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Characterizing Klein–Fock–Gordon–Majorana Particles in (1 + 1) Dimensions

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Abstract Theoretically, in (1 + 1) dimensions, one can have Klein–Fock–Gordon–Majorana (KFGM) particles. More precisely, these are one-dimensional (1D) Klein–Fock–Gordon (KFG) and Majorana particles at the same time. In principle, the wave equations considered to describe such first-quantized particles are the standard 1D KFG equation and/or the 1D Feshbach–Villars (FV) equation, each with a real Lorentz scalar potential and some kind of Majorana condition. The aim of this paper is to analyze the latter assumption fully and systematically; additionally, we introduce specific equations and boundary conditions to characterize these particles when they lie within an interval (or on a line with a tiny hole at a point). In fact, we write first-order equations in the time derivative that do not have a Hamiltonian form. We may refer to these equations as first-order 1D Majorana equations for 1D KFGM particles. Moreover, each of them leads to a second-order equation in time that becomes the standard 1D KFG equation when the scalar potential is independent of time. Additionally, we examine the nonrelativistic limit of one of the first-order 1D Majorana equations.

1 Introduction

In (3 + 1) dimensions, there is the possibility that a spin-0 particle is its own antiparticle. A typical example of this is the neutral pion (or neutral pi meson) Π^0 (although it is not exactly an elementary particle) [1,2]. We may refer to these particles as three-dimensional (3D) Klein–Fock–Gordon–Majorana (KFGM) particles. We recall that, in general, a Majorana particle is its own antiparticle, i.e., it is a strictly neutral particle [3], and the wavefunction that characterizes it is invariant under the respective charge-conjugation operation [4–6] (this is, in principle, within a phase factor). The latter specific fact is what defines a Majorana particle and is called the Majorana condition. Among the known spin-$\frac{1}{2}$ particles, only neutrinos could be of a Majorana nature, i.e., only neutrinos could be Majorana fermions [7]. Similarly, because photons (spin-1) and gravitons (spin-2) are also strictly neutral particles, we may say that they are also of a Majorana nature [7,8]. Incidentally, completely different types of Majorana particles can even be found in certain condensed-matter systems described in the second quantization formalism. These particles emerge as quasiparticles excitations that are their own antiparticles and whose statistics is not fermionic [7,9–12]. Incidentally, these excitations have been called Majorinos [13,14].

Thus, in the first quantization, we may say that the primordial wave equation intended to describe a strictly neutral 3D KFG spin-0 particle, i.e., a 3D KFGM spin-0 particle, is the 3D Klein–Fock–Gordon (KFG) equation in its standard form with a real-valued Lorentz scalar interaction (or potential) [1,2,15–17], but in addition, together with some Majorana conditions. Likewise, the 3D Feshbach–Villars (FV) equation with the scalar potential and the respective Majorana condition may also be used [18]. Naturally, this way of characterizing a 3D KFGM particle can also be implemented to describe a 1D KFGM particle (the latter is a KFGM particle

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living in one space and one time dimension). Thus, in this case, we may also use the 1D KFG equation in its standard form and/or the 1D FV equation, both with a real-valued scalar potential together with their respective Majorana condition.

The subject of neutral 3D KFG particles in the first quantization (as well as in the second quantization) is generally mentioned in books on relativistic quantum mechanics [1–3,19]. Additionally, a neutral 3D KFG particle may not be equal to its antiparticle, for example, a neutral K^0 meson (or neutral kaon) is different from its antiparticle $\overline{\mathrm{K}}^0$. In this case, these two particles carry different internal attributes (in fact, different hypercharges) and can be described by (classical) complex fields or complex solutions of the standard 3D KFG equation [1,20]. However, if a neutral 3D KFG particle is equal to its antiparticle, then there are no internal attributes that distinguish them; consequently, they must be described by (classical) real fields or real solutions of the standard 3D KFG equation, as usual [1,3,7]. Incidentally, we have also seen some references that roughly question the use of real solutions to describe a strictly neutral 3D KFG particle (see, for example, Refs. [19,21–23]).

Actually, there is a well-known connection between the complexity of any solution of the standard 3D KFG equation (which is a Lorentz scalar) and the internal attribute electric charge. That is, if a solution, i.e., ψ, describes the particle, then its complex conjugate ψ^* describes the antiparticle [20]. In fact, the usual densities $\varrho = \varrho(\mathbf{r}, t) = (\mathrm{i}\hbar/2\mathrm{m}c^2)(\psi^* \partial_t \psi - \psi\, \partial_t \psi^*) - (V/\mathrm{m}c^2)\psi^*\psi$ and $\mathbf{j} = \mathbf{j}(\mathbf{r}, t) = -(\mathrm{i}\hbar/2\mathrm{m})(\psi^* \nabla\psi - \psi\, \nabla\psi^*)$, which satisfy a continuity equation (i.e., $\partial_t\varrho + \nabla \cdot \mathbf{j} = 0$), change sign when the replacements $\psi \to \psi^*$, $\psi^* \to \psi$ and $V \to -V$ are made (V is a real potential). Incidentally, the latter result is also valid when there is additionally a (real) Lorentz scalar potential (because ϱ and $\mathbf{j}$ do not depend on this type of potential). Consequently, it is clear that for real solutions, $\psi = \psi^*$ (with $V = 0$), ϱ and $\mathbf{j}$ vanish, i.e., there is no place for a conserved current density four-vector for the strictly neutral 3D KFG particle [3,19]. This conclusion seems to be a general property of other strictly neutral (bosonic) particles [3].

The following is the path that our study follows. We consider the (1+1)-dimensional case only. In the first part of Section 2, we begin by discussing the conditions that must be imposed on the electric and scalar potentials such that the one-component solutions ψ of the standard 1D KFG wave equation can be written as real solutions. We also analyze the existence of complex solutions for this equation, i.e., complex but not pure imaginary solutions. Then, we introduce the 1D KFG wave equation in Hamiltonian form, i.e., the 1D FV wave equation (its solutions Ψ are two-component wavefunctions and the Hamiltonian operator $\hat{\mathrm{h}}$ is a 2×2 matrix), and again, we only include the electric potential and the Lorentz scalar potential. We show that this equation cannot have real solutions regardless of the real or complex nature of the potentials. We also introduce the charge-conjugate wavefunction and the respective Hamiltonian operator for this wavefunction. We show that if Ψ describes a 1D KFG particle in the presence of the potentials V and S, then Ψ_c (its charge-conjugate wavefunction) describes a 1D KFG particle in the potentials $-V^*$ and S^*. We then introduce a first Majorana condition that defines a 1D KFGM particle, namely, $\Psi = \Psi_c$, which suggests that the two components of Ψ are no longer independent. Furthermore, this condition also implies that V must be a purely imaginary potential, i.e., $V = -V^*$ (consequently, V must be zero if it can be considered real), and S must be real. We also obtain the reality condition $\psi = \psi^*$ as a consequence of imposing this Majorana condition. A second Majorana condition that defines a 1D KFGM particle is given by $\Psi = -\Psi_c$, and it imposes the same restrictions on the potentials as the first Majorana condition; however, this time, we have that ψ satisfies the relation $\psi = -\psi^*$, i.e., ψ is purely imaginary but can obviously be written as real by writing $(\psi - \psi^*)/2\mathrm{i} = \psi/\mathrm{i}$. Thus, due to the Majorana conditions, the components of the wavefunction Ψ are always related to each other, which allows us to write equations for only one component and to obtain the other component algebraically. However, these equations are not of the Hamiltonian type, i.e., they do not have the form $(\mathrm{i}\hbar\, \partial_t - \hat{\mathrm{H}})\phi = 0$.

In the second part of Section 2, we show that the further imposition of the formal pseudohermiticity condition on the Hamiltonian $\hat{\mathrm{h}}$ implies that the electric potential satisfies the relation $V = V^*$. The latter formula together with the condition $V = -V^*$ gives us $V = 0$. In fact, if we place a 1D KFGM particle in the interior of an interval, for example, $\Omega = [a, b]$, the operator $\hat{\mathrm{h}}$ with $V = 0$ and a scalar potential $S \in \mathbb{R}$ is a pseudo self-adjoint operator. We show that as a consequence of the latter property, the respective solutions Ψ of the 1D FV wave equation must satisfy any boundary condition that is included in a general set of boundary conditions that depends on three real parameters. Similarly, the respective solutions ψ of the standard 1D KFG equation must satisfy any boundary condition that is included in its own real three-parameter general family of boundary conditions. This is because the solutions ψ must be written as real if they are to describe a 1D KFGM particle. It is worth mentioning that these general sets of boundary conditions are the same for both types of Majorana conditions and the most general for a 1D KFGM particle that is within an interval. In fact, the most general sets of boundary conditions for a 1D KFG particle in an interval were obtained in Ref. [24], and the general sets of boundary conditions for a 1D KFGM particle presented here arise precisely from those. Here, we obtain the most general sets of boundary conditions for each component of the FV wavefunction (these results depend on the type of Majorana condition one chooses). Certainly, each component satisfies its own differential equation, i.e., its own first-order 1D Majorana equation in the time derivative (which we also present here). Additionally, the (complex) solutions of these equations have the following characteristic: If the Majorana condition is given by $\Psi = \Psi_c$ ($\Psi = -\Psi_c$), then the real (imaginary) parts of the solutions satisfy the real second-order 1D KFG equation in time, and the imaginary (real) parts are simply the time derivatives of the real (imaginary) parts. In Section 3, we present a summary and our conclusions. Additionally, in the Appendix A, we show that each component of the 1D FV equation also satisfies its own second-order 1D Majorana differential equation in time; however, these equations become the standard 1D KFG equation when the scalar potential is independent of time. Finally, in the Appendix B, we examine the nonrelativistic limit of one of the first-order 1D Majorana equations.

