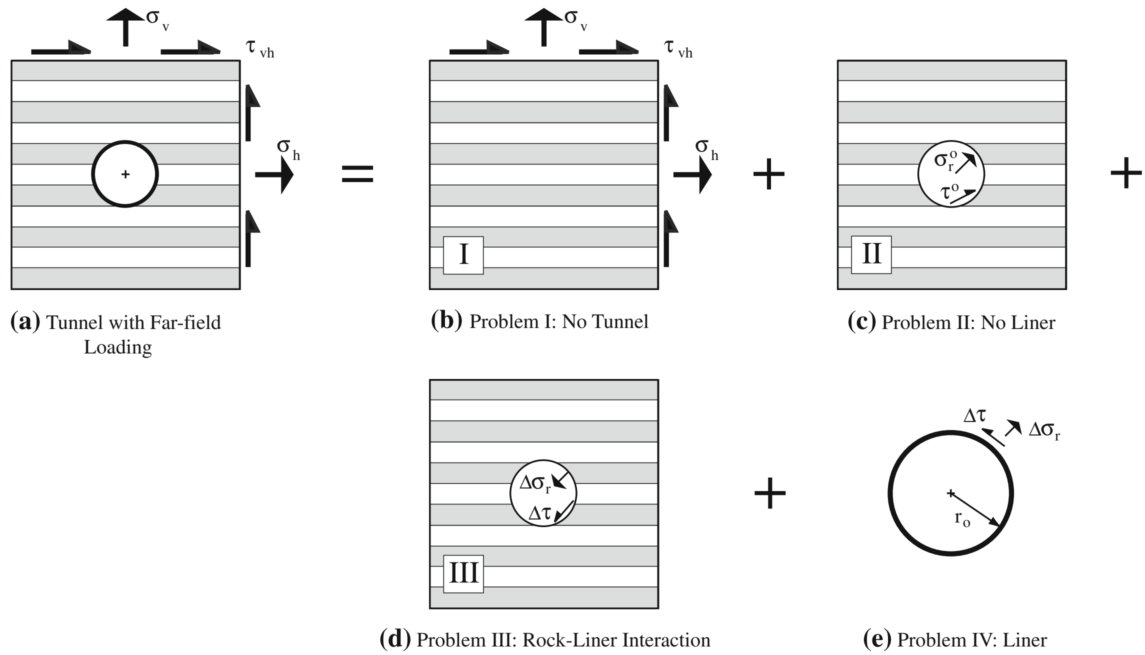

Fig. 2 Problem decomposition, after Bobet (2011)

IV describes the liner response to the stresses at the rock-liner interface, Fig. 1e.

The solution of Problem I is trivial, with stresses equal to those of the far field. The displacements that are obtained from such stress field are not included in the solution of the general problem because these displacements should have occurred before the tunnel construction. In other words, stresses in the rock are given by the sum of the stresses from Problems I, II, and III, while displacements are given by the sum of the displacements from problems II and III. Stresses and displacements of the liner are the result of problem IV.

The full solution must satisfy compatibility of displacements and stresses at the rock-liner interface. The compatibility of stresses is satisfied given the problem definition in Fig. 2. The stresses at the interface can be expressed with the following (Bobet 2011):

$$\begin{aligned}\Delta\sigma_r &= \sigma_o + \sum_{n=2,4,6}^{\infty} \sigma_n^a \cos n\theta + \sum_{n=2,4,6}^{\infty} \sigma_n^b \sin n\theta \\ \Delta\tau &= \sum_{n=2,4,6}^{\infty} \tau_n^a \sin n\theta + \sum_{n=2,4,6}^{\infty} \tau_n^b \cos n\theta,\end{aligned} \tag{8}$$

where $\sigma_o, \sigma_n^a, \sigma_n^b, \tau_n^a$ and τ_n^b are constants that are found by imposing compatibility of displacements at the rock-liner interface, which is given by:

$$\begin{aligned}U_x^{IV} &= U_x^{II} + U_x^{III} \\ U_y^{IV} &= U_y^{II} + U_y^{III}.\end{aligned} \tag{9}$$

