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# Path Integral Methods From the Generalized Displacement Operator, and Some of Their Applications

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**Abstract** The non-relativistic Feynman propagator with position dependent mass (PDM) is formulated by means to introduce the generalized infinitesimal translation operator approach. Which is similar to deformed quantum mechanics based on modified commutation relations. This latter is connected to the Feynman propagator with constant-mass by adopting the coordinates transformation methods, which are $\alpha-$point discretization dependent, to evaluate quantum corrections. In each application, the propagator (or the Green function) is calculated, the wave functions and their associated eigenvalues are obtained, also the curves of energies are illustrated. The limit cases are then deduced for a small parameter.

## 1 Introduction

The physical systems with position-dependent mass (PDM) have attracted great interest in various applied domains of quantum mechanics and, by their importance, they agree with reality and allow us to understand many phenomena of matter physics. Its different expressions appear in the motion of electrons in perturbed periodic lattices [1], in the electronic properties of semiconductors [2], in the quantum dots and quantum wells [3,4], in semiconductors Hetero-structures [5], in supper-lattice band structures [6] and He-Clusters [7] quantum liquids [8], in the solid-state problem in the framework of the Dirac equation [9] and also the dependence of energy gap on the magnetic field semiconductor Nano-scale rings [10].

Moreover, the PDM systems have generated great difficulties at the quantum level due to the ordering ambiguity of the momentum and mass operators in the term of kinetic energy. Lot of authors have discussed and tried to elucidate the problem of this kinetic energy operator order ambiguity, and one can find an excellent review given by Shewell [11]. Following these requirements, several kinetic energy operators were proposed, such as the ordering of von Roos [12,13], in which the following order had been chosen

$$\hat{T}_R = \frac{1}{4}(m^{\alpha}\hat{\mathbf{p}}m^{\beta}\hat{\mathbf{p}}m^{\gamma} + m^{\gamma}\hat{\mathbf{p}}m^{\beta}\hat{\mathbf{p}}m^{\alpha}), \tag{1}$$

where $\hat{\mathbf{p}} = -\imath\hbar\nabla$ is the momentum operator, and the parameters $\alpha$, $\beta$ and $\gamma$ satisfying the condition $\alpha+\beta+\gamma = -1$, also including special cases of values $\alpha$, $\beta$ and $\gamma$. Referring to some of them, Gora and Williams [14] suggested the values $\beta = \gamma = 0$, $\alpha = -1$, the proposal of Zhu and Kroemer [15] is $\alpha = \gamma = -\frac{1}{2}$, $\beta = 0$

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and Ben Daniel *et al* [16] gave the following parameters $\alpha = \gamma = 0$, $\beta = -1$. The other ordering is the Weyl order defined by [17],

$$\hat{T}_W = \frac{1}{6}\left[\frac{1}{m}\hat{\mathbf{p}}^2 + \hat{\mathbf{p}}\frac{1}{m}\hat{\mathbf{p}} + \hat{\mathbf{p}}^2\frac{1}{m}\right], \tag{2}$$

which is used as the prescription to evaluate the Feynman path integral relative to the mid-point interval $(\bar{x}_j = \left(x_j + x_{j-1}\right)/2)$ [18–20]. As well as a more general form than von Roos symmetric order by taking into account of the Weyl prescription [21],

$$\hat{T}_{RW} = \frac{a}{4\,(a+1)}\left[\frac{1}{m}\hat{\mathbf{p}}^2 + \hat{\mathbf{p}}^2\frac{1}{m} + \frac{1}{a}\left(m^{\alpha}\hat{\mathbf{p}}m^{\beta}\hat{\mathbf{p}}m^{\gamma} + m^{\gamma}\hat{\mathbf{p}}m^{\beta}\hat{\mathbf{p}}m^{\alpha}\right)\right], \tag{3}$$

where " $a$ " is an arbitrary parameter, and when $(a = 0)$ we find the von Roos symmetric order. It can also be tuned the $a-$parameter to generate different order prescriptions. All these expressions of kinetic energy operators proposed above are Hermitian and they have attracted much attention for many applications with several methods. We refer to some of them, the bound states for square potential wells [22], the Natanzon potentials in position dependent mass background [23], the PDM momentum operator and minimal coupling [24–26], the path integral formalism in nonrelativistic systems with PDM [27,28], the Lie algebraic approach to effective mass Schrodinger equations [29,30] and the generation of solvable potentials with PDM [31].

While in the past few years, some studies have shown the existence of intimate connections between, the Schrodinger equations based on the use of deformed canonical commutation relations (EUP), or the modification in the underlying space, and the Schrodinger equation with a position-dependent mass [32]. These three problems have in common a generalized translation operator

$$\mathcal{T}_{\lambda}(dx) = \exp\left[-\imath\,\hat{P}_{\lambda}dx\right], \tag{4}$$

which leads the corresponding momentum operator depends on the position. As a consequence, Gosta *et al*. [33,34] and Borges [35] have suggested a new type of EUP with a minimum momentum dispersion, based on a generalized translation operator. In this context, the corresponding infinitesimal displacement operator is related to the $q-$exponential and the $q-$logarithm functions. Wherefore, our aim in this paper is to propose a generalization of the Ref. [33] by the following generalized momentum operator

$$\hat{P}_{\lambda} = -\imath\hbar\left[m_{\lambda}(x)/m_0\right]^{-1/2}\frac{d}{dx}, \tag{5}$$

where $m_{\lambda}(x) = m_0 f_{\lambda}(x)$ plays the same role as that of the particle mass (i.e., it is called a position-dependent mass function). The quantity $f_{\lambda}(x)$ is a dimensionless position-dependent mass and is related to $\lambda-$parameter, with $\lambda$ is a real constant and $m_0$ is a constant mass, taking in mind that we must recover the usual quantum mechanics in the limit $m_{\lambda\to 0}(x) = m_0$ and $\hat{P}_{\lambda\to 0} = -i\hbar d/dx$. The generalized momentum operator in Eq. (5) makes the following canonical commutation relation between the position and momentum operators:

$$\left[\hat{X}_{\lambda},\,\hat{P}_{\lambda}\right] = \imath\hbar\left[m_{\lambda}(x)/m_0\right]^{-1/2},\ \Delta X_{\lambda}\Delta P_{\lambda} \geqslant \frac{\hbar}{2}\left[m_{\lambda}(x)/m_0\right]^{-1/2}. \tag{6}$$

A representation of the commutation relation (6) can be obtained from operators $\hat{x} = x$ and $\hat{p} = -\imath\hbar d/dx$, satisfying canonical commutation relations through the transformation

$$\begin{cases} \hat{X}_{\lambda} = x, \\ \hat{P}_{\lambda} = \left[m_{\lambda}(x)/m_0\right]^{-1/2}\hat{p} + h(x) \end{cases}. \tag{7}$$

with $h(x)$ an arbitrary function.

On the other hand, the EUP can be obtained also from the translation operator acting in a space with a diagonal metric which describes the motion of a quantum particle in the curved space. It is defined by the following metric [36]:

$$ds^2 = \sum_{\mu,\nu} g_{\mu\nu}dx^{\mu}dx^{\nu}, \tag{8}$$

where $g_{\mu\nu}$ is the metric of the curved space and $(\mu,\,\nu) \equiv x,\,y,\,z$.

In one dimension, the Ref. [36] amazes us that if a particle goes from a point $x$ to $x + g_{xx}^{-1/2}dx$ it has to get a translation like $\mathcal{T}_g(dx)|x\rangle = |x + g_{xx}^{-1/2}dx\rangle$. This translation is clearly non-additive and the operator

can be written as $\mathcal{T}_g(dx) = 1 - \imath \hat{P}_g dx$, where $\hat{P}_g = \imath g_{xx}^{-1/2}\partial_x$ is a generalized momentum. It obeys to an stationary equation of motion for a particle $\hat{H}_g\psi = E\psi$, with $\hat{H}_g = -\frac{1}{2m}\mathcal{D}_x^2 + V(x)$ and $\mathcal{D}_x = g_{xx}^{-1/2}\partial_x$. This means that in a 3−dimensions the square of $\mathcal{D}_x$−operator indicates a Laplace-Beltrami operator (i.e., $\mathcal{D}^2 \equiv \frac{1}{\sqrt{g}}\sum_{\mu,\nu}\partial_\mu\sqrt{g}g^{\mu\nu}\partial_\nu$), where $g$ is the determinant of the matrix of components of the metric tensor. Hence, under these considerations, the Heisenberg commutator has to be modified, in one dimension, as follows:

$$\left[\hat{x}, \hat{P}_g\right] = \imath g_{xx}^{-1/2}. \tag{9}$$

From Eqs. (6) and (9) we can extract the following conformity

$$g_{xx} \equiv m_\lambda(x)/m_0. \tag{10}$$

Therefore, the modification of Heisenberg uncertainty principle in Eqs. (6) and (9) guarantees the existence of the minimal momentum $(\Delta p)_{\min} \neq 0$.

