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Hartmann Potential with a Minimal Length and Generalized Recurrence Relations for Matrix Elements

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Abstract In this work we study the Schrödinger equation in the presence of the Hartmann potential with a generalized uncertainty principle. We pertubatively obtain the matrix elements of the hamiltonian at first order in the parameter of deformation β and show that some degenerate states are removed. We give analytic expressions for the solutions of the diagonal matrix elements. Finally, we derive a generalized recurrence formula for the angular average values.

1 Introduction

In recent years many arguments have been suggested to motivate a modified Heisenberg algebra in quantum mechanics such as theories of quantum gravity and string theory, which lead to the existence of a minimal observable length expected to be of order of the Planck length [1–11] . The minimal length can be obtained from the deformed canonical commutation relation between position and momentum operators [12–17]:

$$[X, P] = i\hbar \left(1 + \beta P^2\right), \tag{1}$$

where β is a positive parameter of deformation. This commutation relation implies the following generalized uncertainty principle (GUP):

$$\Delta X \Delta P \geq \frac{\hbar}{2} \left(1 + \beta (\Delta P)^2\right), \tag{2}$$

which corresponds to a minimal length $(\Delta X)_{\min} = \hbar\sqrt{\beta}$.

There has recently been a lot of interest in the study of quantum mechanics problems in the presence of a minimal length with various potentials such as the harmonic oscillator, the Coulomb and Yukawa potentials, the Woods–Saxon potential, the Kratzer potential, ... [18–28]. Other works treat problems of momentum-dependent mass systems [29] non-hermitian systems,...[30]. The first author to use the standard perturbation theory to solve the Schrödinger equation of the central potentials in the presence of a minimal length was Brau in Ref. [21], where energy-level corrections were calculated and splitting of degenerate levels were found to occur. On the other hand the solutions of the Schrödinger equation with Hartmann potential $V(r, \theta) =$

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$\eta\sigma^2\left(\frac{e^2}{r}+\frac{q\hbar^2}{2\mu r^2\sin^2\theta}\right)$, where η and σ are positive real numbers with values ranging from about 1 to 10, and q is a real parameter, is well known [31–37]. This potential has been introduced in [38–40] to describe ring-shaped molecules. When $q=0$ and $\eta\sigma^2=Z$ the Hartmann potential reduces to the Coulomb potential. The purpose of this paper is to study the extension of the Schrödinger equation with the Hartmann potential in the presence of a minimal length. In Sect. 2 we use the first-order perturbation theory to give the general form of the hamiltonian matrix elements and, for a particular case, we show that the degeneracy is completely removed. In Sect. 3 we give an explicit analytical expression of the diagonal matrix elements and show that the splitting of the degenerate energy levels also occurs. In Sect. 4 we provide a general recurrence formula for the angular part. Finally, in the last section, we draw our conclusion.

2 Hamiltonian Matrix Elements

To calculate the hamiltonian matrix elements for a Hartmann potential in the presence of a minimal length, we solve the corresponding Schrödinger equation:

$$\left[\frac{\hat{P}^2}{2\mu}+V\left(\hat{r},\hat{\theta}\right)\right]\psi\left(\vec{r}\right)=E^{(\beta)}\psi\left(\vec{r}\right). \tag{3}$$

We choose to work with the following representation that verifies the relation (1) to first order in β:

$$\hat{X}_i\psi\left(\vec{r}\right)=x_i\psi\left(\vec{r}\right), \tag{4}$$

$$\hat{P}_i\psi\left(\vec{r}\right)=p_i\left(1+\beta p^2\right)\psi\left(\vec{r}\right),\qquad p_i=i\hbar\frac{\partial}{\partial x^i}. \tag{5}$$

To first order in β the Schrödinger equation (3) can be written as:

$$\left[\frac{p^2}{2\mu}+\frac{\beta p^4}{\mu}+V\left(r,\theta\right)\right]\hat{\psi}\left(r,\theta,\varphi\right)=E^{(\beta)}\hat{\psi}\left(r,\theta,\varphi\right). \tag{6}$$

In this equation the Hartmann potential in the presence of a minimal length appears within the perturbation term:

$$\frac{\beta p^4}{\mu} \tag{7}$$

To investigate the correlations we use the first-order perturbation theory. For $\beta=0$ the spectrum of Eq. (6) and the corresponding wave functions are well-known and are given by [31,41,42]:

$$\psi\left(r,\theta,\varphi\right)=\frac{1}{r}R\left(r\right)\Theta\left(\theta\right)\Phi\left(\phi\right), \tag{8}$$

where

$$\Phi\left(\phi\right)=\frac{1}{\sqrt{2\pi}}\exp\left(im\phi\right),\qquad m=0,\pm1,\pm2,\pm3,\ldots$$

$$\Theta_{nm}\left(x\right)=\frac{\Gamma\left(2k+1\right)}{\Gamma\left(k+1\right)}\sqrt{\frac{\left(2n+2k+1\right)}{2^{2k+1}\Gamma\left(n+2k+1\right)}}\left(1-x^2\right)^{k/2}C_n^{(k+1/2)}\left(x\right),\quad x=\cos\theta$$

$$R_{Nnm}\left(r\right)=\left(\frac{\mu\eta\sigma^2e^2}{\hbar^2n'}\right)^{1/2}\left[\frac{\left(n'-l-1\right)!}{n'\Gamma\left(n'+l+1\right)}\right]^{1/2}\left(\frac{2\mu\eta\sigma^2e^2}{\hbar^2n'}r\right)^{l+1}$$
$$\times\exp\left(-\frac{\mu\eta\sigma^2e^2}{\hbar^2n'}r\right)L_N^{(2l+1)}\left(\frac{2\mu\eta\sigma^2e^2}{\hbar^2n'}r\right),$$