2 Characterizing a 1D KFGM Particle

2.1 Preliminaries

Let us begin by writing the 1D KFG wave equation in its standard form or the second order in time KFG equation in one spatial dimension,

$$\left[(\hat{\mathrm{E}} - V)^2 - (c\,\hat{\mathrm{p}})^2 - (\mathrm{m}c^2)^2 - 2\,\mathrm{m}c^2 S\right]\psi = 0, \tag{1}$$

where $\psi = \psi(x,t)$ is a one-component wavefunction, $\hat{\mathrm{E}} = \mathrm{i}\hbar\,\partial/\partial t$ is the energy operator, $\hat{\mathrm{p}} = -\mathrm{i}\hbar\,\partial/\partial x$ is the momentum operator, $V = V(x)$ is the electric potential (energy), and $S = S(x,t) \in \mathbb{R}$ is a real-valued Lorentz scalar potential (energy). Thus, the latter equation can also be written as follows:

$$\left[-\hbar^2\frac{\partial^2}{\partial t^2} - 2V\mathrm{i}\hbar\frac{\partial}{\partial t} + V^2 + \hbar^2 c^2\frac{\partial^2}{\partial x^2} - (\mathrm{m}c^2)^2 - 2\,\mathrm{m}c^2 S\right]\psi = 0. \tag{2}$$

Clearly, the differential operator acting on ψ in Eq. (2) is real when the potential V is purely imaginary (this is somewhat trivial but unexpected). Additionally, this operator is real when $V = 0$, i.e., when there is only a (real) scalar potential (throughout the article, we consider S to be a real potential, unless explicitly stated otherwise). Consequently, in these two cases, the solutions of the time-dependent 1D KFG equation in (2) can always be chosen to be real, i.e., this equation can have solutions *à la* Majorana. In these two cases, the solutions of Eq. (2) need not be real, i.e., complex solutions can also be written (naturally, we refer to complex solutions, i.e., we are not thinking of pure imaginary solutions, unless that is explicitly stated). In fact, if $\hat{\mathrm{L}}$ is a real operator, then ψ and ψ^* are solutions of the differential equation $\hat{\mathrm{L}}\psi = 0$, and its real solutions are given by $(\psi + \psi^*)/2$ and $(\psi - \psi^*)/2\mathrm{i}$ (the superscript * denotes the complex conjugate). Certainly, it is also important to note that whether the electric potential $V \in \mathbb{R}$ is not zero and $S \in \mathbb{R}$ exists (or not), the differential operator acting on ψ in Eq. (2) is complex, and therefore, the solutions of the time-dependent 1D KFG equation in (2) are necessarily complex.

Now, let us introduce the following functions [18]:

$$\varphi + \chi = \psi \tag{3}$$

and

$$\varphi - \chi = \frac{1}{\mathrm{m}c^2}(\hat{\mathrm{E}} - V)\psi. \tag{4}$$

By using Eqs. (3), (4) and (1), we obtain a system of coupled differential equations for φ and χ, namely,

$$\hat{\mathrm{E}}\,\varphi = \left(\frac{\hat{\mathrm{p}}^2}{2\mathrm{m}} + S\right)(\varphi + \chi) + (\mathrm{m}c^2 + V)\varphi, \tag{5}$$

$$\hat{\mathrm{E}}\,\chi = -\left(\frac{\hat{\mathrm{p}}^2}{2\mathrm{m}} + S\right)(\varphi + \chi) - (\mathrm{m}c^2 - V)\chi. \tag{6}$$

The latter system can be written in matrix form, namely,

$$\hat{\mathrm{E}}\,\hat{1}_2\Psi = \hat{\mathrm{h}}\Psi, \tag{7}$$

where

$$\hat{\mathrm{h}} = \frac{\hat{\mathrm{p}}^2}{2\mathrm{m}}(\hat{\tau}_3 + \mathrm{i}\hat{\tau}_2) + \mathrm{m}c^2\hat{\tau}_3 + V\,\hat{1}_2 + S\,(\hat{\tau}_3 + \mathrm{i}\hat{\tau}_2) \tag{8}$$

is the Hamiltonian operator, $\Psi = \Psi(x,t) = [\,\varphi\;\;\chi\,]^{\mathrm{T}} = [\,\varphi(x,t)\;\;\chi(x,t)\,]^{\mathrm{T}}$ is the two-component column state vector (the symbol $^{\mathrm{T}}$ represents the transpose of a matrix), $\hat{\tau}_3 = \hat{\sigma}_z$ and $\hat{\tau}_2 = \hat{\sigma}_y$ are Pauli matrices, and $\hat{1}_2$ is the 2×2 identity matrix. The equation in (7), with $\hat{\mathrm{h}}$ given in Eq. (8), is the 1D FV wave equation with an electric potential and a scalar potential (and is also called the 1D KFG equation in Hamiltonian form) [18]. Although this equation has not received sufficient attention when addressing problems within the 1D KFG theory in (1+1) dimensions, a few papers in which it is considered can be found in Refs. [25–27]. By using Eqs. (3) and (4), we can explicitly write the relation between the one-component wavefunction ψ and the two-component column state vector (or wavefunction) Ψ, namely,

$$\Psi = \begin{bmatrix}\varphi\\ \chi\end{bmatrix} = \frac{1}{2}\begin{bmatrix}\psi + \frac{1}{\mathrm{m}c^2}(\hat{\mathrm{E}} - V)\psi\\ \psi - \frac{1}{\mathrm{m}c^2}(\hat{\mathrm{E}} - V)\psi\end{bmatrix}. \tag{9}$$

Certainly, in this last expression, the scalar potential does not appear. We note that even if ψ is a real function, the components of Ψ, φ and χ, are always complex, i.e., Ψ is inexorably complex. In fact, if one has a wave equation of the form $(\hat{\mathrm{E}} - \hat{\mathrm{H}})\Phi = 0$, then one also has that $(\hat{\mathrm{E}} - \hat{\mathrm{H}})^*\Phi = 0$ if $\mathrm{i}\hat{\mathrm{H}} = (\mathrm{i}\hat{\mathrm{H}})^*$, i.e., if $\mathrm{i}\hat{\mathrm{H}}$ is a real operator. This result tells us that the time-dependent 1D FV wave equation cannot have real solutions. Clearly, the same goes for the time-dependent Schrödinger equation. That is, these time-dependent wave equations cannot have real solutions.

The charge conjugate of Ψ,

$$\Psi_c \equiv \hat{\tau}_1\Psi^*, \tag{10}$$

where $\hat{\tau}_1 = \hat{\sigma}_x$ is a Pauli matrix, satisfies the following equation:

$$\hat{\mathrm{E}}\,\hat{1}_2\Psi_c = \hat{\mathrm{h}}_c\Psi_c, \tag{11}$$

where the respective Hamiltonian operator is given by

$$\hat{\mathrm{h}}_c \equiv \frac{\hat{\mathrm{p}}^2}{2\mathrm{m}}(\hat{\tau}_3 + \mathrm{i}\hat{\tau}_2) + \mathrm{m}c^2\hat{\tau}_3 + V_c\,\hat{1}_2 + S_c\,(\hat{\tau}_3 + \mathrm{i}\hat{\tau}_2). \tag{12}$$

Taking the complex conjugate of Eq. (7), with $\hat{\mathrm{h}}$ given in Eq. (8), also using the results $\hat{\mathrm{E}}^* = -\hat{\mathrm{E}}$ and $(\hat{\mathrm{p}}^2)^* = \hat{\mathrm{p}}^2$, and the facts that $(\hat{\tau}_3 + \mathrm{i}\hat{\tau}_2)^* = (\hat{\tau}_3 + \mathrm{i}\hat{\tau}_2)$, $(\hat{\tau}_3)^* = \hat{\tau}_3$, and $\hat{\tau}_1\hat{\tau}_3 = -\hat{\tau}_3\hat{\tau}_1$, $\hat{\tau}_1\hat{\tau}_2 = -\hat{\tau}_2\hat{\tau}_1$ ($\Rightarrow (\hat{\tau}_3 + \mathrm{i}\hat{\tau}_2)\hat{\tau}_1 = -\hat{\tau}_1(\hat{\tau}_3 + \mathrm{i}\hat{\tau}_2)$), and $\hat{\tau}_1^2 = \hat{1}_2$, and finally, using the definition of Ψ_c in Eq. (10), we obtain the same Eq. (11), but $\hat{\mathrm{h}}_c$ is given by

$$\hat{\mathrm{h}}_c = \frac{\hat{\mathrm{p}}^2}{2\mathrm{m}}(\hat{\tau}_3 + \mathrm{i}\hat{\tau}_2) + \mathrm{m}c^2\hat{\tau}_3 - V^*\,\hat{1}_2 + S\,(\hat{\tau}_3 + \mathrm{i}\hat{\tau}_2). \tag{13}$$

Comparing the latter operator with the operator given in Eq. (12), we obtain the following two relations:

$$V_c = -V^*\,, \quad S_c = S. \tag{14}$$

If we had considered placing a complex scalar potential in the Hamiltonian $\hat{\mathrm{h}}$ given in Eq. (8), then the Hamiltonian $\hat{\mathrm{h}}_c$ would be the one given in Eq. (13) but with the replacement $S \to S^*$, and therefore, the corresponding relation in Eq. (14) would be $S_c = S^*$.

Equation (7) describes via Ψ a 1D KFG particle in the presence of the potentials V and S. Likewise, Eq. (11) describes via Ψ_c a 1D KFG particle in the presence of the potentials $-V^*$ and S. For example, if one sets $V \in \mathbb{R}$ and $S \in \mathbb{R}$, then one has that $\hat{\mathrm{E}}\,\hat{1}_2\Psi = \hat{\mathrm{h}}(V)\Psi$ (see Eq. 8) and $\hat{\mathrm{E}}\,\hat{1}_2\Psi_c = \hat{\mathrm{h}}_c(V)\Psi_c = \hat{\mathrm{h}}(-V)\Psi_c$ (see

Eq. 13), i.e., Ψ describes a 1D KFG particle with one sign of electric charge, and Ψ_c describes the 1D KFG particle with the opposite sign of electric charge (i.e., its antiparticle).