In the following, we derive the necessary relations based on systems whose masses depend on position. So our strategy works on subspace $L^2(\Lambda, d_\lambda x)$ instead of all Hilbert space $L^2(\mathbb{R}, dx)$, since the generalized momentum operator in (5) is not Hermitian in all Hilbert space $L^2(\mathbb{R}, dx)$. This leads to the deformed measure $d_\lambda x$ defined by:

$$d_\lambda x = [m_\lambda(x)/m_0]^{1/2}\, dx. \tag{11}$$

Thus, some algebraic considerations will make the modified scalar product of two functions $\psi(x)$ and $\varphi(x)$ as follows:

$$\langle \varphi \mid \psi \rangle = \int \varphi^*(x)\, \psi(x)\, d_\lambda x. \tag{12}$$

Under these assumptions, the closure relation can be generalized in the following way:

$$\int d_\lambda x \;\; |x\rangle \langle x| = 1. \tag{13}$$

This means that the corresponding projection relation becomes as

$$\begin{aligned} \langle x \mid x' \rangle_\lambda &= \int_{-\infty}^{+\infty} \frac{dp}{2\pi\hbar} [m_\lambda(x)/m_0]^{-1/2} \exp\left[-\frac{\imath p\left(x - x'\right)}{\hbar}\right] \\ &= [m_\lambda(x)/m_0]^{-1/2}\, \delta\left(x - x'\right). \end{aligned} \tag{14}$$

In effect, we have

$$\begin{aligned} \langle x \mid x' \rangle_\lambda &= \int d_\lambda x'' \langle x \mid x'' \rangle_\lambda \langle x'' \mid x' \rangle_\lambda \\ &= [m_\lambda(x)/m_0]^{-1/2} \int dx'' \left[m_\lambda(x'')/m_0\right]^{1/2} \left[m_\lambda(x'')/m_0\right]^{-1/2} \delta\left(x - x''\right) \delta\left(x'' - x'\right) \\ &= [m_\lambda(x)/m_0]^{-1/2}\, \delta\left(x - x'\right). \end{aligned} \tag{15}$$

In the case $\lambda = 0$, we recover the usual projection relation and the well-known by Dirac delta function $\langle x \mid x' \rangle_{\lambda\to 0} = \delta\left(x - x'\right)$. Hence, the element matrix for a generalized momentum operator under the EUP is calculated by

$$\left\langle x \left| \hat{P}_\lambda \right| x' \right\rangle_\lambda = \int_{-\infty}^{+\infty} \frac{dp}{2\pi\hbar} [m_\lambda(x)/m_0]^{-1/2} \exp\left(-\frac{\imath p\left(x - x'\right)}{\hbar}\right) \left[-p + \imath\hbar\frac{m_\lambda(x')}{2m_\lambda(x)}\right]. \tag{16}$$

In this framework of the deformed displacement operator $\mathcal{T}_\lambda(\delta x)$, many applications have been treated, and for instance, we find the harmonic oscillator potential in one dimension [34,37,38], the particle in a square well [33] and the inverse square plus Coulomb-like potential [39]. In addition, the Schrodinger equation with PDM provides an interesting and useful model for the description of many physical problems, especially in the physics of semiconductor nanostructures. For this, our task in this paper is similar to that used in our previous

work [40], but by generalizing it to arbitrary position-dependent masses function ($m_\lambda(x) = m_0 f_\lambda(x)$). We will present a general method to use the coordinate transformation technique at different usual point intervals and the difference resulting from the use of EUP algebra will be matched to the case of generalized uncertainly principle (GUP) (see, Ref. [41]).

Our strategy is based on modeling the path integral formalism of non-relativistic systems by introducing the generalized infinitesimal translation operator which corresponds to a modified Heisenberg uncertainty relation stipulated in below equation (6). Moreover, a similar study of this model was discussed in the generalized uncertainty principle in one dimension momentum space to treat the spinless relativistic particle [42] and extended later to include the relativistic 1/2 spinning particle [41,43] and also in two dimensions [44]. It is remarkable that in this case, the effective potential depends on the point discretization interval.

In the next section, we generalize the solutions to the PDM problem [40] by using the path integral formulation. We will formulate the path integral representation of the transition amplitude for nonrelativistic particle in the presence of potential $V(x)$ in one dimension. This approach is formulated by means the introduction of the generalized infinitesimal translation operator which is equivalent to a deformed quantum mechanics based on modified commutation relations. In Sect. 3, we will validate the accuracy of $\alpha-$points discretization to have the exact solution. In Sect. 4, we distinguish three particular cases of PDM with choice of some scalar potentials. We will determine the energy spectrum and the corresponding wave functions. The last section is reserved for the conclusion.

## 2 Path Integral Formulation

The transition amplitude of non-relativistic quantum dynamics corresponding to a position-dependent effective mass will be formulated by means of the generalized displacement operator approach [33,34] where we follow the canonical steps in [45], then we have,

$$K\left(x_b, t_b; x_a, t_a\right) = \lim_{N\to\infty} < x_b \mid \prod_{J=1}^{N+1} U\left(t_J, t_{J-1}\right) \mid x_a >_\lambda, \tag{17}$$

where $U\left(t_J, t_{J-1}\right) = \exp(-\imath\varepsilon\hat{H}/\hbar)$ represents the infinitesimal evolution operator, with $\hat{H}$ is the standard Hamiltonian form and $\varepsilon = (t_b - t_a)/(N+1)$, while $t_b$ and $t_a$ are the final and the initial time, respectively. Inserting the closure relation for the position states given by Eq. (13) between each pair of infinitesimal evolution operators, we easily obtain the following expression given by

$$K(x_b, x_a, T) = \lim_{N\to\infty} \prod_{J=1}^{N} \left[\int d_\lambda x_J\right] \prod_{J=1}^{N+1} < x_J | e^{-\frac{\imath\varepsilon}{\hbar}\left[\frac{\hat{P}_\lambda^2}{2m_0} + V(\hat{X}_\lambda)\right]} | x_{J-1} >_\lambda, \tag{18}$$

where $x_0 = x_a$ and $x_{N+1} = x_b$.

The appearance of the term $d_\lambda x_J$ in the measure part is due to the extended uncertainty principle algebra (6). To extract the path integral representation of the transition amplitude for a particle with nonzero minimum momentum uncertainty under the scalar potential $V(x)$, firstly, we must calculate the matrix element for the square of the generalized momentum operator $\hat{P}_\lambda^2$ and is related to the relations (5), (14) and (16) as,

$$\begin{aligned}\left\langle x_J \left| \hat{P}_\lambda^2 \right| x_{J-1} \right\rangle_\lambda = \int_{-\infty}^{+\infty} \frac{dp_J}{2\pi\hbar} \left[m_\lambda(x_J)/m_0\right]^{-1/2} e^{-\frac{\imath p_J \Delta x_J}{\hbar}} \left[\left[m_\lambda(x_J)/m_0\right]^{-1} p_J^2 \right.\\ \left. -\imath\hbar p_J \frac{3m_0 m_\lambda'(x_J)}{2m_\lambda^2(x_J)} - \frac{\hbar^2 m_0}{2}\left(2\frac{(m_\lambda^{'}(x_J))^2}{m_\lambda^3(x_J)} - \frac{m_\lambda^{''}(x_J)}{m_\lambda^2(x_J)}\right)\right].\end{aligned} \tag{19}$$