The stress functions and their derivatives for Problem II, following the same process as Bobet (2011), are

$$\begin{aligned}\phi_1 &= \frac{1}{2}\frac{r_o}{\mu_1+\bar{\mu}_1}\left[\overline{(1-i\mu_1)}\,\tau_{vh} - \bar{\mu}_1\sigma_v - i\,\sigma_h\right]\frac{1}{\varsigma_1} \\ \phi_2 &= -\frac{1}{2}\frac{r_o}{\mu_1+\bar{\mu}_1}\left[(1-i\mu_1)\,\tau_{vh} + \mu_1\sigma_v - i\,\sigma_h\right]\frac{i}{\varsigma_2} \\ \phi_1' &= -\frac{1}{\mu_1+\bar{\mu}_1}\,\frac{\overline{(1-i\,\mu_1)}\,\tau_{vh} - \bar{\mu}_1\sigma_v - i\,\sigma_h}{(1-i\,\mu_1)\,\varsigma_1^2 - (1+i\,\mu_1)} \\ \phi_2' &= \frac{1}{\mu_1+\bar{\mu}_1}\,\frac{(1-i\,\mu_1)\,\tau_{vh} + \mu_1\sigma_v - i\,\sigma_h}{\overline{(1-i\,\mu_1)}\,\varsigma_2^2 - (1-i\,\bar{\mu}_1)},\end{aligned} \tag{10}$$

where ζ_k is a complex variable such that $z_k = \frac{1}{2}(1-\mu_k i)\, r_o\,\varsigma_k + \frac{1}{2}(1+\mu_k i)\, r_o\,\varsigma_k^{-1}$.

The displacements at the tunnel perimeter, i.e. at $r = r_o$ are

$$\begin{aligned}U_x^{II} &= -\left[\left(a^2+b^2\right)A_1 - A_2\right]\sigma_v r_o \cos\theta + 2bA_1\,\sigma_h r_o \cos\theta \\ &\quad + \left[\left(a^2+b^2+2b\right)A_1 - A_2\right]\tau_{vh} r_o \sin\theta \\ U_y^{II} &= 2\left(a^2+b^2\right)bA_1\,\sigma_v r_o \sin\theta - \left[\left(a^2+b^2\right)A_1 - A_2\right] \\ &\quad \times \sigma_h r_o \sin\theta + \left[\left(a^2+b^2\right)(1+2b)A_1 - A_2\right]\tau_{vh} r_o \cos\theta\end{aligned} \tag{11}$$

where $A_1 = \alpha_1$ and $A_2 = \alpha_2$, and θ is the angle measured counterclockwise from the x-axis.

For Problem III:

$$
\begin{aligned}
\phi_1 =& \frac{1}{4}\frac{r_o}{\mu_1+\bar{\mu}_1}\left\{\left[2\overline{(\mu_1-i)}\,\sigma_o-\overline{(\mu_1+i)}\,(\sigma_2^a-\tau_2^a)-\overline{(1-i\mu_1)}\,(\sigma_2^b+\tau_2^b)\right]\frac{1}{\varsigma_1}\right.\\
&\left.+\sum_{n=3,5,7}^{\infty}\frac{1}{n}\left[\overline{(\mu_1-i)}(\sigma_{n-1}^a+\tau_{n-1}^a)-\overline{(1+i\mu_1)}(\sigma_{n-1}^b-\tau_{n-1}^b)-\overline{(\mu_1+i)}(\sigma_{n+1}^a-\tau_{n+1}^a)-\overline{(1-i\mu_1)}(\sigma_{n+1}^b+\tau_{n+1}^b)\right]\frac{1}{\varsigma_1^n}\right\}\\
\phi_2 =& \frac{1}{4}\frac{r_o}{\mu_1+\bar{\mu}_1}\left\{\left[2(\mu_1-i)\,\sigma_o-(\mu_1+i)\,(\sigma_2^a-\tau_2^a)+(1-i\mu_1)\,(\sigma_2^b+\tau_2^b)\right]\frac{1}{\varsigma_2}\right.\\
&\left.+\sum_{n=3,5,7}^{\infty}\frac{1}{n}\left[(\mu_1-i)(\sigma_{n-1}^a+\tau_{n-1}^a)+(1+i\mu_1)(\sigma_{n-1}^b-\tau_{n-1}^b)-(\mu_1+i)(\sigma_{n+1}^a-\tau_{n+1}^a)+(1-i\mu_1)(\sigma_{n+1}^b+\tau_{n+1}^b)\right]\frac{1}{\varsigma_2^n}\right\}\\
\phi_1' =& -\frac{1}{2}\frac{1}{(\mu_1+\bar{\mu}_1)\left[(1-i\,\mu_1)\,\varsigma_1^2-(1+i\,\mu_1)\right]}\left\{2\overline{(\mu_1-i)}\,\sigma_o-\overline{(\mu_1+i)}\,(\sigma_2^a-\tau_2^a)-\overline{(1-i\mu_1)}\,(\sigma_2^b+\tau_2^b)\right.\\
&\left.+\sum_{n=3,5,7}^{\infty}\left[\overline{(\mu_1-i)}(\sigma_{n-1}^a+\tau_{n-1}^a)-\overline{(1+i\mu_1)}(\sigma_{n-1}^b-\tau_{n-1}^b)-\overline{(\mu_1+i)}(\sigma_{n+1}^a-\tau_{n+1}^a)-\overline{(1-i\mu_1)}(\sigma_{n+1}^b+\tau_{n+1}^b)\right]\frac{1}{\varsigma_1^{n-1}}\right\}\\
\phi_2' =& -\frac{1}{2}\frac{1}{(\mu_1+\bar{\mu}_1)\left[\overline{(1-i\,\mu_1)}\,\varsigma_2^2-(1-i\,\bar{\mu}_1)\right]}\left\{2(\mu_1-i)\,\sigma_o-(\mu_1+i)\,(\sigma_2^a-\tau_2^a)+(1-i\mu_1)\,(\sigma_2^b+\tau_2^b)\right.\\
&\left.+\sum_{n=3,5,7}^{\infty}\left[(\mu_1-i)(\sigma_{n-1}^a+\tau_{n-1}^a)+(1+i\mu_1)(\sigma_{n-1}^b-\tau_{n-1}^b)-(\mu_1+i)(\sigma_{n+1}^a-\tau_{n+1}^a)+(1-i\mu_1)(\sigma_{n+1}^b+\tau_{n+1}^b)\right]\frac{1}{\varsigma_2^{n-1}}\right\}
\end{aligned}
\tag{12}
$$