Now, let us explore the possibility that a 1D KFG particle is its own antiparticle; therefore, it must be an electrically and strictly neutral particle. The condition that defines a particle of this type is customarily given by

$$\Psi = \Psi_c. \tag{15}$$

We refer to this relation as the standard Majorana condition. The latter condition imposes the following relation between the components of Ψ: $\varphi = \varphi_c = \chi^*$ ($\Leftrightarrow \chi = \chi_c = \varphi^*$). Hence, $\Psi = \left[\, \chi^* \;\; \chi \,\right]^{\mathrm{T}}$ ($\Leftrightarrow \Psi = \left[\, \varphi \;\; \varphi^* \,\right]^{\mathrm{T}}$), i.e., φ and χ are not independent. Then, comparing the 1D FV wave equations for Ψ (see Eqs. 7 and 8) and Ψ_c $(= \Psi)$ (see Eqs. 11 and 12), and by using the relations in Eq. (14), we obtain

$$V = -V^*. \tag{16}$$

That is, the complex potential V must be purely imaginary, but if V had been chosen as real, then V must be zero (because $V = -V$). We already know that the potential S is real valued, but if from the beginning we had decided to place a complex potential S, then, in addition to the relation given in Eq. (16), we can obtain $S = S^*$ (because $S_c = S^*$ and we also used the Majorana condition). That is, the Majorana condition imposes that S be a real scalar potential (see the comment following Eq. (14)). Thus, in principle, the 1D FV wave equation describing a 1D KFG particle that is also a one-dimensional Majorana particle, i.e., a 1D KFGM particle, can be written as follows:

$$\hat{\mathrm{E}}\, \hat{1}_2 \Psi = \hat{\mathrm{h}} \Psi = \left[\frac{\hat{\mathrm{p}}^2}{2\mathrm{m}} (\hat{\tau}_3 + \mathrm{i}\hat{\tau}_2) + \mathrm{m}c^2 \hat{\tau}_3 + V\, \hat{1}_2 + S\, (\hat{\tau}_3 + \mathrm{i}\hat{\tau}_2) \right] \Psi, \tag{17}$$

where if $V \in \mathbb{C}$, then it must be imaginary; and if $V \in \mathbb{R}$, then it must be zero. Likewise, the Lorentz scalar potential S must be real. Additionally, the wavefunction Ψ must have the form $\Psi = \left[\, \chi^* \;\; \chi \,\right]^{\mathrm{T}}$ or $\Psi = \left[\, \varphi \;\; \varphi^* \,\right]^{\mathrm{T}}$.

Equivalently, it should be noted that the 1D FV wave equation given in Eqs. (7) and (8), or better, the coupled system of equations given in Eqs. (5) and (6), is invariant under the following substitution:

$$\Psi = [\, \varphi \;\; \chi \,]^{\mathrm{T}} \;\rightarrow\; \Psi_c = \left[\, \chi^* \;\; \varphi^* \,\right]^{\mathrm{T}}, \tag{18}$$

but the conditions $V = -V^*$ and $S = S^*$ must be satisfied. In other words, if the latter conditions are satisfied, then Ψ and Ψ_c satisfy the same equation (the latter is the equation for the 1D KFGM particle, namely, Eq. 17).

Now, let us study the consequences of imposing the Majorana condition given in Eq. (15) on the two-component column vector Ψ given in Eq. (9). Because $\hat{\mathrm{E}}^* = -\hat{\mathrm{E}}$, and $V^* = -V$ for the 1D KFGM particle (see Eq. 16), it follows that

$$\Psi = \frac{1}{2} \begin{bmatrix} \psi + \frac{1}{\mathrm{m}c^2} (\hat{\mathrm{E}} - V) \psi \\ \psi - \frac{1}{\mathrm{m}c^2} (\hat{\mathrm{E}} - V) \psi \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \frac{1}{2} \begin{bmatrix} \psi^* - \frac{1}{\mathrm{m}c^2} (\hat{\mathrm{E}} - V) \psi^* \\ \psi^* + \frac{1}{\mathrm{m}c^2} (\hat{\mathrm{E}} - V) \psi^* \end{bmatrix} = \hat{\tau}_1 \Psi^* = \Psi_c,$$

from which one immediately obtains the result

$$\psi = \psi^*, \tag{19}$$

which is the reality condition for the wavefunction ψ and can be considered the standard Majorana condition for these solutions, i.e., $\psi = \psi^* \equiv \psi_c$, where ψ_c is the charge conjugate of ψ [28]. The latter relation also arises immediately when using Eq. (3) and the Majorana condition in terms of the components of Ψ (i.e., $\varphi = \chi^*$), namely, $\psi = \varphi + \chi = \chi^* + \varphi^* = (\chi + \varphi)^* = \psi^*$. Thus, if the solutions of the standard 1D KGF wave equation given in Eq. (2) describe a 1D KFGM particle; then, they must be real. However, V must be an imaginary potential (see Eq. 16), or zero (if $V \in \mathbb{R}$), and S must be a real potential. Consistently, when the latter conditions on the potentials are imposed on the operator acting on ψ in Eq. (2), the operator is real, and the solutions ψ can be written real. Obviously, the solutions of the time-dependent 1D FV wave equation (17), Ψ, do not have to be real to describe a 1D KFGM particle; as we know, these solutions are not real even when $\psi = \psi^*$ (see the comment that follows Eq. (9)). The latter is a situation somewhat similar to that which occurs in Dirac theory in (1+1) dimensions. Indeed, only in the Majorana representation can the solutions of the Dirac equation describing a 1D Dirac-Majorana particle be real valued, but in any other representation, the solutions of the Dirac equation describing this particle are complex valued. Certainly, all these solutions satisfy the Majorana condition [29].

Similarly, the 1D FV wave equation is invariant under the substitution

$$\Psi = [\,\varphi \;\; \chi\,]^{\mathrm{T}} \;\rightarrow\; -\Psi_c = \left[\,-\chi^* \;\; -\varphi^*\,\right]^{\mathrm{T}}, \tag{20}$$

and again, the conditions $V = -V^*$ and $S = S^*$ must be satisfied. That is, Ψ and $-\Psi_c$ satisfy the same equation, but this time, we obtain the result $\psi = -\psi^*$ (to prove this, one can use the same procedure that led us to Eq. (19)). Thus, in this case, the solutions ψ are imaginary, but they can be written as real simply by writing $(\psi - \psi^*)/2\mathrm{i} = \psi/\mathrm{i}$. In conclusion, the Majorana condition, which relates the two-component wavefunction Ψ to its charge-conjugate state Ψ_c (or Ψ_c to its charge-conjugate state Ψ), appears here in two forms, one standard form, $\Psi = \Psi_c$, and, say, one nonstandard form, $\Psi = -\Psi_c$. In both cases, the one-component solution ψ can (and must) be written real, but additionally, the potentials must satisfy the conditions $V = -V^*$ and $S = S^*$. Thus, the Majorana condition $\Psi = -\Psi_c$ imposes the following relation between the components of Ψ: $\varphi = -\varphi_c = -\chi^*$ ($\Leftrightarrow \chi = -\chi_c = -\varphi^*$); hence, $\Psi = \left[\,-\chi^* \;\; \chi\,\right]^{\mathrm{T}}$ ($\Leftrightarrow \Psi = \left[\,\varphi \;\; -\varphi^*\,\right]^{\mathrm{T}}$). Using the relation $\varphi = -\chi^*$ and Eq. (3), the condition $\psi = -\psi^*$ can also be obtained, namely, $\psi = \varphi + \chi = -\chi^* - \varphi^* = -(\chi + \varphi)^* = -\psi^*$. In principle, the existence of two Majorana conditions defines two specific and different types of 1D KFGM particles (this is the case for 3D KFGM particles). When $\Psi = \Psi_c$, we have $+\Psi = \Psi_c = \hat{\tau}_1 \Psi^* \equiv \hat{\mathrm{C}}\Psi$, i.e., $\hat{\mathrm{C}}\Psi = +\Psi$, and when $-\Psi = \Psi_c$, we have $-\Psi = \Psi_c = \hat{\tau}_1 \Psi^* \equiv \hat{\mathrm{C}}\Psi$, i.e., $\hat{\mathrm{C}}\Psi = -\Psi$. Then, for one of these particles, its respective wavefunction is an eigenfunction of the charge conjugation transformation (or operator) $\hat{\mathrm{C}}$ with the eigenvalue $+1$, and for the other particle, its wavefunction is an eigenfunction of $\hat{\mathrm{C}}$ with the eigenvalue -1 [2,18]. For example, the wavefunction corresponding to the 3D KFGM neutral pion Π^0 is an eigenfunction of $\hat{\mathrm{C}}$ with eigenvalue $+1$, i.e., the so-called C-parity of this particle is $+1$ [18].

Thus far, the wave equation that we have considered to describe a 1D KFGM particle is Eq. (17). Because the components φ and χ of Ψ are not independent, one can write an equation for only one of these components and can obtain the other component algebraically. In effect, Eq. (17) is the system of equations given in Eqs. (5) and (6) with $V = -V^*$ and $S = S^*$. Then, if we take Eq. (5) and use the Majorana condition $\Psi = \Psi_c$, namely, $\chi = \varphi^*$, we obtain the following wave equation for the 1D KFGM particle:

$$\hat{\mathrm{E}}\,\varphi = \left(\frac{\hat{\mathrm{p}}^2}{2\mathrm{m}} + S\right)(\varphi + \varphi^*) + (\mathrm{m}c^2 + V)\varphi. \tag{21}$$

From the solution φ of the latter equation, the two-component wavefunction Ψ can be written immediately, namely, $\Psi = [\,\varphi \;\; \chi\,]^{\mathrm{T}} = \left[\,\varphi \;\; \varphi^*\,\right]^{\mathrm{T}} = \Psi_c$. Alternatively, if we take Eq. (6) and use the Majorana condition $\varphi = \chi^*$, we obtain an equation for the one-component wavefunction χ, namely,

$$\hat{\mathrm{E}}\,\chi = -\left(\frac{\hat{\mathrm{p}}^2}{2\mathrm{m}} + S\right)(\chi + \chi^*) - (\mathrm{m}c^2 - V)\chi, \tag{22}$$

and from the solution χ of the latter equation, we can write $\Psi = [\,\varphi \;\; \chi\,]^{\mathrm{T}} = \left[\,\chi^* \;\; \chi\,\right]^{\mathrm{T}} = \Psi_c$. Certainly, it is sufficient to solve at least one of these one-component equations because χ and φ are algebraically related. Thus, it can be said that Eq. (21) alone, or Eq. (22) alone, describe this kind of 1D KFGM particle. Similarly, if we now use the Majorana condition $\Psi = -\Psi_c$, i.e., $\varphi = -\chi^*$ (or $\chi = -\varphi^*$), we can write the following wave equation for this 1D KFGM particle:

$$\hat{\mathrm{E}}\,\varphi = \left(\frac{\hat{\mathrm{p}}^2}{2\mathrm{m}} + S\right)(\varphi - \varphi^*) + (\mathrm{m}c^2 + V)\varphi, \tag{23}$$

in which case the respective wavefunction Ψ is given by $\Psi = [\,\varphi \;\; \chi\,]^{\mathrm{T}} = \left[\,\varphi \;\; -\varphi^*\,\right]^{\mathrm{T}} = -\Psi_c$. Alternatively, we can write the following wave equation:

$$\hat{\mathrm{E}}\,\chi = -\left(\frac{\hat{\mathrm{p}}^2}{2\mathrm{m}} + S\right)(\chi - \chi^*) - (\mathrm{m}c^2 - V)\chi, \tag{24}$$

in which case the respective wavefunction Ψ is given by $\Psi = [\,\varphi \;\; \chi\,]^{\mathrm{T}} = \left[\,-\chi^* \;\; \chi\,\right]^{\mathrm{T}} = -\Psi_c$. Certainly, any of these last two equations models the 1D KFGM particle which is characterized by the condition $\Psi = -\Psi_c$. Incidentally, we note that none of these last four equations for the 1D KFGM particle is of the form $(\hat{\mathrm{E}} - \hat{\mathrm{H}})\phi = 0$.