The primes in Eq. (19) denote derivatives with respect to $x_J$. Substituting (19) into (18), and performing the Gaussian integrations over $p_J$, the transition amplitude for the particle in question will simplify to:

$$K(x_b, x_a, T) = \lim_{N\to\infty} \prod_{J=1}^{N} \int d_\lambda x_J \prod_{J=1}^{N+1} \sqrt{\frac{m_0}{2\pi\imath\hbar\varepsilon}} \exp\left\{\frac{\imath}{\hbar} \sum_{J=1}^{N+1}\left[\frac{m_\lambda(x_J)}{2\varepsilon}(\Delta x_J)^2\right.\right.$$

$$-3\imath\hbar\frac{m'_\lambda(x_J)}{4m_\lambda(x_J)}\Delta x_J-\frac{\varepsilon\hbar^2}{4}\left(-\frac{7}{8}\frac{(m'_\lambda(x_J))^2}{m^3_\lambda(x_J)}+\frac{m''_\lambda(x_J)}{m^2_\lambda(x_J)}\right)-\varepsilon V(x_J)\Bigg]\Bigg\}. \tag{20}$$

It is remarkable that our system is similar to the relativistic quantum mechanics with non-zero minimum momentum uncertainty (see Refs. [41,43]). Otherwise, if the $d_\lambda x-$measure term contains a singularity problem, this problem will not worry us and for motivation see Ref. [40]. In addition, from Eq. (20), it is possible to resort to the Schrodinger equation of the studied system and verify that the action fits our problem that concerns the position-dependent mass. As we know the relation between the wave function and the corresponding propagator is

$$\Psi(x_J,t+\varepsilon)=\sqrt{\frac{m_0}{2\pi\imath\hbar\varepsilon}}\int\sqrt{\frac{m_\lambda(x_{J-1})}{m_0}}\exp\left\{\frac{\imath}{\hbar}\mathcal{S}(J,J-1)\right\}\Psi(x_{J-1},t)dx_{J-1}, \tag{21}$$

with

$$\mathcal{S}(J,J-1)=\frac{m_\lambda(x_J)}{2\varepsilon}(\Delta x_J)^2-\frac{\varepsilon\hbar^2}{4}\left(-\frac{7}{8}\frac{(m'_\lambda(x_J))^2}{m^3_\lambda(x_J)}+\frac{m''_\lambda(x_J)}{m^2_\lambda(x_J)}\right)$$
$$-3\imath\hbar\frac{m'_\lambda(x_J)}{4m_\lambda(x_J)}\Delta x_J-\varepsilon V(x_J), \tag{22}$$

where $\mathcal{S}(J,J-1)$ is the infinitesimal action. By substituting $x_{J-1}$ by $x_J-y$, we expand $\Psi(x_J-y,t)$ and $\sqrt{m_\lambda(x_J-y)}$ around $y=0$ as follows,

$$\sqrt{m_\lambda(x_J-y)}=\sqrt{m_\lambda(x_j)}\left(1-\frac{y}{2}m'_\lambda(x_j)+\frac{y^2}{2}\left[\frac{m''_\lambda(x_j)}{2m_\lambda(x_j)}-\frac{(m'(x_j))^2}{(m_\lambda(x_j))^2}\right]\right), \tag{23}$$

and

$$\Psi(x_J-y,t)=\Psi\left(x_j,t\right)-y\frac{\partial\Psi}{\partial x_j}+\frac{1}{2}y^2\frac{\partial^2\Psi}{\partial x_j^2}. \tag{24}$$

Substituting (23) and (24) into (21), we can rewrite the expression (21) as:

$$\Psi(x_J,t+\varepsilon)=\sqrt{\frac{m_\lambda(x_j)}{2\pi\imath\hbar\varepsilon}}e^{-\frac{\imath\varepsilon}{\hbar}\left[\frac{\hbar^2}{4}\left(\frac{m''_\lambda(x_J)}{m^2_\lambda(x_J)}-\frac{7}{8}\frac{(m'_\lambda(x_J))^2}{m^3_\lambda(x_J)}\right)+V(x_J)\right]}$$
$$\times\int dy\,e^{\frac{\imath}{\hbar}\left(\frac{m_\lambda(x_J)}{2\varepsilon}y^2-3\imath\hbar\frac{m'_\lambda(x_J)}{4m_\lambda(x_J)}y\right)}\left[\Psi\left(x_j,t\right)-y\frac{\partial\Psi}{\partial x_j}+\frac{1}{2}y^2\frac{\partial^2\Psi}{\partial x_j^2}\right]$$
$$\times\left[1-\frac{y}{2}m'_\lambda(x_j)+\frac{y^2}{2}\left(\frac{m''_\lambda(x_j)}{2m_\lambda(x_j)}-\frac{(m'(x_j))^2}{(m_\lambda(x_j))^2}\right)\right]. \tag{25}$$

Then, using all the Gaussian integrations over $y$ and doing calculations to first order in $\varepsilon$, we get:

$$i\hbar\frac{\partial}{\partial t}\Psi(x_J,t)=\left[-\frac{\hbar^2}{2m_\lambda(x_J)}\frac{\partial^2}{\partial x_J^2}+\frac{\hbar^2}{4}\frac{m'_\lambda(x_J)}{m^2_\lambda(x_J)}\frac{\partial}{\partial x_J}+V(x_J)\right]\Psi(x_J,t). \tag{26}$$

This equation is the same Schrodinger equation of position-dependent mass with its Hamiltonian operator is

$$\hat{H}=-\frac{\hbar^2}{2m_0}\sqrt{\frac{m_0}{m_\lambda(x)}}\frac{\partial}{\partial x}\left(\sqrt{\frac{m_0}{m_\lambda(x)}}\frac{\partial}{\partial x}\right)+V(x). \tag{27}$$

Moreover, in order to convert the expression (20) to the standard form of the path integral formulation of quantum mechanics, we implement the coordinate transformation method [20,45] in the next section. It is self-evident that we are faced with the problem of determining the appropriate interval point prescription to calculate the exact quantum corrections.

## 3 Quantum Corrections Evaluation

It would be an exploit if all problems with position-dependent mass could be solved in one dimension using the generalized displacement operator for non-relativistic quantum dynamics, with the path integral formalism. Therefore, in order to make some specific applications in the next section, we will first examine the $\alpha$-point discretization interval value by comparing it to the Schrodinger equation. We define the $\alpha$-point discretization interval as:

$$\bar{x}_J^{(\alpha)} = \alpha x_J + (1-\alpha)\, x_{J-1}. \tag{28}$$

We convert $x_J$ and $x_{J-1}$ to $\bar{x}_J^{(\alpha)}$, there are three quantum corrections obtained in expression (20),

- the first is related to the action $C_{act}^{(1)}$
- the second correction is related to the measure $C_{mes}^{(1)}$
- and the third correction is related to the $f-$factor (the $C_f^T-$prefactor).

The basic idea is to expand the kinetic term in powers of the $\alpha-$point prescription $\bar{x}_J^{(\alpha)}$, we get

$$\exp\left(\frac{\imath}{\hbar}\sum_{J=1}^{N+1}\left(\frac{m_\lambda(x_J)}{2\varepsilon}\left(\Delta x_J\right)^2\right)\right) = \exp\left[\frac{\imath}{\hbar}\sum_{J=1}^{N+1}\frac{m_\lambda(\bar{x}_J^{(\alpha)})}{2\varepsilon}\left(\Delta x_J\right)^2\right]\left(1+C_{act}^{(1)}\right), \tag{29}$$

where $C_{act}^{(1)}$ is the first quantum correction related to the action:

$$C_{act}^{(1)} = \frac{\imath}{2\hbar\varepsilon}\left[\frac{2\,(1-\alpha)\,m'_\lambda(\bar{x}_J^{(\alpha)})}{2}\left(\Delta x_J\right)^3 + (1-\alpha)^2\,\frac{m''_\lambda(\bar{x}_J^{(\alpha)})}{2}\left(\Delta x_J\right)^4\right] - \frac{2\,(1-\alpha)^2}{(2\hbar\varepsilon)^2}\,\frac{\left(m'_\lambda(\bar{x}_J^{(\alpha)})/m_0\right)^2}{4}\left(\Delta x_J\right)^6, \tag{30}$$

here $m'_\lambda(\bar{x}_J^{(\alpha)})$ and $m''_\lambda(\bar{x}_J^{(\alpha)})$ are the abbreviated derivatives function $m_\lambda(\bar{x}_J^{(\alpha)})$ at the point $\bar{x}_J^{(\alpha)}$. Also, the measure term contains corrections, will be developed as

$$\sqrt{\frac{m_\lambda(x_J)}{m_0}} = \sqrt{\frac{m_\lambda(\bar{x}_J^{(\alpha)})}{m_0}}\left(1+C_{mes}^{(1)}\right), \tag{31}$$

where $C_{mes}^{(1)}$ is the second correction related to measurement

$$C_{mes}^{(1)} = (1-\alpha)\,\frac{m'_\lambda(\bar{x}_J^{(\alpha)})}{2m_\lambda(\bar{x}_J^{(\alpha)})}\Delta x_J + \frac{(1-\alpha)^2}{2}\left[\frac{m''_\lambda(\bar{x}_J^{(\alpha)})}{2m_\lambda(\bar{x}_J^{(\alpha)})} - \frac{(m'_\lambda(\bar{x}_J^{(\alpha)}))^2}{4(m_\lambda(\bar{x}_J^{(\alpha)}))^2}\right]\left(\Delta x_J^2\right). \tag{32}$$