The displacements at the rock-liner interface are

$$
\begin{aligned}
U_x^{\mathrm{III}} &= \frac{1}{2}r_o\left\{\begin{array}{l}\left[(a^2+b^2-2b)A_1-A_2\right]2\sigma_o\cos\theta-\left[(a^2+b^2+2b)A_1-A_2\right]\left[(\sigma_2^a-\tau_2^a)\cos\theta+(\sigma_2^b+\tau_2^b)\sin\theta\right]+\\ \displaystyle\sum_{n=3,5,7}^{\infty}\frac{1}{n}\left[\begin{array}{l}\left[(a^2+b^2-2b)A_1-A_2\right]\left[(\sigma_{n-1}^a+\tau_{n-1}^a)\cos n\theta+(\sigma_{n-1}^b-\tau_{n-1}^b)\sin n\theta\right]-\\ \quad\left[(a^2+b^2+2b)A_1-A_2\right]\left[(\sigma_{n+1}^a-\tau_{n+1}^a)\cos n\theta+(\sigma_{n+1}^b+\tau_{n+1}^b)\sin n\theta\right]\end{array}\right]\end{array}\right\}\\
U_y^{\mathrm{III}} &= \frac{1}{2}r_o\left\{\begin{array}{l}\left[(a^2+b^2)(1-2b)A_1-A_2\right]2\sigma_o\sin\theta+\left[(a^2+b^2)(1+2b)A_1-A_2\right]\left[(\sigma_2^a-\tau_2^a)\sin\theta-(\sigma_2^b+\tau_2^b)\cos\theta\right]+\\ \displaystyle\sum_{n=3,5,7}^{\infty}\frac{1}{n}\left[\begin{array}{l}\left[(a^2+b^2)(1-2b)A_1-A_2\right]\left[(\sigma_{n-1}^a+\tau_{n-1}^a)\sin n\theta-(\sigma_{n-1}^b-\tau_{n-1}^b)\cos n\theta\right]+\\ \quad\left[(a^2+b^2)(1+2b)A_1-A_2\right]\left[(\sigma_{n+1}^a-\tau_{n+1}^a)\sin n\theta-(\sigma_{n+1}^b+\tau_{n+1}^b)\cos n\theta\right]\end{array}\right]\end{array}\right\}.
\end{aligned}
\tag{13}
$$

where $A_1 = \alpha_1$ and $A_2 = \alpha_2$.

For Problem IV

The axial force T^s, moment M^s and displacements of the liner obey the following relations (Flügge 1966):

$$
\begin{aligned}
& r_o\frac{dT^s}{d\theta}-\frac{dM^s}{d\theta}=-r_o^2\,\Delta\tau\\
& r_o\,T^s+\frac{d^2M^s}{d\theta^2}=r_o^2\,\Delta\sigma_r\\
& \frac{d^2U_\theta^{\mathrm{IV}}}{d\theta^2}+\frac{dU_r^{\mathrm{IV}}}{d\theta}=-\frac{(1-\nu_s^2)}{E_s\,A_s}\,r_o^2\,\Delta\tau\\
& \frac{dU_\theta^{\mathrm{IV}}}{d\theta}+U_r^{\mathrm{IV}}+\frac{I_s}{r_o^2A_s}\left(\frac{d^4U_r^{\mathrm{IV}}}{d\theta^4}+2\frac{d^2U_r^{\mathrm{IV}}}{d\theta^2}+U_r^{\mathrm{IV}}\right)\\
& =\frac{(1-\nu_s^2)}{E_s\,A_s}\,r_o^2\,\Delta\sigma_r
\end{aligned}
\tag{14}
$$

where A_s and I_s are the cross-sectional area and moment of inertia of the liner, respectively, E_s and ν_s are the Young's modulus and Poisson's ratio of the liner, and r_o is the radius of the tunnel. Given the interface stresses in (8), the following expressions are obtained:

$$T^{\mathrm{s}} = \sigma_{\mathrm{o}} r_{\mathrm{o}} - \sum_{n=2,4,6}^{\infty} \left\{ \left[\frac{\sigma_n^a - n\tau_n^a}{n^2-1} \right] r_{\mathrm{o}} \cos n\theta + \left[\frac{n\,\sigma_n^b + \tau_n^b}{n(n^2-1)} + \frac{\tau_n^b}{n} \right] r_{\mathrm{o}} \sin n\theta \right\}$$

$$M^{\mathrm{s}} = - \sum_{n=2,4,6}^{\infty} \left\{ \left[\frac{n\,\sigma_n^a - \tau_n^a}{n(n^2-1)} \right] r_{\mathrm{o}}^2 \cos n\theta + \left[\frac{n\,\sigma_n^b + \tau_n^b}{n(n^2-1)} \right] r_{\mathrm{o}}^2 \sin n\theta \right\}$$