2.2 A 1D KFGM Particle in an Interval

It is important to mention that up to this point, we have not imposed any particular or specific condition on the Hamiltonian $\hat{\mathrm{h}}$, for example, we have not yet imposed on $\hat{\mathrm{h}}$ the condition of formal pseudohermiticity, i.e., $\hat{\mathrm{h}}_{\mathrm{adj}} \equiv \hat{\tau}_3\,\hat{\mathrm{h}}^{\dagger}\,\hat{\tau}_3 = \hat{\mathrm{h}}$ (the symbol † denotes the usual Hermitian conjugate of a matrix and an operator). Effectively, the generalized Hermitian conjugate, or the formal generalized adjoint of $\hat{\mathrm{h}}$, that is, $\hat{\mathrm{h}}_{\mathrm{adj}}$, is defined as

$$\hat{\mathrm{h}}_{\mathrm{adj}} \equiv \hat{\tau}_3\,\hat{\mathrm{h}}^{\dagger}\,\hat{\tau}_3, \tag{25}$$

and is given by

$$\hat{\mathrm{h}}_{\mathrm{adj}} = \frac{\hat{\mathrm{p}}^2}{2\mathrm{m}}(\hat{\tau}_3 + \mathrm{i}\hat{\tau}_2) + \mathrm{m}c^2\hat{\tau}_3 + V^*\,\hat{1}_2 + S\,(\hat{\tau}_3 + \mathrm{i}\hat{\tau}_2). \tag{26}$$

In fact, we introduce the following pseudo inner product [1]:

$$\langle\langle \Psi, \Phi \rangle\rangle \equiv \int_{\Omega} \mathrm{d}x\; \Psi^{\dagger}\hat{\tau}_3\Phi, \tag{27}$$

where $\Omega = [a, b]$ (an interval), and $\Psi = [\,\varphi \;\; \chi\,]^{\mathrm{T}}$ and $\Phi = [\,\zeta \;\; \xi\,]^{\mathrm{T}}$, we can verify that the definition given in Eq. (25) can also be formally written as follows:

$$\langle\langle \hat{\mathrm{h}}_{\mathrm{adj}}\Psi, \Phi \rangle\rangle = \langle\langle \Psi, \hat{\mathrm{h}}\Phi \rangle\rangle. \tag{28}$$

Specifically, this last relation requires only the definitions of $\hat{\mathrm{h}}_{\mathrm{adj}}$ (see Eq. 25) and the scalar product given in Eq. (27). Actually, the relation in Eq. (28) defines the generalized Hermitian conjugate, or the generalized adjoint $\hat{\mathrm{h}}_{\mathrm{adj}}$ on an indefinite inner product space. Then, the Hamiltonian operator in Eq. (17) is formally pseudo-Hermitian or formally generalized Hermitian because it satisfies the following formal relation:

$$\hat{\mathrm{h}}_{\mathrm{adj}} = \hat{\mathrm{h}}. \tag{29}$$

Consequently, the potential V must be real (compare $\hat{\mathrm{h}}$ in Eq. (17) with $\hat{\mathrm{h}}_{\mathrm{adj}}$ in Eq. (26)), i.e., $V = V^*$, and because we want to characterize a 1D KFGM particle (then $V = -V^*$), V must be zero. Indeed, a real or complex electric interaction does not seem to affect a particle that is strictly neutral.

Thus, the 1D FV wave equation that describes a 1D KFGM particle is given by Eq. (17) with $V = 0$ and $S \in \mathbb{R}$, namely,

$$\hat{\mathrm{E}}\,\hat{1}_2\Psi = \hat{\mathrm{h}}\Psi = \left[\frac{\hat{\mathrm{p}}^2}{2\mathrm{m}}(\hat{\tau}_3 + \mathrm{i}\hat{\tau}_2) + \mathrm{m}c^2\hat{\tau}_3 + S\,(\hat{\tau}_3 + \mathrm{i}\hat{\tau}_2)\right]\Psi, \tag{30}$$

where $\Psi = \left[\,\varphi \;\; \varphi^*\,\right]^{\mathrm{T}}$ or $\Psi = \left[\,\chi^* \;\; \chi\,\right]^{\mathrm{T}}$ (with $\Psi = \Psi_c$ being the Majorana condition); equivalently, Eq. (21), or Eq. (22), with $V = 0$ and $S \in \mathbb{R}$, also describes this 1D KFGM particle. Equally, Eq. (30) describes a 1D KFGM particle with $\Psi = \left[\,\varphi \;\; -\varphi^*\,\right]^{\mathrm{T}}$ or $\Psi = \left[\,-\chi^* \;\; \chi\,\right]^{\mathrm{T}}$ (where $\Psi = -\Psi_c$ is the Majorana condition); equivalently, Eq. (23), or Eq. (24), with $V = 0$ and $S \in \mathbb{R}$, also describes this 1D KFGM particle.

By virtue of two integrations by parts, the Hamiltonian operator $\hat{\mathrm{h}}$ in Eq. (30) and its formal generalized adjoint $\hat{\mathrm{h}}_{\mathrm{adj}}$ (which acts as the operator $\hat{\mathrm{h}}$) satisfy the following relation:

$$\langle\langle \hat{\mathrm{h}}_{\mathrm{adj}}\Psi, \Phi \rangle\rangle = \langle\langle \Psi, \hat{\mathrm{h}}\Phi \rangle\rangle - \frac{\hbar^2}{2\mathrm{m}}\frac{1}{2}\left[\left((\hat{\tau}_3 + \mathrm{i}\hat{\tau}_2)\Psi_x\right)^{\dagger}(\hat{\tau}_3 + \mathrm{i}\hat{\tau}_2)\Phi - \left((\hat{\tau}_3 + \mathrm{i}\hat{\tau}_2)\Psi\right)^{\dagger}(\hat{\tau}_3 + \mathrm{i}\hat{\tau}_2)\Phi_x\right]\Big|_a^b, \tag{31}$$

where $[\,g\,]|_a^b \equiv g(b, t) - g(a, t)$, and $\Psi_x \equiv \partial\Psi/\partial x$, etc. It is worth mentioning that the latter relation is also valid if the Hamiltonians $\hat{\mathrm{h}}$ and $\hat{\mathrm{h}}_{\mathrm{adj}}$ contain, in addition to a real scalar potential S, a real electric potential V. Certainly, in this specific case, we are not using the Majorana condition, i.e., we are not considering a 1D

KFGM particle inside an interval (because the latter case requires that $V = 0$), but only a 1D KFG particle in an interval [24].

Then, if the boundary conditions imposed on Ψ and Φ at the ends of interval Ω lead to the vanishing of the boundary term in Eq. (31), the Hamiltonian $\hat{\mathrm{h}}$, formally satisfying Eq. (29) (i.e., $\hat{\mathrm{h}}_{\mathrm{adj}} = \hat{\mathrm{h}}$), is effectively pseudo-Hermitian (or generalized Hermitian). The most general family of boundary conditions leading to the cancellation of the boundary term in Eq. (31) was obtained in Ref. [24]. For all the boundary conditions inside this family, $\hat{\mathrm{h}}$ is a pseudo-Hermitian operator, but it is also a pseudo self-adjoint operator [24], that is, $\hat{\mathrm{h}}$ satisfies the relation

$$\langle\langle \hat{\mathrm{h}}\Psi, \Phi\rangle\rangle = \langle\langle \Psi, \hat{\mathrm{h}}\Phi\rangle\rangle. \tag{32}$$

Thus, the functions belonging to the domains of $\hat{\mathrm{h}}$ and $\hat{\mathrm{h}}_{\mathrm{adj}}$ obey the same boundary conditions, and $\hat{\mathrm{h}}_{\mathrm{adj}} = \hat{\mathrm{h}}$ (in this case, the latter is not just a formal equality). Then, the most general set of pseudo self-adjoint boundary conditions for the Hamiltonian $\hat{\mathrm{h}}$ is given by (for clarity, we omit the variable t in the boundary conditions hereinafter)

$$\begin{bmatrix} (\hat{\tau}_3 + \mathrm{i}\hat{\tau}_2)(\Psi - \mathrm{i}\lambda\Psi_x)(b) \\ (\hat{\tau}_3 + \mathrm{i}\hat{\tau}_2)(\Psi + \mathrm{i}\lambda\Psi_x)(a) \end{bmatrix} = \hat{\mathrm{U}}_{(4\times4)} \begin{bmatrix} (\hat{\tau}_3 + \mathrm{i}\hat{\tau}_2)(\Psi + \mathrm{i}\lambda\Psi_x)(b) \\ (\hat{\tau}_3 + \mathrm{i}\hat{\tau}_2)(\Psi - \mathrm{i}\lambda\Psi_x)(a) \end{bmatrix}, \tag{33}$$

where $\hat{\mathrm{U}}_{(4\times4)}$ is a 4×4 unitary matrix which can be written as follows:

$$\hat{\mathrm{U}}_{(4\times4)} = \hat{\mathrm{S}}^{\dagger} \begin{bmatrix} \hat{\mathrm{U}}_{(2\times2)} & \hat{0} \\ \hat{0} & \hat{\mathrm{U}}_{(2\times2)} \end{bmatrix} \hat{\mathrm{S}}, \tag{34}$$

with

$$\hat{\mathrm{S}} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \tag{35}$$

($\hat{\mathrm{S}}^{\dagger} = \hat{\mathrm{S}}^{-1}$), and as we will see immediately, $\hat{\mathrm{U}}_{(2\times2)}$ is a 2×2 unitary matrix that depends on three real parameters.