Also, in the second term of action (20) there is the prefactor term and it will be developed to second order in $\Delta x_J$:

$$\exp\left(\frac{3}{4}\frac{m'_\lambda(x_J)}{m_\lambda(x_J)}\Delta x_J\right) = 1 + C_f^T, \tag{33}$$

where the quantum correction related to the $f-$factor is obtained as follows:

$$C_f^T = \tfrac{3}{4}\frac{m'_\lambda(\bar{x}_J^{(\alpha)})}{m_\lambda(\bar{x}_J^{(\alpha)})}\Delta x_J + \tfrac{9}{32}\frac{(m'_\lambda(\bar{x}_J^{(\alpha)}))^2}{(m_\lambda(\bar{x}_J^{(\alpha)}))^2}\left(\Delta x_J\right)^2 + \tfrac{3}{4}\,(1-\alpha)\left(\frac{m''_\lambda(\bar{x}_J^{(\alpha)})}{m_\lambda(\bar{x}_J^{(\alpha)})} - \frac{(m'_\lambda(\bar{x}_J^{(\alpha)}))^2}{(m_\lambda(\bar{x}_J^{(\alpha)}))^2}\right)\left(\Delta x_J\right)^2. \tag{34}$$

Moreover, in all the expressions of quantum correction proposed above, we have retained only the terms which are all of order $\varepsilon$. In order to convert the path integral defined in the Eq. (20) to the usual form of the Feynman path integral, we must change the kinetic term to the conventional one, namely, with a constant mass. Carrying out the coordinate transformation over $x$ by $x_J = g(\eta_J)$. This transformation generates two corrections:

– the first is related to the action $C_{act}^{(2)}$
– and the other correction is related to the measure $C_m^{(2)}$.

The $\alpha$-point expansion of $\Delta x_J$ reads at each $(J)$

$$\begin{aligned}\Delta x_J &= g(\eta_J) - g(\eta_{J-1}) \\ &= \Delta\eta_J \bar{g}_J^{(\alpha)\prime}\left(1 + \frac{(1-2\alpha)}{2!}\frac{\bar{g}_J^{(\alpha)\prime\prime}}{\bar{g}_J^{(\alpha)\prime}}\Delta\eta_J + \frac{(1-\alpha)^3+\alpha^3}{3!}\frac{\bar{g}_J^{(\alpha)\prime\prime\prime}}{\bar{g}_J^{(\alpha)\prime}}\Delta\eta_J^2\right),\end{aligned} \tag{35}$$

where $\bar{g}_J^{\alpha} = g(\bar{\eta}_J^{\alpha})$, and all the derivatives of the function $g(\bar{\eta}_J^{\alpha})$ in Eq. (35) denote derivatives with respect to the new coordinate $\bar{\eta}_J^{(\alpha)}$ (i.e., $\bar{\eta}_J^{(\alpha)} = \alpha\eta_J + (1-\alpha)\eta_{J-1}$). It would be useful to express the function $g(\eta)$ by this condition

$$\frac{dg\,(\eta)}{d\eta} = [m_\lambda(x)/m_0]^{-1/2}. \tag{36}$$

Thereafter, we develop the exponential of the kinetic term by expanding $g(\eta_J)$ and $g(\eta_{J-1})$ up to sixth order in $\Delta\eta_J$, so that

$$\exp\left[\frac{\imath}{\hbar}\sum_{J=1}^{N+1}\left(\frac{m_\lambda(x_J)}{2\varepsilon}\left(\Delta x_J\right)^2\right)\right] = \exp\left\{\frac{\imath}{\hbar}\sum_{J=1}^{N+1}\left[\frac{m_0\Delta\eta_J^2}{2\varepsilon}\right]\right\}\left[1 + C_{act}^{(1)}\right]\left[1 + C_{act}^{(2)}\right], \tag{37}$$

where $C_{act}^{(2)}$ is the second correction on the action and is obtained as,

$$\begin{aligned}C_{act}^{(2)} = \Bigg\{\frac{\imath m_0}{2\hbar\varepsilon}\Bigg[&(1-2\alpha)\frac{\bar{g}_J^{(\alpha)\prime\prime}}{\bar{g}_J^{(\alpha)\prime}}\Delta\eta_J^3 \\ &+\left[\frac{(1-2\alpha)^2}{4}\left(\frac{\bar{g}_J^{(\alpha)\prime\prime}}{\bar{g}_J^{(\alpha)\prime}}\right)^2 + \frac{(1-\alpha)^3+\alpha^3}{3}\frac{\bar{g}_J^{(\alpha)\prime\prime\prime}}{\bar{g}_J^{(\alpha)\prime}}\right]\Delta\eta_J^4\Bigg] \\ &-\frac{(1-2\alpha)^2}{2(2\hbar\varepsilon/m_0)^2}\left(\frac{\bar{g}_J^{(\alpha)\prime\prime}}{\bar{g}_J^{(\alpha)\prime}}\right)^2\Delta\eta_J^6 + \ldots\Bigg\}.\end{aligned} \tag{38}$$

Following the Kleinert method [45], the measure induces also a correction, we find

$$\prod_{J=1}^{N}\int dx_J = \frac{\partial\Delta x_J}{\partial\Delta\eta_J} = \bar{g}_J^{(\alpha)\prime}\left(1 + C_m^{(2)}\right), \tag{39}$$

with $C_m^{(2)}$ is the second correction on the measure

$$C_m^{(2)} = (1-2\alpha)\frac{\bar{g}_J^{(\alpha)\prime\prime}}{\bar{g}_J^{(\alpha)\prime}}\Delta\eta_J + \frac{(1-\alpha)^3+\alpha^3}{2}\frac{\bar{g}_J^{(\alpha)\prime\prime\prime}}{\bar{g}_J^{(\alpha)\prime}}\Delta\eta_J^2. \tag{40}$$

We can remove the terms with odd order $(\Delta\eta_J)^{2\ell+1}$ and for the even terms, we can use the following expectation values [20]:

$$\left\langle(\Delta\eta)^{2\ell}\right\rangle = (\imath\hbar\varepsilon/m_0)^{\ell}\,(2\ell-1)!!. \tag{41}$$

So by combining all the corrections and performing substitution given by the Eq. (41), the $C_T$ − total correction becomes as,

$$C_T = \frac{\imath\hbar\varepsilon}{2m_0}\left[\frac{9}{4}\frac{\left(g''(\eta)\right)^2}{\left(g'(\eta)\right)^2} - \left(1 + \alpha - 2\alpha^2\right)\left[\frac{g'''(\eta)}{g'(\eta)}\right]\right]. \tag{42}$$

Thereafter, from the above results, we remark that in the deformed Heisenberg algebra (6), the calculation of quantum correction via path integral approach is dependent on the $\alpha$-point discretization interval, and it is the same remark obtained in the previous works [41,43,44]. To confirm the same results obtained previously, we will determine the $\alpha$ values by comparing them with a Schrodinger equation method. Hence, when inserting the

Eq. (42) to the transition amplitude (20) for a particle with PDM subject in the potential $V(x)$, the infinitesimal propagator takes the form

$$K(\eta_J, \eta_{J-1}, \varepsilon) = \sqrt{\frac{m_0}{2\pi\imath\hbar\varepsilon}} \exp\left\{\frac{\imath\varepsilon}{\hbar}\left[\frac{m_0}{2\varepsilon^2}(\Delta\eta_J)^2 - \frac{\hbar^2}{2m_0}\frac{g'''(\eta)}{g'(\eta)}\left(\alpha - 2\alpha^2\right) - V(\eta_J)\right]\right\}. \tag{43}$$