$$\begin{aligned} U_x^{\mathrm{IV}} = {} & \frac{1-\nu_{\mathrm{s}}^2}{E_{\mathrm{s}}\left(I_{\mathrm{s}} + r_{\mathrm{o}}^2 A_{\mathrm{s}}\right)} r_{\mathrm{o}}^4 \sigma_{\mathrm{o}} \cos\theta - C \sin\theta + \\ & \frac{1}{2}\frac{1-\nu_{\mathrm{s}}^2}{E_{\mathrm{s}} I_{\mathrm{s}}} r_{\mathrm{o}}^2 \left\{ \frac{1}{12}\left[\left(2\sigma_2^a - \tau_2^a\right) r_{\mathrm{o}}^2 - 3\frac{I_{\mathrm{s}}}{A_{\mathrm{s}}} \tau_2^a \right] \cos\theta + \frac{1}{12}\left[\left(2\sigma_2^b + \tau_2^b\right) r_{\mathrm{o}}^2 + 3\frac{I_{\mathrm{s}}}{A_{\mathrm{s}}} \tau_2^b \right] \sin\theta + \right. \\ & \sum_{n=3,5,7}^{\infty} \left[\left(\frac{(n-1)\sigma_{n-1}^a - \tau_{n-1}^a}{n^2(n-1)^2(n-2)} + \frac{(n+1)\sigma_{n+1}^a - \tau_{n+1}^a}{n^2(n+1)^2(n+2)} \right) r_{\mathrm{o}}^2 + \frac{I_s}{A_s}\left(\frac{\tau_{n-1}^a}{(n-1)^2} - \frac{\tau_{n+1}^a}{(n+1)^2} \right) \right] \cos n\theta + \\ & \left. \sum_{n=3,5,7}^{\infty} \left[\left(\frac{(n-1)\sigma_{n-1}^b + \tau_{n-1}^b}{n^2(n-1)^2(n-2)} + \frac{(n+1)\sigma_{n+1}^b + \tau_{n+1}^b}{n^2(n+1)^2(n+2)} \right) r_{\mathrm{o}}^2 - \frac{I_s}{A_s}\left(\frac{\tau_{n-1}^b}{(n-1)^2} - \frac{\tau_{n+1}^b}{(n+1)^2} \right) \right] \sin n\theta \right\} \end{aligned} \tag{15}$$

$$\begin{aligned} U_y^{\mathrm{IV}} = {} & \frac{1-\nu_{\mathrm{s}}^2}{E_{\mathrm{s}}\left(I_{\mathrm{s}} + r_{\mathrm{o}}^2 A_{\mathrm{s}}\right)} r_{\mathrm{o}}^4 \sigma_{\mathrm{o}} \sin\theta + C \cos\theta + \\ & \frac{1}{2}\frac{1-\nu_{\mathrm{s}}^2}{E_{\mathrm{s}} I_{\mathrm{s}}} r_{\mathrm{o}}^2 \left\{ -\frac{1}{12}\left[\left(2\sigma_2^a - \tau_2^a\right) r_{\mathrm{o}}^2 - 3\frac{I_{\mathrm{s}}}{A_{\mathrm{s}}} \tau_2^a \right] \sin\theta + \frac{1}{12}\left[\left(2\sigma_2^b + \tau_2^b\right) r_{\mathrm{o}}^2 + 3\frac{I_{\mathrm{s}}}{A_{\mathrm{s}}} \tau_2^b \right] \cos\theta + \right. \\ & \sum_{n=3,5,7}^{\infty} \left[\left(\frac{(n-1)\sigma_{n-1}^a - \tau_{n-1}^a}{n^2(n-1)^2(n-2)} - \frac{(n+1)\sigma_{n+1}^a - \tau_{n+1}^a}{n^2(n+1)^2(n+2)} \right) r_{\mathrm{o}}^2 + \frac{I_s}{A_s}\left(\frac{\tau_{n-1}^a}{(n-1)^2} + \frac{\tau_{n+1}^a}{(n+1)^2} \right) \right] \sin n\theta - \\ & \left. \sum_{n=3,5,7}^{\infty} \left[\left(\frac{(n-1)\sigma_{n-1}^b + \tau_{n-1}^b}{n^2(n-1)^2(n-2)} - \frac{(n+1)\sigma_{n+1}^b + \tau_{n+1}^b}{n^2(n+1)^2(n+2)} \right) r_{\mathrm{o}}^2 - \frac{I_s}{A_s}\left(\frac{\tau_{n-1}^b}{(n-1)^2} + \frac{\tau_{n+1}^b}{(n+1)^2} \right) \right] \cos n\theta \right\} \end{aligned}$$