The boundary term in Eq. (31) can also be written in terms of the one-component wavefunctions corresponding to the two-component column vectors $\Psi = [\,\varphi \;\; \chi\,]^{\mathrm{T}}$ and $\Phi = [\,\zeta \;\; \xi\,]^{\mathrm{T}}$, namely, $\psi = \varphi + \chi$ and $\phi = \zeta + \xi$ (see Eq. 3), evaluated at the endpoints of the interval Ω. Certainly, the relation given in Eq. (31) can be written as follows:

$$\langle\langle \hat{\mathrm{h}}_{\mathrm{adj}}\Psi, \Phi\rangle\rangle = \langle\langle \Psi, \hat{\mathrm{h}}\Phi\rangle\rangle - \frac{\hbar^2}{2\mathrm{m}} \left[\, \psi_x^* \phi - \psi^* \phi_x \,\right]\Big|_a^b . \tag{36}$$

The boundary term in the latter relation is proportional to the total derivative with respect to time of the pseudo scalar product given in Eq. (27), where Ψ and Φ are solutions of the 1D FV wave equation, that is,

$$-\frac{\hbar^2}{2\mathrm{m}} \left[\, \psi_x^* \phi - \psi^* \phi_x \,\right]\Big|_a^b = \frac{\hbar}{\mathrm{i}} \frac{\mathrm{d}}{\mathrm{d}t} \langle\langle \Psi, \Phi\rangle\rangle. \tag{37}$$

It is also important to note that the pseudo inner product $\langle\langle \Psi, \Phi\rangle\rangle$ in Eq. (37) does not depend on the Lorentz scalar potential S, but it depends on the electric potential V (here, we have $V = 0$). However, its time derivative is independent of the two potentials (provided they are real valued). Then, because $\hat{\mathrm{h}}$ is a pseudo self-adjoint operator, the boundary term in Eq. (36) is zero. The most general family of pseudo self-adjoint boundary conditions for $\hat{\mathrm{h}}$, which is similar for the operator $\hat{\mathrm{h}}_{\mathrm{adj}}$ (because $\hat{\mathrm{h}}_{\mathrm{adj}}$ is equal to $\hat{\mathrm{h}}$, i.e., their actions and domains are equal) and consistent with the cancellation of the boundary term in Eq. (36), is given by

$$\begin{bmatrix} \psi(b) - \mathrm{i}\lambda\psi_x(b) \\ \psi(a) + \mathrm{i}\lambda\psi_x(a) \end{bmatrix} = \hat{\mathrm{U}}_{(2\times2)} \begin{bmatrix} \psi(b) + \mathrm{i}\lambda\psi_x(b) \\ \psi(a) - \mathrm{i}\lambda\psi_x(a) \end{bmatrix}, \tag{38}$$

where $\hat{\mathrm{U}}_{(2\times2)}$ is precisely the 2×2 unitary matrix that appears in Eq. (34) [24]. Let us note that if $\hat{\mathrm{U}}_{(2\times2)}$ is known, by using Eqs. (34) and (35), the matrix $\hat{\mathrm{U}}_{(4\times4)}$ can be calculated immediately. Thus, the pseudo inner product $\langle\langle\Psi, \Phi\rangle\rangle$ in Eq. (37) is constant, i.e., for all the corresponding solutions ψ and ϕ of the standard 1D KFG wave equation that satisfy any of the boundary conditions included in Eq. (38) (see Eqs. 36 and 37).

Naturally, the standard 1D KFG wave equation also describes a 1D KFGM particle when $V = 0$ and $S \in \mathbb{R}$, namely,

$$\left[-\hbar^2 \frac{\partial^2}{\partial t^2} + \hbar^2 c^2 \frac{\partial^2}{\partial x^2} - (\mathrm{m}c^2)^2 - 2\,\mathrm{m}c^2 S\right]\psi = 0 \tag{39}$$

(see Eq. 2), where $\psi = \psi^*$ ($\Psi = \Psi_c$ is the Majorana condition). That is, the solutions of Eq. (39) must be written as real. Furthermore, we can impose the Majorana condition $\Psi = -\Psi_c$, and therefore, $\psi = -\psi^*$; however, these purely imaginary solutions can and should also be written as real. Consequently, when the Majorana condition is given by $\Psi = \Psi_c$, it follows that both ψ and ψ^* satisfy the general boundary condition in Eq. (38), and when the Majorana condition is $\Psi = -\Psi_c$, it follows that both ψ and $-\psi^*$ satisfy it. In these two cases, the matrix $\hat{\mathrm{U}}_{(2\times2)}$ satisfies the following condition:

$$\hat{\mathrm{U}}^{\mathrm{T}}_{(2\times2)} = \hat{\mathrm{U}}_{(2\times2)}, \tag{40}$$

that is, $\hat{\mathrm{U}}_{(2\times2)}$ must additionally be a (complex) symmetric matrix. If we choose the following general expression for $\hat{\mathrm{U}}_{(2\times2)}$:

$$\hat{\mathrm{U}}_{(2\times2)} = \mathrm{e}^{\mathrm{i}\,\mu}\begin{bmatrix} \mathrm{m}_0 - \mathrm{i}\,\mathrm{m}_3 & -\mathrm{m}_2 - \mathrm{i}\,\mathrm{m}_1 \\ \mathrm{m}_2 - \mathrm{i}\,\mathrm{m}_1 & \mathrm{m}_0 + \mathrm{i}\,\mathrm{m}_3 \end{bmatrix}, \tag{41}$$

where $\mu \in [0, \pi)$, and the real quantities m_k ($k = 0, 1, 2, 3$) satisfy $(\mathrm{m}_0)^2 + (\mathrm{m}_1)^2 + (\mathrm{m}_2)^2 + (\mathrm{m}_3)^3 = 1$, and impose on it the condition given in Eq. (40), we obtain the result $\mathrm{m}_2 = 0$. Thus, the most general set of pseudo self-adjoint boundary conditions for a 1D KFGM particle, or for the ultimately real solutions of the 1D KFG wave equation in Eq. (39), is given by

$$\begin{bmatrix} \psi(b) - \mathrm{i}\lambda\psi_x(b) \\ \psi(a) + \mathrm{i}\lambda\psi_x(a) \end{bmatrix} = \mathrm{e}^{\mathrm{i}\,\mu}\begin{bmatrix} \mathrm{m}_0 - \mathrm{i}\,\mathrm{m}_3 & -\mathrm{i}\,\mathrm{m}_1 \\ -\mathrm{i}\,\mathrm{m}_1 & \mathrm{m}_0 + \mathrm{i}\,\mathrm{m}_3 \end{bmatrix}\begin{bmatrix} \psi(b) + \mathrm{i}\lambda\psi_x(b) \\ \psi(a) - \mathrm{i}\lambda\psi_x(a) \end{bmatrix}, \tag{42}$$

and depends on three real parameters. Indeed, the square matrix $\hat{\mathrm{U}}_{(2\times2)}$ in Eq. (42) is the one that determines the matrix $\hat{\mathrm{U}}_{(4\times4)}$ that appears in the more general set of pseudo self-adjoint boundary conditions for the 1D KFGM particle (see Eqs. 33 and 34). The latter general family of boundary conditions is for the solutions of the 1D FV wave equation in Eq. (30), which can also describe a 1D KFGM particle. Incidentally, because the matrix $\hat{\mathrm{S}}$ in Eq. (35) is real, one has that $\hat{\mathrm{S}}^\dagger = \hat{\mathrm{S}}^{\mathrm{T}}$, but in addition, $\hat{\mathrm{U}}_{(2\times2)}$ satisfies Eq. (40), consequently, the matrix $\hat{\mathrm{U}}_{(4\times4)}$ in Eq. (34) is also a (complex) symmetric matrix, i.e., $\hat{\mathrm{U}}^{\mathrm{T}}_{(4\times4)} = \hat{\mathrm{U}}_{(4\times4)}$. Actually, if $\hat{\mathrm{U}}_{(2\times2)}$ is the matrix given in Eq. (41) with $\mathrm{m}_2 = 0$, then $\hat{\mathrm{U}}_{(4\times4)}$ is given by

$$\hat{\mathrm{U}}_{(4\times4)} = \mathrm{e}^{\mathrm{i}\,\mu}\begin{bmatrix} (\mathrm{m}_0 - \mathrm{i}\,\mathrm{m}_3)\hat{1}_2 & -\mathrm{i}\,\mathrm{m}_1\hat{1}_2 \\ -\mathrm{i}\,\mathrm{m}_1\hat{1}_2 & (\mathrm{m}_0 + \mathrm{i}\,\mathrm{m}_3)\hat{1}_2 \end{bmatrix}. \tag{43}$$

Some of the boundary conditions included in the general set of pseudo self-adjoint boundary conditions for a 1D particle KFGM given in Eq. (42) are the following: (i) $\psi(a) = \psi(b) = 0$ ($\mathrm{m}_0 = +1$, $\mathrm{m}_1 = \mathrm{m}_3 = 0$ and $\mu = \pi$); (ii) $\psi_x(a) = \psi_x(b) = 0$ ($\mathrm{m}_0 = +1$, $\mathrm{m}_1 = \mathrm{m}_3 = 0$ and $\mu = 0$); (iii) $\psi(a) - \lambda\psi_x(a) = 0$ and $\psi(b) + \lambda\psi_x(b) = 0$ ($\mathrm{m}_0 = +1$, $\mathrm{m}_1 = \mathrm{m}_3 = 0$ and $\mu = \pi/2$); (iv) $\psi(a) = \psi(b)$ and $\psi_x(a) = \psi_x(b)$ ($\mathrm{m}_0 = \mathrm{m}_3 = 0$, $\mathrm{m}_1 = +1$ and $\mu = \pi/2$); (v) $\psi(a) = -\psi(b)$ and $\psi_x(a) = -\psi_x(b)$ ($\mathrm{m}_0 = \mathrm{m}_3 = 0$, $\mathrm{m}_1 = -1$ and $\mu = \pi/2$). In this short list, we have distinguished boundary conditions: (i) is the Dirichlet boundary condition, (ii) is the Neumann condition, (iii) is a kind of Robin boundary condition, (iv) is the periodic condition and (v) is the antiperiodic condition. As discussed above, a 1D KFGM particle supports only those boundary conditions arising from the unitary matrix $\hat{\mathrm{U}}_{(2\times2)}$ in Eq. (41) with $\mathrm{m}_2 = 0$. For example, some boundary conditions that are not suitable for a 1D KFGM particle but are suitable for a 1D KFG particle ($\mathrm{m}_2 \neq 0$) are the following: (vi) $\psi(a) = \pm\mathrm{i}\psi(b)$ and $\psi_x(a) = \pm\mathrm{i}\psi_x(b)$ ($\mathrm{m}_0 = \mathrm{m}_1 = \mathrm{m}_3 = 0$, $\mathrm{m}_2 = \pm1$ and $\mu = \pi/2$); (vii) $\psi(a) = \pm\mathrm{i}\lambda\psi_x(b)$ and $\psi(b) = \pm\mathrm{i}\lambda\psi_x(a)$ ($\mathrm{m}_0 = \mathrm{m}_1 = \mathrm{m}_3 = 0$, $\mathrm{m}_2 = \pm1$ and $\mu = 0$).