From Eq. (43), we have successfully converted the kinetic term defined in Eq. (29) to the conventional form by using the following coordinate transformation $x \longrightarrow \eta$, which is known local path integral representation. It is worth noting that the term obtained in the Kernel expression is that the effective potential is a function of the $\alpha$ values. In order to determine these values, we make the proceeding Schrodinger equation method. Further it can be readily find from relationship between this propagation and wave function, where the corresponding wave function must satisfy for infinitesimal $\varepsilon$ by the following relation [46],

$$\Psi(\eta_J, t+\varepsilon) = \int K(\eta_J, \eta_{J-1}, \varepsilon)\Psi(\eta_{J-1}, t)d\eta_{J-1}, \tag{44}$$

$K(\eta_J, \eta_{J-1}, \varepsilon)$ is the infinitesimal propagator. By substituting $\eta_{J-1}$ by $\eta_J - y$ and expanding $\Psi(\eta_J - y, t)$ around $y = 0$, we find

$$\begin{aligned}\Psi(\eta_J, t+\varepsilon) &= \sqrt{\frac{m_0}{2\pi\imath\hbar\varepsilon}} e^{-\frac{\imath\varepsilon}{\hbar}\left[\frac{\hbar^2}{2m_0}\frac{g'''(\eta)}{g'(\eta)}(\alpha-2\alpha^2)+V(\eta_J)\right]} \\ &\times \int dy\, e^{\frac{\imath\varepsilon}{\hbar}\frac{m_0}{2\varepsilon^2}y^2}\left[\Psi(\eta_J, t) - y\frac{\partial\Psi(\eta_J, t)}{\partial\eta_J} + \frac{y^2}{2}\frac{\partial^2\Psi(\eta_J, t)}{\partial\eta_J^2}\right].\end{aligned} \tag{45}$$

Then using the Gaussian integration over $y$, we get:

$$\Psi(\eta_J, t+\varepsilon) = e^{-\frac{\imath\varepsilon}{\hbar}\left(\frac{\hbar^2}{2m_0}\frac{g'''(\eta)}{g'(\eta)}(\alpha-2\alpha^2)+V(\eta_j)\right)}\left[\Psi(\eta_J, t) + \frac{\imath\hbar\varepsilon}{2m_0}\frac{\partial^2\Psi(\eta_J, t)}{\partial\eta_J^2}\right]. \tag{46}$$

Expanding both sides in powers of $\varepsilon$ up to the first order, and convert $\eta$ to $x$, we find:

$$\begin{aligned}\imath\hbar\frac{\partial}{\partial t}\Psi(x_J, t) &= -\frac{\hbar^2}{2}\left[m_\lambda(x_J)\right]^{-1/2}\frac{\partial}{\partial x_J}\left[m_\lambda(x_J)\right]^{-1/2}\frac{\partial}{\partial x_J}\Psi(x_J, t) \\ &+V\left(x_J\right)\Psi(x_J, t) - \frac{\hbar^2}{2}\left(\alpha - 2\alpha^2\right)\left[-\frac{(m'_\lambda(x_J))^2}{(m_\lambda(x_J))^3} + \frac{m''_\lambda(x_J)}{2(m_\lambda(x_J))^2}\right]\Psi(x_J, t).\end{aligned} \tag{47}$$

On other hand, the Schrodinger equation for the deformed algebra (6) is given in Eq. (26). Therefore, Eq. (47) is similar to the Schrodinger equation for the system corresponding to the PDM particle when $\alpha = 0$ or $1/2$. In addition, when we use the standard model [20], there is a difference in correction between the two methods, which is generated by the measure term. Consequently, this leads to different total quantum correction and gives the different $\alpha-$points discretization interval which takes two values from the domain $\left[x_{J-1}, x_J\right]$, are $\alpha = (1 \pm 1/\sqrt{2})/\sqrt{2}$. Moreover, these $\alpha-$values have been confirmed in our previous works [41,44,47]. This also gives the guarantee on the need to change of $\alpha-$values of point discretization interval in the existence of a EUP or GUP compared to usual Heisenberg algebra.

Accordingly, for these considerations and by replacing $\alpha$ by 0 or $1/2$, the Feynman path integral in non-relativistic case with PDM and in the presence of potential $V(x)$ becomes as:

$$K(\eta_b, \eta_a, T) = \lim_{N\to\infty}\prod_{J=1}^{N}\int d\eta_J \prod_{J=1}^{N+1}\sqrt{\frac{m_0}{2\pi\imath\hbar\varepsilon}}\exp\left\{\frac{\imath\varepsilon}{\hbar}\left[\sum_{J=1}^{N+1}\frac{m_0}{2\varepsilon^2}(\Delta\eta_J)^2 - V(\eta_J)\right]\right\}, \tag{48}$$

where $\eta_a = \eta_0$, $\eta_b = \eta_{N+1}$. The use of the relation (36) ends up at the Feynman propagator with a constant mass where the potential $V(g(\eta))$ is simply replaced by $V_{eff} \equiv V(g(\eta))$ without additional potentials.

In the next section we treat some examples for the quantum systems with different position-dependent masses having exact solutions in the formulation of position-dependent infinitesimal translation operator.

## 4 Specific Cases

Now, let us consider some explicit examples which have analytical solutions in this path integral for PDM systems given by (48). In each case, the energy values and their corresponding wave functions are correctly obtained.

### 4.1 The First Case

In this case, we consider the scalar harmonic oscillator potential ($V(x) = \frac{m_0\omega^2}{2}x^2$) with the PDM defined by:

$$m_\lambda(x) = \frac{m_0}{(1+\lambda x^2)^2}, \tag{49}$$

where $m_0$, $\omega$ and $\lambda$ are constant and positive parameters to indicate mass, frequency, and deformation coefficients respectively, with dimension of $\lambda$ as $[\text{length}]^{-2}$. This example is similar to a new type of EUP whose modified Heisenberg relation includes a term of the linear function in $x^2$ (i.e., $\left[\hat{X}_\lambda, \hat{P}_\lambda\right] = \imath\hbar\left(1+\lambda x^2\right)$), where the EUP guarantees the existence of the minimum momentum $(\Delta p)_{\min} = \hbar\sqrt{\lambda}$. In order to convert the kinetic term to the conventional form, we use the following coordinate transformation:

$$\sqrt{\lambda}x = \tan(\sqrt{\lambda}\eta), \text{ where } \eta \in \left]-\pi/2\sqrt{\lambda}, +\pi/2\sqrt{\lambda}\right[. \tag{50}$$

This provides an exact path integral representation of the transition amplitude of a point particle moving in the symmetric Poschl-Teller potential. Thus, for the one-dimensional harmonic oscillator potential (49), the transformation $\sqrt{\lambda}x = \tan(\sqrt{\lambda}\eta)$ leads to the new propagator,

$$K(\eta_b, \eta_a, T) = \lim_{N\to\infty}\prod_{J=1}^{N}\int d\eta_J \prod_{J=1}^{N+1}\sqrt{\frac{m_0}{2\pi\imath\hbar\varepsilon}}$$
$$\times \exp\left\{\frac{\imath\varepsilon}{\hbar}\sum_{J=1}^{N+1}\left[\frac{m_0}{2\varepsilon^2}(\Delta\eta_J)^2 - \frac{m_0\omega^2}{2\lambda^2}(1+\tan^2(\sqrt{\lambda}\eta))\right]\right\}. \tag{51}$$

The solution of the spectral decomposition of the transition amplitude for the one-dimensional harmonic oscillator with the PDM particle defined in Eq. (51) simplifies to

$$K(\eta_b, \eta_a; T) = \sum_{n=0}^{+\infty}\sqrt{\lambda}\Gamma(\alpha)^2\left[\frac{2^{2\alpha-1}n!(n+\alpha)}{\pi\Gamma(n+2\alpha)}\right]\exp\left[-\frac{\imath}{\hbar}\left(\frac{\hbar^2\lambda T}{2m_0}\left(n^2+(2n+1)\alpha\right)\right)\right]$$
$$\times\left(\cos\left(\sqrt{\lambda}\eta_b\right)\right)^\alpha C_n^\alpha\left(\sin\left(\sqrt{\lambda}\eta_b\right)\right)\left(\cos\left(\sqrt{\lambda}\eta_a\right)\right)^\alpha C_n^\alpha\left(\sin\left(\sqrt{\lambda}\eta_a\right)\right), \tag{52}$$

where $C_n^\alpha(x)$ are Gegenbauer polynomials and $\alpha$ is a constant parameter given by:

$$\alpha = \frac{1}{2} + \frac{\sqrt{1+4m_0^2\omega^2/\lambda^2\hbar^2}}{2}. \tag{53}$$