where C is a constant to be determined in the same manner as the other constants.

In (9) the conditions are imposed term-by-term, i.e. for the constant term, for the terms with $\sin\theta$, $\cos\theta$, $\sin 2\theta$, $\cos 2\theta$, and so forth. The result is a system of equations that, when solved, provides the values of the constants $\sigma_{\mathrm{o}}, \sigma_n^a, \sigma_n^b, \tau_n^a$ and τ_n^b. Note that the constants $\sigma_{\mathrm{o}}, \sigma_n^a$ and τ_n^a depend on the far-field stresses σ_v and σ_h, while the constants σ_n^b and τ_n^b depend on the far-field shear stress τ_{vh}.

As shown by Bobet (2011), the above formulation also applies to tunnels below the water table when the far-field stresses are input in total stresses.

3 Tunnel in Saturated Rock Subjected to Undrained Loading

The tunnel is subjected to a far-field undrained loading with the excess pore pressures not allowed to dissipate. The following expressions for strains apply, after Bobet (2011):

$$\begin{aligned} \varepsilon_x &= \left(\alpha_1 - \frac{\beta_1^2}{\beta_3}\right)\sigma_x - \left(\alpha_2 + \frac{\beta_1\beta_2}{\beta_3}\right)\sigma_y \\ \varepsilon_y &= -\left(\alpha_2 + \frac{\beta_1\beta_2}{\beta_3}\right)\sigma_x + \left(\alpha_3 - \frac{\beta_2^2}{\beta_3}\right)\sigma_y \\ \gamma &= \frac{\tau_{xy}}{G_{xy}} \\ \beta_3 &= \frac{1}{M} + \alpha_x\beta_1 + \alpha_y\beta_2\,. \\ u &= -\frac{1}{\beta_3}\left(\beta_1\sigma_x + \beta_2\sigma_y\right) \end{aligned} \tag{16}$$

where M is Biot's modulus, defined as the increase of the amount of fluid per unit volume of rock as a result of a unit increase of pore pressure under constant volumetric strain.

The compatibility equation and the characteristic equation now take the following form:

$$\begin{aligned} &\left(\alpha_1 - \frac{\beta_1^2}{\beta_3}\right)\frac{\partial^4 F}{\partial y^4} + \left(\alpha_3 - \frac{\beta_2^2}{\beta_3}\right)\frac{\partial^4 F}{\partial x^4} + \left(\frac{1}{G_{xy}} - 2\alpha_2 - 2\frac{\beta_1\beta_2}{\beta_3}\right)\frac{\partial^4 F}{\partial x^2 \partial y^2} = 0 \\ &\left(\alpha_1 - \frac{\beta_1^2}{\beta_3}\right)\mu_k^4 + \left(\frac{1}{G_{xy}} - 2\alpha_2 - 2\frac{\beta_1\beta_2}{\beta_3}\right)\mu_k^2 + \left(\alpha_3 - \frac{\beta_2^2}{\beta_3}\right) = 0 \end{aligned} \tag{17}$$