Due to the Majorana condition, the components of the wave function Ψ in Eq. (30) are not independent. When this condition is given by $\Psi = \Psi_c$, then $\chi = \varphi^*$, and therefore, $\psi = \varphi + \chi = \varphi + \varphi^* = 2\,\mathrm{Re}(\varphi)$ and $\psi_x = 2\,(\mathrm{Re}(\varphi))_x$. Thus, the equation in (21) with $V = 0$ and $S \in \mathbb{R}$, namely,

$$\hat{\mathrm{E}}\,\varphi = \left(\frac{\hat{\mathrm{p}}^2}{2\mathrm{m}} + S\right)(\varphi + \varphi^*) + \mathrm{m}c^2\varphi, \tag{44}$$

describes a 1D KFGM particle (the one for which $\Psi = \Psi_c$), and its solutions must satisfy any of the boundary conditions included in the general set of boundary conditions given in Eq. (42) but written only in terms of φ, namely,

$$\begin{bmatrix} (\mathrm{Re}(\varphi))(b) - \mathrm{i}\lambda(\mathrm{Re}(\varphi))_x(b) \\ (\mathrm{Re}(\varphi))(a) + \mathrm{i}\lambda(\mathrm{Re}(\varphi))_x(a) \end{bmatrix} = \mathrm{e}^{\mathrm{i}\,\mu} \begin{bmatrix} \mathrm{m}_0 - \mathrm{i}\,\mathrm{m}_3 & -\mathrm{i}\,\mathrm{m}_1 \\ -\mathrm{i}\,\mathrm{m}_1 & \mathrm{m}_0 + \mathrm{i}\,\mathrm{m}_3 \end{bmatrix} \begin{bmatrix} (\mathrm{Re}(\varphi))(b) + \mathrm{i}\lambda(\mathrm{Re}(\varphi))_x(b) \\ (\mathrm{Re}(\varphi))(a) - \mathrm{i}\lambda(\mathrm{Re}(\varphi))_x(a) \end{bmatrix}. \tag{45}$$

Let us note the simultaneous presence of φ and φ^* in Eq. (44); however, from this equation, it follows that the real part of φ satisfies the 1D KFG equation, namely,

$$\left[-\hbar^2\frac{\partial^2}{\partial t^2} + \hbar^2 c^2\frac{\partial^2}{\partial x^2} - (\mathrm{m}c^2)^2 - 2\,\mathrm{m}c^2 S\right]\mathrm{Re}(\varphi) = 0, \tag{46}$$

and the imaginary part of φ can be obtained by taking the time derivative of the real part, namely,

$$\mathrm{Im}(\varphi) = \frac{\hbar}{\mathrm{m}c^2}\frac{\partial}{\partial t}\mathrm{Re}(\varphi). \tag{47}$$

Clearly, if the scalar potential depends explicitly on time, the imaginary part of φ does not satisfy Eq. (46) (see Appendix A). Finally, the solutions of Eq. (44) are simply given by $\varphi = \mathrm{Re}(\varphi) + \mathrm{i}\,\mathrm{Im}(\varphi)$ (and the component χ of Ψ is obtained from the Majorana condition, i.e., $\chi = \varphi^*$). As discussed above, in this same case ($\Psi = \Psi_c$), the equation in (22) can alternatively be used with $V = 0$ and $S \in \mathbb{R}$, namely,

$$\hat{\mathrm{E}}\,\chi = -\left(\frac{\hat{\mathrm{p}}^2}{2\mathrm{m}} + S\right)(\chi + \chi^*) - \mathrm{m}c^2\chi. \tag{48}$$

However, in addition, in Eq. (42), the relations $\psi = \varphi + \chi = \chi^* + \chi = 2\,\mathrm{Re}(\chi)$ and $\psi_x = 2\,(\mathrm{Re}(\chi))_x$ must be used, namely,

$$\begin{bmatrix} (\mathrm{Re}(\chi))(b) - \mathrm{i}\lambda(\mathrm{Re}(\chi))_x(b) \\ (\mathrm{Re}(\chi))(a) + \mathrm{i}\lambda(\mathrm{Re}(\chi))_x(a) \end{bmatrix} = \mathrm{e}^{\mathrm{i}\,\mu} \begin{bmatrix} \mathrm{m}_0 - \mathrm{i}\,\mathrm{m}_3 & -\mathrm{i}\,\mathrm{m}_1 \\ -\mathrm{i}\,\mathrm{m}_1 & \mathrm{m}_0 + \mathrm{i}\,\mathrm{m}_3 \end{bmatrix} \begin{bmatrix} (\mathrm{Re}(\chi))(b) + \mathrm{i}\lambda(\mathrm{Re}(\chi))_x(b) \\ (\mathrm{Re}(\chi))(a) - \mathrm{i}\lambda(\mathrm{Re}(\chi))_x(a) \end{bmatrix}. \tag{49}$$

In this case, it can be shown that $\mathrm{Re}(\chi)$ satisfies the same Eq. (46) and any of the boundary conditions in Eq. (49). The imaginary part of χ is obtained from the following relation:

$$\mathrm{Im}(\chi) = -\frac{\hbar}{\mathrm{m}c^2}\frac{\partial}{\partial t}\mathrm{Re}(\chi). \tag{50}$$

Thus, the solutions of Eq. (48) are simply given by $\chi = \mathrm{Re}(\chi) + \mathrm{i}\,\mathrm{Im}(\chi)$ (and the component φ of Ψ is obtained from the Majorana condition, i.e., $\varphi = \chi^*$).

Similarly, when the Majorana condition is given by $\Psi = -\Psi_c$, $\chi = -\varphi^*$, and therefore, $\psi = \varphi + \chi = \varphi - \varphi^* = 2\mathrm{i}\,\mathrm{Im}(\varphi)$ and $\psi_x = 2\mathrm{i}\,(\mathrm{Im}(\varphi))_x$. Thus, the equation in (23) with $V = 0$ and $S \in \mathbb{R}$, namely,

$$\hat{\mathrm{E}}\,\varphi = \left(\frac{\hat{\mathrm{p}}^2}{2\mathrm{m}} + S\right)(\varphi - \varphi^*) + \mathrm{m}c^2\varphi, \tag{51}$$

characterizes a 1D KFGM particle (the one for which $\Psi = -\Psi_c$) and its solutions must satisfy some of the boundary conditions given in Eq. (42), but the latter equation written in terms of φ only, specifically,

$$\begin{bmatrix} (\mathrm{Im}(\varphi))(b) - \mathrm{i}\lambda(\mathrm{Im}(\varphi))_x(b) \\ (\mathrm{Im}(\varphi))(a) + \mathrm{i}\lambda(\mathrm{Im}(\varphi))_x(a) \end{bmatrix} = \mathrm{e}^{\mathrm{i}\,\mu} \begin{bmatrix} \mathrm{m}_0 - \mathrm{i}\,\mathrm{m}_3 & -\mathrm{i}\,\mathrm{m}_1 \\ -\mathrm{i}\,\mathrm{m}_1 & \mathrm{m}_0 + \mathrm{i}\,\mathrm{m}_3 \end{bmatrix} \begin{bmatrix} (\mathrm{Im}(\varphi))(b) + \mathrm{i}\lambda(\mathrm{Im}(\varphi))_x(b) \\ (\mathrm{Im}(\varphi))(a) - \mathrm{i}\lambda(\mathrm{Im}(\varphi))_x(a) \end{bmatrix}. \tag{52}$$

From Eq. (51), the imaginary part of φ satisfies the 1D KFG equation, as expected, that is,

$$\left[-\hbar^2 \frac{\partial^2}{\partial t^2} + \hbar^2 c^2 \frac{\partial^2}{\partial x^2} - (\mathrm{m}c^2)^2 - 2\,\mathrm{m}c^2 S \right] \mathrm{Im}(\varphi) = 0, \tag{53}$$

with the boundary conditions given in Eq. (52). The real part of φ is obtained from the relation

$$\mathrm{Re}(\varphi) = -\frac{\hbar}{\mathrm{m}c^2} \frac{\partial}{\partial t} \mathrm{Im}(\varphi). \tag{54}$$

Finally, $\varphi = \mathrm{Re}(\varphi) + \mathrm{i}\,\mathrm{Im}(\varphi)$ is the solution of Eq. (51) (and the component χ of Ψ is obtained from the Majorana condition, i.e., $\chi = -\varphi^*$). Equivalently (because $\Psi = -\Psi_c$ is still valid), the equation in (24) can also be used with $V = 0$ and $S \in \mathbb{R}$, namely,

$$\hat{\mathrm{E}}\,\chi = -\left(\frac{\hat{\mathrm{p}}^2}{2\mathrm{m}} + S \right) (\chi - \chi^*) - \mathrm{m}c^2 \chi. \tag{55}$$