From Eq. (52), the normalized eigenfunctions $\Psi_n(x)$ and $E_n$ have the expressions,

$$\Psi_n(x) = \Gamma(\alpha)\left[\frac{2^{2\alpha-1}n!(n+\alpha)\sqrt{\lambda}}{\pi\Gamma(n+2\alpha)}\right]^{\frac{1}{2}}\left(\cos\left(\arctan\left(\sqrt{\lambda}x\right)\right)\right)^\alpha$$
$$\times C_n^\alpha\left(\sin\left(\arctan(\sqrt{\lambda}x)\right)\right), \tag{54}$$

and

$$E_n = \hbar\omega\left[\left(n+\frac{1}{2}\right)\sqrt{1+\frac{\lambda^2\hbar^2}{4m_0^2}} + \frac{\lambda\hbar}{2m_0}\left(n^2+n+\frac{1}{2}\right)\right]. \tag{55}$$

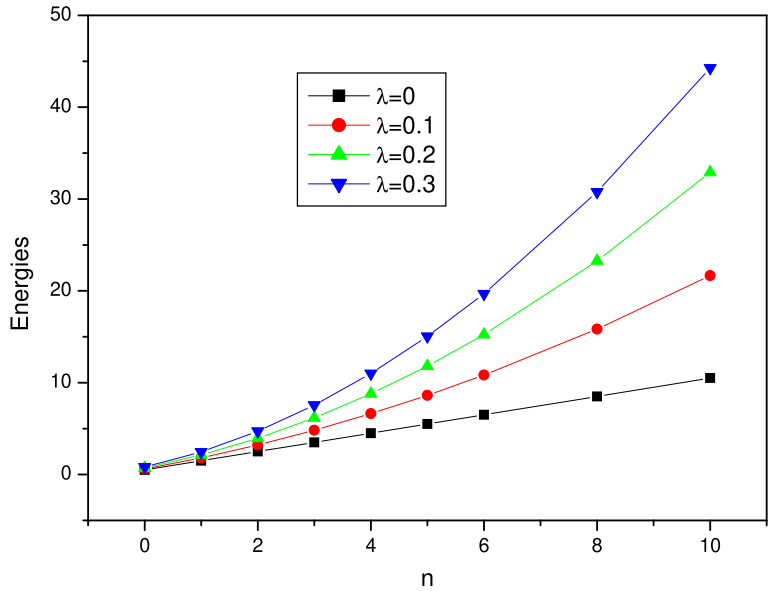


**Fig. 1** The energy $E_n$ as a function of $n$ for several values of $\lambda$ and $n \leq 10$

The energy $E_n$ is presented as a function of $n$ for several values of $\lambda$ and $n \leq 10$, with $\hbar = 1$, $\omega = 1$ and $2m_0 = 1$ in the Fig. 1.

Following this example, it is remarkable that given a small parameter of $\lambda$, the form of (55) can be easily expanded in terms of $\lambda$ and we obtain the corrections to the energy spectrum of the usual $1D$ harmonic oscillator, can be expressed as

$$E_n = \hbar\omega\left(n + \frac{1}{2}\right) + \lambda\frac{\hbar^2\omega}{2m_0}\left(n^2 + n + \frac{1}{2}\right) + \lambda^2\frac{\hbar^3\omega}{8m_0^2}\left(n + \frac{1}{2}\right). \tag{56}$$

From (56) we can see that the dependence of mass on spatial coordinate makes the energy levels for the harmonic oscillator potential depend on $n^2$. In the limit $\lambda \to 0$ we recover the wave functions and corresponding energy spectrum of the 1D harmonic oscillator potential in usual Heisenberg algebra [48].

Beside this example $m_\lambda(x) = m_0/(1 + \lambda x^2)^2$, there is also other case for position-dependent mass that are similar to the type chosen in this section, defined by the formula $m_\lambda(x) = m_0/(1 + \lambda x + \lambda^2 x^2)^2$ with a harmonic oscillator perturbed by a time-independent force which can give accurate results.

### 4.2 The Second Case

In the second example, we consider the mass of a particle and the scalar potential with an exponential form, where their expressions can be defined as

$$m_\lambda(x) = m_0 e^{\lambda x},\ V(x) = Ae^{\lambda x} + Be^{-\lambda x} + Ce^{-\frac{\lambda}{2}x}. \tag{57}$$

$A$, $B$, $C$ and $\lambda$ are positive constant, $m_0$ is the mass of particle in the case $\lambda = 0$ and $\lambda$ is a deformation coefficient with the length of an inverse unit. Recent studies indicate that the exponential variation of mass with position is useful to study the confined energy states of carriers in semiconductor quantum well structures, where we can often describe the motion of electrons by the envelope function effective-mass Schrodinger equation [49,50]. Where the absolute $\lambda-$value is proportion to the inverse of the quantum-well width.

In that case, the corresponding EUP modified Heisenberg relation includes the exponential function in $x$ (i.e., $\left[\hat{X}_\lambda, \hat{P}_\lambda\right] = i\hbar\exp(-\lambda x/2)$), where the EUP guarantees the existence of the minimum momentum $(\Delta p)_{\min}$ as a function of $\lambda$. By performing the same steps from the previous case, we find the following coordinate transformation

$$\eta = \frac{2}{\lambda}\exp(\frac{\lambda x}{2}) \text{ with } \eta \in ]0, +\infty[\,. \tag{58}$$

The new propagator with the above $\eta-$coordinate is defined in the following equation,

$$K(\eta_b,\eta_a,T)=\lim_{N\to\infty}\prod_{J=1}^{N}\int_0^\infty d\eta_J\prod_{J=1}^{N+1}\sqrt{\frac{m_0}{2\pi\imath\hbar\varepsilon}}$$
$$\times\exp\left\{\frac{\imath\varepsilon}{\hbar}\sum_{J=1}^{N+1}\left[\frac{m_0}{2\varepsilon^2}(\Delta\eta_J)^2-\frac{A\lambda^2\eta^2}{4}-\frac{4B}{\lambda^2\eta^2}-\frac{2C}{\lambda\eta}\right]\right\}. \tag{59}$$

This latter contains with three potentials and it is difficult to find an exact solution. Consequently, we will put some conditions on the parameters $A$, $B$ and $C$.

For the particular case when $A\neq 0$ and $B=C=0$, the expression (59) for $K(\eta_b,\eta_a,T)$ simplifies to (see Ref. [51]),

$$K\left(\eta_b,t_b;\eta_a,t_a\right)=\sum_{n=0}^{\infty}\frac{2m_0\omega}{\hbar}\sqrt{\eta_a\eta_b}\frac{n!}{\Gamma(n+\frac{3}{2})}\left(\frac{m_0\omega}{\hbar}\eta_a\eta_b\right)^{\frac{1}{2}}$$
$$\times e^{-\frac{m_0\omega}{2\hbar}(\eta_a^2+\eta_b^2)}L_n^{(\frac{1}{2})}\left(\frac{m_0\omega}{\hbar}\eta_a^2\right)L_n^{(\frac{1}{2})}\left(\frac{m_0\omega}{\hbar}\eta_b^2\right)e^{-i\omega T\left(2n+\frac{3}{2}\right)}, \tag{60}$$

which coincides exactly with the normalized wave functions $\Phi_n(x)$ reads as [52],

$$\Phi_n(x)=(-\tfrac{1}{4})^n\sqrt{\frac{\sqrt{m_0\omega/\hbar}}{2n!\Gamma(n+\frac{3}{2})}}e^{-\frac{2m_0\omega}{\hbar\lambda^2}\exp(\lambda x)}H_{2n+1}\left(2\sqrt{\frac{m_0\omega}{\lambda^2\hbar}}\exp(\frac{\lambda x}{2})\right), \tag{61}$$

and the corresponding energy levels is

$$E_n=\hbar\omega(2n+\frac{3}{2}),\omega=\sqrt{\frac{A}{2m_0}}\lambda, \tag{62}$$

where $H_{2n+1}(x)$ is defined as [53]

$$H_{2n+1}(x)=(-1)^n2^{2n+1}n!xL_n^{(1/2)}(x^2), \tag{63}$$

$H_n(x)$ represent the Hermit polynomials and $L_n^{(\alpha)}$ are called the generalized Laguerre polynomials.