Total stresses are given by (5) with $F_o = 0$, and displacements by

$$
\begin{aligned}
U_x &= 2\,\mathrm{Re}\left\{\left[\left(\alpha_1 - \frac{\beta_1^2}{\beta_3}\right)\mu_1^2 - \left(\alpha_2 + \frac{\beta_1\beta_2}{\beta_3}\right)\right]\phi_1(z_1)\right.\\
&\quad \left. + \left[\left(\alpha_1 - \frac{\beta_1^2}{\beta_3}\right)\mu_2^2 - \left(\alpha_2 + \frac{\beta_1\beta_2}{\beta_3}\right)\right]\phi_2(z_2)\right\}\\
U_y &= 2\,\mathrm{Re}\left\{\left[-\left(\alpha_2 + \frac{\beta_1\beta_2}{\beta_3}\right)\mu_1 + \left(\alpha_3 - \frac{\beta_2^2}{\beta_3}\right)\frac{1}{\mu_1}\right]\phi_1(z_1)\right.\\
&\quad \left. + \left[-\left(\alpha_2 + \frac{\beta_1\beta_2}{\beta_3}\right)\mu_2 + \left(\alpha_3 - \frac{\beta_2^2}{\beta_3}\right)\frac{1}{\mu_2}\right]\phi_2(z_2)\right\}
\end{aligned}
\tag{18}
$$

As with the case of dry ground, Bobet (2011) found the solution of the problem when the roots of the characteristic equation in (17) are pure imaginary numbers. Similar to the previous case, a solution of the form $\mu_1 = a + ib$, $\mu_2 = -\bar{\mu}_1$ exists. The following provides the formulation for this solution.

Figure 2 can also be used to decompose the problem into four different problems. Compatibility of displacements at the rock-liner interface requires

$$
\begin{aligned}
U_x^{\mathrm{IV}} &= U_x^{\mathrm{I}} + U_x^{\mathrm{II}} + U_x^{\mathrm{III}}\\
U_y^{\mathrm{IV}} &= U_y^{\mathrm{I}} + U_y^{\mathrm{II}} + U_y^{\mathrm{III}}
\end{aligned}
\tag{19}
$$

Problem I has the solution:

$$
\begin{aligned}
\sigma_x &= \sigma_h\\
\sigma_y &= \sigma_v\\
\tau_{xy} &= \tau_{vh}\\
u &= -\frac{1}{\beta_3}\left(\beta_1\sigma_h + \beta_2\sigma_v\right)\\
U_x &= \left[\left(\alpha_1 - \frac{\beta_1^2}{\beta_3}\right)\sigma_h - \left(\alpha_2 + \frac{\beta_1\beta_2}{\beta_3}\right)\sigma_v\right]x + \frac{1}{2}\frac{\tau_{vh}}{G_{xy}}y\\
U_y &= \left[-\left(\alpha_2 + \frac{\beta_1\beta_2}{\beta_3}\right)\sigma_h + \left(\alpha_3 - \frac{\beta_2^2}{\beta_3}\right)\sigma_v\right]y + \frac{1}{2}\frac{\tau_{vh}}{G_{xy}}x
\end{aligned}
\tag{20}
$$

The solution of Problem II is given by Eqs. (10) and (11) with $A_1 = \alpha_1 - \frac{\beta_1^2}{\beta_3}$; $A_2 = \alpha_2 + \frac{\beta_1\beta_2}{\beta_3}$. The solution of Problems III and IV is given by Eqs. (12), (13) and (15).

4 Summary

The Technical Note complements the work done by Bobet (2011) who provided closed-form solutions for a deep circular tunnel in an elastic transversely anisotropic rock. The following assumptions also apply: thin elastic isotropic liner; tied liner-rock interface; simultaneous excavation and support; and plane strain conditions on sections perpendicular to the tunnel axis. The original formulation was developed for pure imaginary roots of the characteristic equation obtained from imposing strain compatibility. The characteristic equation has another solution of the form $\mu_1 = a + ib$, $\mu_2 = -\bar{\mu}_1$. New formulations for stresses and displacements of the rock and liner have been provided that complete those provided earlier. Both formulations, that of Bobet (2011) and the one presented here, complement each other and cover all possible cases.

References

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