However, now, in Eq. (42), the relations $\psi = \varphi + \chi = -\chi^* + \chi = 2\mathrm{i}\,\mathrm{Im}(\chi)$ and $\psi_x = 2\mathrm{i}\,(\mathrm{Im}(\chi))_x$ must be used. Then, we can write

$$\begin{bmatrix} (\mathrm{Im}(\chi))(b) - \mathrm{i}\lambda(\mathrm{Im}(\chi))_x(b) \\ (\mathrm{Im}(\chi))(a) + \mathrm{i}\lambda(\mathrm{Im}(\chi))_x(a) \end{bmatrix} = \mathrm{e}^{\mathrm{i}\,\mu} \begin{bmatrix} \mathrm{m}_0 - \mathrm{i}\,\mathrm{m}_3 & -\mathrm{i}\,\mathrm{m}_1 \\ -\mathrm{i}\,\mathrm{m}_1 & \mathrm{m}_0 + \mathrm{i}\,\mathrm{m}_3 \end{bmatrix} \begin{bmatrix} (\mathrm{Im}(\chi))(b) + \mathrm{i}\lambda(\mathrm{Im}(\chi))_x(b) \\ (\mathrm{Im}(\chi))(a) - \mathrm{i}\lambda(\mathrm{Im}(\chi))_x(a) \end{bmatrix}. \tag{56}$$

The latter boundary conditions are for the solutions of the wave equation in (55), which can also describe the 1D KFGM particle for which $\Psi = -\Psi_c$. In this case, it can be shown that $\mathrm{Im}(\chi)$ also satisfies Eq. (53) and any of the boundary conditions in Eq. (56) (which are certainly the same boundary conditions that are satisfied by $\mathrm{Im}(\varphi)$), and the real part of χ is obtained from the relation

$$\mathrm{Re}(\chi) = \frac{\hbar}{\mathrm{m}c^2} \frac{\partial}{\partial t} \mathrm{Im}(\chi). \tag{57}$$

Finally, the solutions of Eq. (55) are obtained from $\chi = \mathrm{Re}(\chi) + \mathrm{i}\,\mathrm{Im}(\chi)$ (and the component φ of Ψ is obtained from the Majorana condition, i.e., $\varphi = -\chi^*$).

We may refer to Eqs. (44), (48), (51) and (55) as the first-order (non-Hamiltonian) 1D Majorana equations in time for the 1D KFGM particle, and their solutions are complex. On the other hand, all the examples of the boundary conditions we presented above for the wavefunction ψ have identical counterparts for $\mathrm{Re}(\varphi)$ and $\mathrm{Re}(\chi)$, and $\mathrm{Im}(\varphi)$ and $\mathrm{Im}(\chi)$. This is because the families of boundary conditions given in Eqs. (45), (49), and (52), (56), are similar in form. Incidentally, each of the first-order 1D Majorana equations leads to a second-order 1D Majorana equation. None of the latter equations is the standard 1D KFG equation; however, when the scalar potential does not explicitly depend on time, each equation becomes the standard 1D KFG equation. This is exhibited in the Appendix A.

Last, if we set $\Psi = \Phi$ in Eq. (36), and therefore, $\psi = \phi$, we obtain

$$\langle\langle \hat{\mathrm{h}}_{\mathrm{adj}} \Psi, \Psi \rangle\rangle = \langle\langle \Psi, \hat{\mathrm{h}} \Psi \rangle\rangle - \frac{\hbar}{\mathrm{i}} \left[\, j \,\right]\Big|_a^b , \tag{58}$$

where

$$j = j(x,t) = \frac{\mathrm{i}\hbar}{2\mathrm{m}} \left(\psi_x^* \psi - \psi^* \psi_x \right) = \frac{\hbar}{\mathrm{m}} \mathrm{Im}\left(\psi^* \psi_x \right) \tag{59}$$

is the 1D KFG "probability" current density (although it is certainly not correct to interpret it as a quantity representing a probability). Similarly, because the pseudo inner product $\langle\langle \Psi, \Phi \rangle\rangle$ in Eq. (37) is independent of time, $\langle\langle \Psi, \Psi \rangle\rangle \equiv \int_\Omega \mathrm{d}x\, \Psi^\dagger \hat{\tau}_3 \Psi = \int_\Omega \mathrm{d}x\, \varrho$ is also a constant quantity, where

$$\varrho = \varrho(x,t) = \Psi^\dagger \hat{\tau}_3 \Psi = |\varphi|^2 - |\chi|^2 = \frac{1}{2\mathrm{m}c^2} \left[\psi^* (\hat{\mathrm{E}}\psi) - (\hat{\mathrm{E}}\psi^*)\, \psi \right] \tag{60}$$

is the 1D KFG "probability" density, but we also have that $[\, j \,]|_a^b = 0$. Therefore, $j(b,t) = j(a,t)$ (see Eq. 37). These two quantities, $j = j(x,t)$ and $\varrho = \varrho(x,t)$, also satisfy the continuity equation in this situation where a real scalar potential exists, namely, $\partial\varrho/\partial t = -\partial j/\partial x$ (certainly, the latter equation is valid for the solutions of the 1D KFG equation with a scalar potential, for example, the solutions ψ of Eq. (39)). Let us note that if ϱ and j are nonzero, the continuity equation can be integrated in x, which leads to the expected result $\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega}\mathrm{d}x\,\varrho = -\,[\,j\,]|_a^b$. Then, because $[\,j\,]|_a^b = 0$, we have that $\int_{\Omega}\mathrm{d}x\,\varrho = \text{const}$.

Clearly, when the solutions of Eq. (39) are definitely real-valued functions, the 1D KFGM current density j and density ϱ cease to exist. That is, when $\psi = \psi^*$, the expressions for j and ϱ given in Eqs. (59) and (60) are identically zero; however, when $\psi = -\psi^*$, the result is the same, namely,

$$j = j(x,t) = 0 \quad \text{and} \quad \varrho = \varrho(x,t) = 0. \tag{61}$$

Thus, ultimately, the reality condition for the wavefunction ψ also automatically leads to the so-called impenetrability condition at the extremes of the interval Ω, i.e., $j(b,t) = j(a,t) = 0$. This situation contrasts with the case of the 1D Dirac wave equation in the Majorana representation (by considering the Dirac theory as a single-particle theory). There, the Majorana condition is given by $\Psi_{\mathrm{D}} = \Psi_{\mathrm{D}}^*$, so that only real solutions describe the 1D Dirac-Majorana particle; however, the corresponding probability current density is not automatically zero [30]. In conclusion, regardless of which boundary condition one takes from the general set in Eq. (42), which is satisfied by a real solution ψ, this solution does, by necessity, always verify the mathematical condition of impenetrability at the walls of the interval.

In any case, it is important to remember that for a 1D KFG particle moving in a finite interval (see Eq. (2), where V and S are real-valued potentials, and therefore, their solutions ψ are always complex-valued functions), we can recognize impenetrability boundary conditions (or confining boundary conditions) and nonconfining boundary conditions [24]. In general, the confining boundary conditions satisfy $j(b,t) = j(a,t) = 0$, and nonconfining boundary conditions simply satisfy $j(b,t) = j(a,t)$ (for which it is certainly necessary that the solutions ψ are complex). However, when the solutions of Eq. (2) are real-valued functions (which can occur when $V = 0$); that is, when the solutions of the equation describing a 1D KFGM particle are real (see Eq. 39), the distinction between confining and nonconfining boundary conditions is not feasible, at least if the current density j is considered. Certainly, the characterization of these two types of boundary conditions would require the use of some other current density. We hope to study this issue in a forthcoming paper.

3 Summary and Conclusions

In the first quantization, the wave equations considered to describe a strictly neutral 1D KFGM particle are the standard 1D KFG equation and/or the 1D FV equation, both with a real Lorentz scalar potential plus their respective Majorana conditions. Unexpectedly, one finds that the Majorana condition appears here in two specific forms, say, one standard and one nonstandard. Specifically, we showed that the imposition of the standard (nonstandard) Majorana condition on the solutions of the 1D FV equation implies that the solutions of the second order 1D KFG equation in the time must be real (imaginary; however, they can also be written real, as expected). Additionally, both Majorana conditions determine that the scalar potential must be real. In any case, we found that the solutions of the time-dependent 1D FV equation cannot be real. We also showed that the additional imposition of the formal pseudohermiticity condition on the Hamiltonian that is present in this equation together with a Majorana condition determines that the electric potential must be zero. In addition, if we place a 1D KFGM particle in a finite interval, then the corresponding Hamiltonian is a pseudo self-adjoint operator. As a consequence of this property, one has a three-parameter general set of boundary conditions for the solutions of the 1D FV equation and another for the respective real solutions of the standard 1D KFG equation. We found that these two general sets of boundary conditions are the same for the two types of Majorana conditions. Because of the Majorana condition, the components of the wavefunction for the 1D FV equation are not independent; hence, we wrote first-order equations in time for each of these components and obtained the general sets of pseudo self-adjoint boundary conditions that they must obey. Incidentally, these equations do not have a Hamiltonian form, but any of them alone can model a 1D KFGM particle (in fact, if one of the complex components of the solution of the 1D FV equation is known, the other component can be obtained algebraically via the Majorana condition). Thus, we refer to these equations as the first-order 1D Majorana equations for the 1D KFGM particle. Additionally, we wrote second-order 1D Majorana equations in time for each of the components of the 1D FV equation. These equations become the standard 1D KFG equation when the scalar potential does not explicitly depend on time (see Appendix A).

As shown in Appendix B, the nonrelativistic limit of the first-order 1D Majorana equation given in Eq. (44) yields the partial differential equation given in Eq. (B11). The latter equation is not the Schrödinger equation because the term enclosed in the bracket does not have to be zero. Having said that, if it is assumed that the solution of this nonrelativistic equation $\varphi_{\rm NR}$ and its complex conjugate $\varphi^{*}_{\rm NR}$ can be treated as independent solutions, then $\varphi_{\rm NR}$ satisfies the Schrödinger equation and $\varphi^{*}_{\rm NR}$ satisfies the equation which is the complex conjugate of the Schrödinger equation. A similar situation arises when studying the nonrelativistic limit of certain real scalar field theories (see, for example, Refs. [31–35]). There, the (classical) relativistic field is real, i.e., $\psi = \psi^{*}$, but the nonrelativistic $\psi_{\rm NR}$ is complex; thus, in that case, by taking the nonrelativistic limit, the typical ansatz we used in Appendix B must be modified. Apropos of this, the first-order Majorana equations we introduce here describe strictly neutral particles, and their solutions are always complex; thus, the ansatz can be the usual one. Finally, in each of those field theories (and only in certain cases), the nonrelativistic Schrödinger equation and its respective complex conjugate equation could be obtained if the solutions of these two equations are assumed to be independent. Incidentally, the latter assumption has been questioned (see, for example, Ref. [31]). In closing, our results can also be extended to the problem of a 1D KFGM particle in a real line with a tiny hole at a point, for example, at $x = 0$ (i.e., $\Omega = \mathbb{R} - \{0\}$). Indeed, the general sets of pseudo self-adjoint boundary conditions for this problem can be obtained from those corresponding to the particle within the interval $\Omega = [a, b]$ by identifying the ends of the interval with the two sides of the hole, namely, $x = a \to 0+$ and $x = b \to 0-$.