The Fig. 2 represents the energy levels $E_n$ for $B=C=0$, $A=2m_0=\hbar=1$, and is a function of $n$ for several values of $\lambda$.

A Kratzer potential is represented by the choice of the parameters $A=0$, $B\neq 0$ and $C\neq 0$, therefore, according to Ref. [51], we get

$$K(\eta_b,\eta_a,T)=\sum_n\left(\frac{\sqrt{\lambda}}{n+\sigma+1/2}\right)^2\frac{n!}{a\Gamma(n+2\sigma+1)}e^{\frac{\imath T}{\hbar}\frac{2m_0C^2}{\hbar^2\lambda^2(n+\sigma+1/2)^2}}$$
$$\times\left(\frac{2\lambda\eta_b}{a(n+\sigma+1/2)}\right)^{\lambda}e^{-\frac{\lambda\eta_b}{a(n+\sigma+1/2)}}L_n^{(2\sigma)}\left(\frac{2\lambda\eta_b/a}{(n+\sigma+1/2)}\right)$$
$$\times\left(\frac{2\lambda\eta_a}{a(n+\sigma+1/2)}\right)^{\lambda}e^{-\frac{\lambda\eta_a}{a(n+\sigma+1/2)}}L_n^{(2\sigma)}\left(\frac{2\lambda\eta_a/a}{(n+\sigma+1/2)}\right)+\int_{-\infty}^{+\infty}dk\ldots \tag{64}$$

with $\sigma=\sqrt{\frac{8m_0B}{\lambda^2\hbar^2}+\frac{1}{4}}$ and $a=-\frac{\lambda^2\hbar^2}{2m_0C}$.

Consequently, we can easily extract the energies and the corresponding wave functions,

$$E_n=-\frac{2m_0C^2}{\hbar^2\lambda^2(n+\sigma+\frac{1}{2})^2}, \tag{65}$$

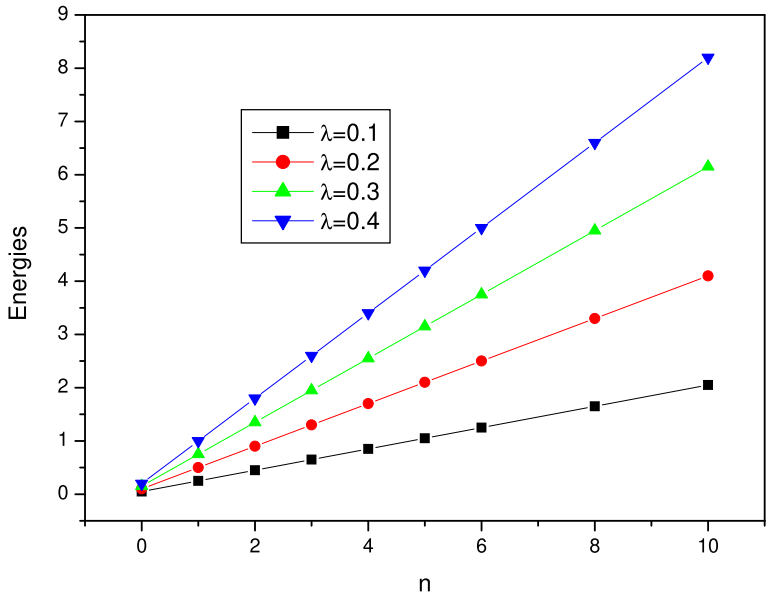


**Fig. 2** The energy $E_n$ as a function of $n$ for several values of $\lambda$ and $n \leq 10$, $B = C = 0$ and $A \neq 0$

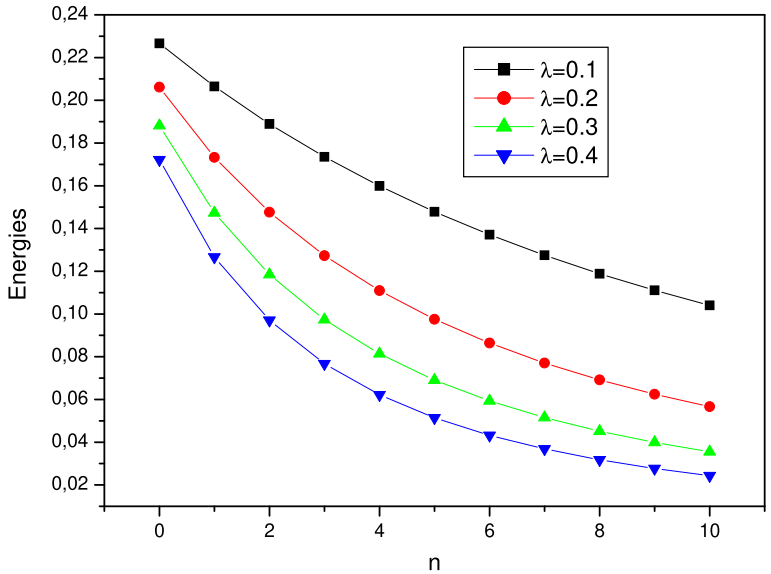


**Fig. 3** The energy $E_n$ as a function of $n$ for several values of $\lambda$ and $n \leq 10$, $B \neq 0$, $C \neq 0$ and $A = 0$

and

$$\Phi_n(x) = \sqrt{\frac{\lambda n!}{a\left(n+\sigma+\frac{1}{2}\right)^2 \Gamma(n+2\sigma+1)}} \left(\frac{\exp(\frac{\lambda x}{2})}{a(n+\sigma+\frac{1}{2})}\right)^{\lambda} \exp\left[-\frac{\exp(\frac{\lambda x}{2})}{2a(n+\sigma+\frac{1}{2})}\right] \times L_n^{(2\sigma)}\left(\frac{\exp(\frac{\lambda x}{2})}{a(n+\sigma+\frac{1}{2})}\right). \tag{66}$$

The energy for this special case is represented in Fig. 3, with $A = 0$, $B = C = 2m_0 = \hbar = 1$.

We note that when $B = 0$, the Eq. (65) coincides exactly with the following expression

$$E_n = -\frac{2m_0 C^2}{\hbar^2 \lambda^2 (n+1)^2}. \tag{67}$$

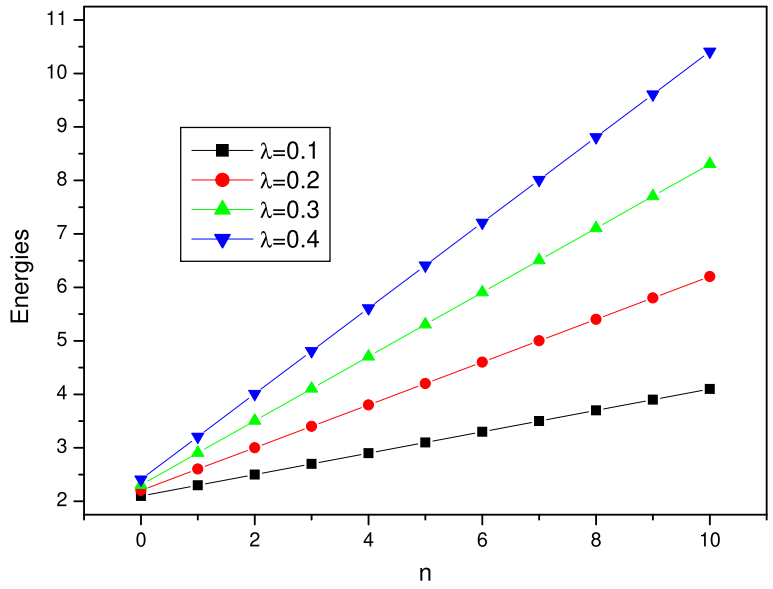


**Fig. 4** The energy $E_n$ as a function of $n$ for several values of $\lambda$ and $n \leq 10$, $B \neq 0$, $C = 0$ and $A \neq 0$

A general radial harmonic oscillator is represented by the choice of the parameters $A \neq 0$, $B \neq 0$ and $C = 0$. For this case we obtain

$$K(\eta_b, \eta_a, T) = \sum_{n=0}^{\infty} \frac{(2m_0\omega)\, n!\sqrt{\eta_a\eta_b}}{\hbar\Gamma(n+\kappa+1)} \left(\frac{m_0\omega}{\hbar}\eta_a\eta_b\right)^{\kappa} \exp\left(-\frac{m_0\omega}{2\hbar}(\eta_a^2+\eta_b^2)\right) \times L_n^{(\kappa)}\left(\frac{m_0\omega}{\hbar}\eta_b^2\right) L_n^{(\kappa)}\left(\frac{m_0\omega}{\hbar}\eta_a^2\right) \exp\left(-i\omega T(2n+\kappa+1)\right). \tag{68}$$