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Declarations

Conflicts of interest The author declares no conflicts of interest.

Appendix A

As we have seen, we can write $2+2=4$ first-order (non-Hamiltonian) 1D Majorana equations in time for the 1D KFGM particles, namely, Eqs. (44) and (48) (by using the standard Majorana condition), and (51) and (55) (by using the nonstandard Majorana condition). Likewise, we can also write four second-order 1D Majorana equations in time for these particles, i.e., four second-order equations for the components φ and χ of Ψ. In effect, applying the operator $\hat{\rm E}$ to both sides of Eq. (44), and using the relation $\hat{\rm E}^{*} = -\hat{\rm E}$, gives the following equation:

$$\left[\hat{\rm E}^2 - (c\,\hat{\rm p})^2 - ({\rm m}c^2)^2 - 2\,{\rm m}c^2 S\right]\varphi = (\hat{\rm E}\,S)(\varphi + \varphi^{*}). \tag{A1}$$

Similarly, from Eq. (48), the following equation is obtained:

$$\left[\hat{\rm E}^2 - (c\,\hat{\rm p})^2 - ({\rm m}c^2)^2 - 2\,{\rm m}c^2 S\right]\chi = -(\hat{\rm E}\,S)(\chi + \chi^{*}). \tag{A2}$$

These two equations correspond to the Majorana condition $\Psi = \Psi_c$, that is, $\psi = \psi^{*}$. If we add Eqs. (A1) and (A2) and use the relations given in Eqs. (3) and (4) (the latter with $V = 0$), it is confirmed that $\psi = \varphi + \chi$ satisfies the 1D KFG equation (i.e., Eq. 39), namely,

$$\left[\hat{\rm E}^2 - (c\,\hat{\rm p})^2 - ({\rm m}c^2)^2 - 2\,{\rm m}c^2 S\right]\psi = 0, \tag{A3}$$

as expected. Note that only when the scalar potential does not explicitly depend on time, the complex components φ and χ of Ψ also satisfy this equation. If this is not the case, only the functions ${\rm Re}(\varphi)$ and ${\rm Re}(\chi)$ can satisfy the 1D KFG equation (see the discussion following Eq. (45) through Eq. (50)).
Similarly, applying the operator $\hat{\rm E}$ to both sides of Eq. (51), and using the relation $\hat{\rm E}^{*} = -\hat{\rm E}$, gives the equation

$$\left[\hat{\rm E}^2 - (c\,\hat{\rm p})^2 - ({\rm m}c^2)^2 - 2\,{\rm m}c^2 S\right]\varphi = (\hat{\rm E}\,S)(\varphi - \varphi^{*}). \tag{A4}$$

In the same manner, applying $\hat{\rm E}$ to Eq. (55) gives the following equation:

$$\left[\hat{\rm E}^2 - (c\,\hat{\rm p})^2 - ({\rm m}c^2)^2 - 2\,{\rm m}c^2 S\right]\chi = -(\hat{\rm E}\,S)(\chi - \chi^{*}). \tag{A5}$$

The latter two equations correspond to the Majorana condition $\Psi = -\Psi_c$, that is, $\psi = -\psi^*$. If we add Eqs. (A4) and (A5) and use the relations given in Eqs. (3) and (4) (the latter with $V = 0$), it is again found that $\psi = \varphi + \chi$ satisfies the 1D KFG equation (i.e., Eq. A3), as expected. Clearly, if $(\hat{\mathrm{E}}\, S) = 0$, then φ and χ also satisfy the 1D KFG equation. If $(\hat{\mathrm{E}}\, S) \neq 0$, then only the functions $\mathrm{Im}(\varphi)$ and $\mathrm{Im}(\chi)$ can satisfy this equation (see the discussion following Eq. (53) through Eq. (57)).

Appendix B

Let us examine the nonrelativistic approximation of one of the first-order 1D Majorana equations in time, for example, Eq. (44). As we know, the latter equation for $\varphi \in \mathbb{C}$ is completely equivalent to Eq. (46), namely,

$$\left[-\hbar^2 \frac{\partial^2}{\partial t^2} + \hbar^2 c^2 \frac{\partial^2}{\partial x^2} - (\mathrm{m}c^2)^2 - 2\,\mathrm{m}c^2 S \right] \mathrm{Re}(\varphi) = 0, \tag{B1}$$

plus the relation given in Eq. (47), namely,

$$\mathrm{Im}(\varphi) = \frac{\hbar}{\mathrm{m}c^2} \frac{\partial}{\partial t} \mathrm{Re}(\varphi). \tag{B2}$$

Let us note that Eq. (B1) can also be written as follows:

$$\mathrm{Re}\left[\left(-\hbar^2 \frac{\partial^2}{\partial t^2} + \hbar^2 c^2 \frac{\partial^2}{\partial x^2} - (\mathrm{m}c^2)^2 - 2\,\mathrm{m}c^2 S \right) \varphi \right] = 0. \tag{B3}$$

Now, we choose the typical ansatz that connects φ to its nonrelativistic approximation φ_{NR}, namely,

$$\varphi = \varphi_{\mathrm{NR}}\, \mathrm{e}^{-\mathrm{i}\frac{\mathrm{m}c^2}{\hbar} t}, \tag{B4}$$

and therefore,

$$\varphi_t = \left[(\varphi_{\mathrm{NR}})_t - \mathrm{i}\frac{\mathrm{m}c^2}{\hbar}\, \varphi_{\mathrm{NR}} \right] \mathrm{e}^{-\mathrm{i}\frac{\mathrm{m}c^2}{\hbar} t} \tag{B5}$$

and

$$\varphi_{tt} = \left[(\varphi_{\mathrm{NR}})_{tt} - \mathrm{i}\frac{2\mathrm{m}c^2}{\hbar}\, (\varphi_{\mathrm{NR}})_t - \frac{(\mathrm{m}c^2)^2}{\hbar^2}\, \varphi_{\mathrm{NR}} \right] \mathrm{e}^{-\mathrm{i}\frac{\mathrm{m}c^2}{\hbar} t}. \tag{B6}$$

In the nonrelativistic approximation, we have that

$$|\, \mathrm{i}\hbar\, (\varphi_{\mathrm{NR}})_t \,| \ll \mathrm{m}c^2\, |\, \varphi_{\mathrm{NR}} \,| \;\Rightarrow\; |\, (\varphi_{\mathrm{NR}})_t \,| \ll \frac{\mathrm{m}c^2}{\hbar}\, |\, \varphi_{\mathrm{NR}} \,| \tag{B7}$$

and

$$|\, \mathrm{i}\hbar\, (\varphi_{\mathrm{NR}})_{tt} \,| \ll \mathrm{m}c^2\, |\, (\varphi_{\mathrm{NR}})_t \,| \;\Rightarrow\; |\, (\varphi_{\mathrm{NR}})_{tt} \,| \ll \frac{\mathrm{m}c^2}{\hbar}\, |\, (\varphi_{\mathrm{NR}})_t \,|\,. \tag{B8}$$

Consequently, in this regime, the relations given in Eqs. (B5) and (B6) can be written as follows:

$$\varphi_t = -\mathrm{i}\frac{\mathrm{m}c^2}{\hbar}\, \varphi_{\mathrm{NR}}\, \mathrm{e}^{-\mathrm{i}\frac{\mathrm{m}c^2}{\hbar} t} \tag{B9}$$

and

$$\varphi_{tt} = \left[-\mathrm{i}\frac{2\mathrm{m}c^2}{\hbar}\, (\varphi_{\mathrm{NR}})_t - \frac{(\mathrm{m}c^2)^2}{\hbar^2}\, \varphi_{\mathrm{NR}} \right] \mathrm{e}^{-\mathrm{i}\frac{\mathrm{m}c^2}{\hbar} t}. \tag{B10}$$

Substituting the latter expression into Eq. (B3), we obtain the following result:

$$\mathrm{Re}\left[\,\mathrm{e}^{-\mathrm{i}\frac{\mathrm{m}c^2}{\hbar}t}\left(-\hat{\mathrm{E}}+\frac{\hat{\mathrm{p}}^2}{2\mathrm{m}}+S\right)\varphi_{\mathrm{NR}}\right]=0. \tag{B11}$$

Clearly, this is not the Schrödinger equation with the scalar interaction, i.e., φ_{NR} in Eq. (B11) does not necessarily obey this equation.
Similarly, the relation that gives the imaginary part of φ (Eq. B2) can also be written as follows:

$$\mathrm{Im}(\varphi)=\mathrm{Re}\left(\frac{\hbar}{\mathrm{m}c^2}\,\varphi_t\right). \tag{B12}$$

Substituting Eqs. (B4) and (B9) into Eq. (B12), we obtain the result

$$\mathrm{Im}\left(\varphi_{\mathrm{NR}}\,\mathrm{e}^{-\mathrm{i}\frac{\mathrm{m}c^2}{\hbar}t}\right)=\mathrm{Re}\left(-\mathrm{i}\,\varphi_{\mathrm{NR}}\,\mathrm{e}^{-\mathrm{i}\frac{\mathrm{m}c^2}{\hbar}t}\right), \tag{B13}$$

which is always true because $\mathrm{Im}(z)=\mathrm{Re}(-\mathrm{i}z)$, for all $z\in\mathbb{C}$. Thus, nothing new is obtained from Eq. (B2) and the nonrelativistic limit of Eq. (44) reduces to Eq. (B11). Finally, φ_{NR} is obtained from Eq. (B11), (φ is given in Eq. (B4) and $\chi=\varphi^*$).

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