Hence, we get the spectral energies

$$E_n = \hbar\omega(2n+\kappa+1), \tag{69}$$

and the corresponding wave functions

$$\psi_n(x) = \left(\tfrac{2}{\lambda}\right)^{\frac{1}{2}+\kappa} e^{\frac{\lambda x}{2}(\kappa+\frac{1}{2})} \sqrt{\frac{2n!}{\Gamma(n+\kappa+1)}} \left(\frac{m_0\omega}{\hbar}\right)^{\frac{\kappa+1}{2}} \times \exp\left(-\frac{2m_0\omega}{\hbar\lambda^2}\exp(\lambda x)\right) L_n^{(\kappa)}\left(\frac{4m_0\omega}{\hbar\lambda^2}\exp(\lambda x)\right), \tag{70}$$

with $\omega = \sqrt{\frac{A}{2m_0}}\lambda$ and $\kappa = \sqrt{\frac{8m_0B}{\hbar^2\lambda^2} + \frac{1}{4}}$.

The Fig. 4 shows the energy $E_n$ varies with $n$ for different values of $\lambda$, and with $C = 0$, $A = B = 2m_0 = \hbar = 1$.

Finally when $\lambda \to 0$, it may be noted that the above results can revert to the case of the discrete levels energy are disappear, therefore only the continuous energy levels remain which entails a constant potential problem.

### 4.3 The Third Case

In this example, we can handle the Mathews-Lakshmanan oscillator corresponding to mass and scalar potential as follows [54]:

$$V(x) = \frac{m_0\omega^2x^2}{1+\lambda^2x^2}, \quad m_\lambda(x) = \frac{m_0}{1+\lambda^2x^2}. \tag{71}$$

Also this example, it is similar to a new type of EUP whose modified Heisenberg relation takes the following expression $[\hat{X}_\lambda, \hat{P}_\lambda] = \imath\hbar(1+\lambda^2 x^2)^{\frac{1}{2}}$. Under this consideration the choice of $g(\eta)$ is fixed by condition $x = g(\eta) = \sinh(\lambda\eta)/\lambda$. So the corresponding propagator is the Feynman path integral for the modified Poschl-Teller potential, which evolves to

$$K(\eta_b,\eta_a,T) = \lim_{N\to\infty}\prod_{J=1}^{N}\int d\eta_J \prod_{J=1}^{N+1}\sqrt{\frac{m_0}{2\pi\imath\hbar\varepsilon}}$$
$$\times \exp\left\{\frac{\imath\varepsilon}{\hbar}\sum_{J=1}^{N+1}\left[\frac{m_0}{2\varepsilon^2}(\Delta\eta_J)^2 + \frac{m_0\omega^2}{2\lambda^2\cosh^2(\lambda\eta)} - \frac{m_0\omega^2}{2\lambda^2}\right]\right\}. \tag{72}$$

According to Ref. [51], $K(\eta_b,\eta_a,T)$ of Eq. (72) simplifies to

$$K(\eta_b,\eta_a,T) = \sum_{n=0}\left(\frac{\lambda(n-\gamma-\frac{1}{2})\Gamma(2\gamma-n)}{n!}\right) e^{-\frac{iT}{\hbar}\left(\frac{m_0\omega^2}{2\lambda^2}-\frac{\hbar^2(n-\gamma+\frac{1}{2})^2}{2M}\right)}$$
$$\times P_{\gamma-1/2}^{(n-\gamma+1/2)}(\tanh(\lambda\eta_b))P_{\gamma-1/2}^{(n-\gamma+1/2)}(\tanh(\lambda\eta_a)). \tag{73}$$

Thus, the eigenfunctions corresponding to this propagator can be evolved as:

$$\Psi_n(x) = \sqrt{(n-\gamma-\frac{1}{2})\frac{\lambda\Gamma(2\gamma-n)}{n!}}\,P_{\gamma-1/2}^{(n-\gamma+1/2)}\left(\frac{\lambda x}{\sqrt{1+\lambda^2x^2}}\right). \tag{74}$$

We can write energy spectrum as

$$E_n = \frac{m_0\omega^2}{2\lambda^2} - \hbar^2\frac{(n-\gamma+\frac{1}{2})^2}{2M}, \tag{75}$$

where $P_\alpha^{(\beta)}(x)$ denotes the Legendre polynomials. $M$ and $\gamma$ are given as

$$M = \frac{m_0}{\lambda^2},\ \gamma = \sqrt{\frac{m_0^2\omega^2}{\lambda^4\hbar^2}+\frac{1}{4}}. \tag{76}$$

By using the Eqs. (76), the energy spectrum can be written as

$$E_n = \hbar\omega_\lambda(n+\frac{1}{2}) - \frac{\hbar^2\lambda^2}{2m_0}(n+\frac{1}{2})^2 - \frac{\hbar^2\lambda^2}{8m_0}, \tag{77}$$

where $\omega_\lambda = \omega\sqrt{1+\lambda^4\hbar^2/4m_0^2\omega^2}$.

These results (74) and (77) are coincide with Ref. [55]. Moreover, the representation of energy for this example is shown in Fig. 5, with $\omega = \hbar = 2m_0 = 1$.

The validity of our results in case $\lambda \to 0$ is compatible with the usual results of the 1D harmonic oscillator with mass constant.

## 5 Conclusion

In summary, we have designed the path integral method of the generalized infinitesimal translation operator in non-relativistic quantum mechanics and without spin term. It is compatible with position dependent mass systems. Under these considerations, we found the non-local Feynman's formalism of which kinetic term is dependent on position coordinate as well as for the measure term. This requires using a coordinates transformation method $x = g(\eta)$ at $\alpha-$points discretization interval. The main result is that the calculation depends on the $\alpha$-point discretization interval, and we conclude that the problem of discretization is definitively settled using the path integral framework. This lends credence to this paper about the necessity of changing $\alpha$-values of point discretization interval in the existence of a EUP or GUP compared to usual quantum mechanics. So as to obtain results very accurately through the examples presented in this paper, where the Green functions and

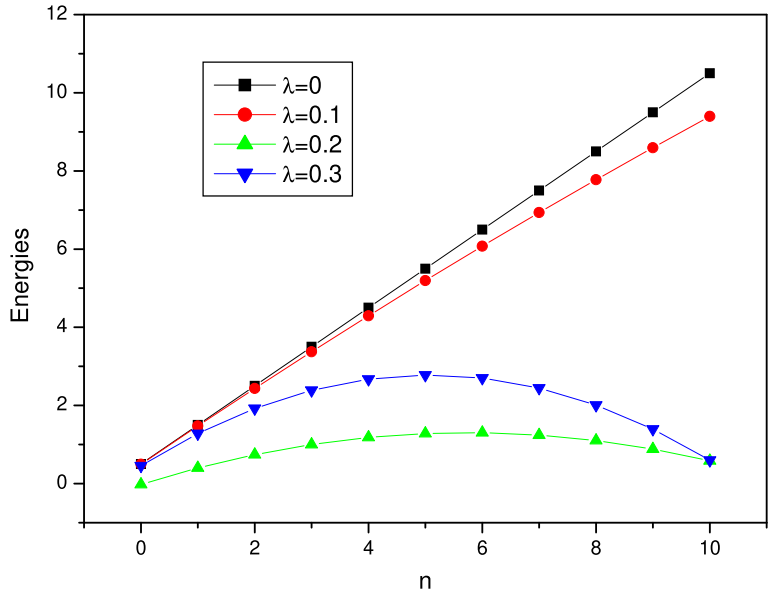


**Fig. 5** The energy $E_n$ as a function of $n$ for several values of $\lambda$ and $n \leq 10$

transition amplitude of each case are converted to the effective potential such as; the Poschl-Teller potential, Morse potentials, and the modified Poschl-Teller potential. The exact spectral energies and the corresponding wave functions have been obtained with curves of energies for each case. In the limit case by taking the parameter $\lambda = 0$, this means that $m_\lambda(x) = m_0$, we recover the same results obtained for physical systems in usual Heisenberg algebra.

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