$$k=\sqrt{m^2+\frac{q\eta\hbar^2\sigma^2}{2\mu}},\quad n'=N+l+1,\quad l=n+k,\quad N,n=0,1,2,3,\ldots \tag{9}$$

with N being the radial quantum number, $L_n^{(\nu)}(x)$ and $C_n^{(\nu)}(x)$ respectively stand for the associated Laguerre and the Gegenbauer (ultraspherical) polynomials. The orthogonality conditions for these functions are:

$$\int_{-1}^{1} dx\left(1-x^2\right)^{\nu-1/2}\left[C_n^{(\nu)}(x)\right]^2=\frac{\pi 2^{1-2\nu}\Gamma(n+2\nu)}{n!(n+\nu)[\Gamma(\nu)]^2},$$
$$\int_{-1}^{1} dx e^{-x}x^{\nu}\left[L_n^{(\nu)}(x)\right]^2=\frac{\Gamma(\nu+n+1)}{n!}. \tag{10}$$

The energy eigenvalues are:

$$E_{Nnm}^{(0)}=-\frac{\mu\left(\eta\sigma^2\right)^2e^4}{2\hbar^2}\left[N+n+\sqrt{m^2+\frac{q\eta\sigma^2\hbar^2}{2\mu}}+1\right]^{-2}. \tag{11}$$

From (6) and for $\beta=0$, we can write:

$$\langle N_1n_1m_1|\,p^2\,|N_2n_2m_2\rangle=E_{N_2n_2m_2}^{(0)}\delta_{N_1N_2}\delta_{n_1n_2}\delta_{m_1m_2}+\langle N_1n_1m_1|\,V(r,\theta)\,|N_2n_2m_2\rangle \tag{12}$$

Thus, the first-order perturbation theory gives the matrix element of the hamiltonian operator (6) up to first order in β as follows [21]:

$$\frac{\beta}{\mu}\langle N_1n_1m_1|\,p^4\,|N_2n_2m_2\rangle=4\mu\beta\left[\left(E_{N_2n_2m_2}^{(0)}\right)^2\delta_{N_1N_2}\delta_{n_1n_2}\delta_{m_1m_2}\right.$$
$$\left.-2E_{N_2n_2m_2}^{(0)}\langle N_1n_1m_1|\,V(r,\theta)\,|N_2n_2m_2\rangle+\langle N_1n_1m_1|\,(V(r,\theta))^2\,|N_2n_2m_2\rangle\right], \tag{13}$$

where

$$V(r,\theta)=\eta\sigma^2\left(\frac{e^2}{r}+\frac{q\hbar^2}{2\mu r^2\sin^2\theta}\right). \tag{14}$$

Each of these terms can be written as

$$\langle N_1n_1m_1|\,\frac{1}{r^s\sin^{2t}\theta}\,|N_2n_2m_2\rangle=\int_0^{\infty}R_{N_1n_1m_1}(r)\,R_{N_2n_2m_2}(r)\,r^{-s}dr$$
$$\times\int_{-1}^{1}dx\left(1-x^2\right)^{-t}\Theta_{n_1m_1}(x)\,\Theta_{n_2m_2}(x) \tag{15}$$

For the radial part, the first integral has been evaluated in [32] and its expression is given by:

$$\langle N_1n_1m_1|\,r^s\,|N_2n_2m_2\rangle=\int_0^{\infty}R_{N_1n_1m_1}(r)\,R_{N_2n_2m_2}(r)\,r^s dr$$
$$=\frac{\eta\sigma^2}{n_1n_2}\sqrt{\frac{N_1!N_2!}{\Gamma(2l_1+N+2)\,\Gamma(2l_2+N+2)}}\left(\frac{2\eta\sigma^2}{n_1'}\right)^{l_1+1}\left(\frac{2\eta\sigma^2}{n_2'}\right)^{l_2+1}$$
$$\times\left[\eta\sigma^2\left(1/n_1'+1/n_2'\right)\right]^{-s-l_1-l_2-3}\sum_{m_1}^{N}\sum_{m_2}^{N}\frac{(-1)^{2N+m_2}}{m_1!m_2!}\left(\frac{1/n_1'-1/n_2'}{1/n_1'+1/n_2'}\right)^{m_1+m_2}$$
$$\times\Gamma(l_1+l_2+s+m_1+m_2+3)\sum_{m_3=0}^{t}\binom{l_1+l_2+s+m_2+1}{N-m_1-m_3}$$
$$\times\binom{l_1+l_2+s+m_1+1}{N-m_2-m_3}\binom{l_1+l_2+s+m_1+m_2+m_3+2}{m_3}, \tag{16}$$

with $t = \min(N - m_1, N - m_2)$ and $s < l_1 + l_2 + 3$. The only values of s which contribute in the calculation of the matrix elements given in the expression (13) are $s = \{-1, -2, -3, -4\}$.

For the angular part, the two integrals which contribute in (13) are given as:

$$\int_{-1}^{1} dx \left(1 - x^2\right)^{-1} \Theta_{n_1 m_1}(x)\, \Theta_{n_2 m_2}(x) = \frac{\Gamma(2k_1 + 1)\,\Gamma(2k_2 + 1)}{\Gamma(k_1 + 1)\,\Gamma(k_2 + 1)} \sqrt{\frac{(2n_1 + 2k_1 + 1)(2n_2 + 2k_2 + 1)}{2^{2k_1 + k_2 + 2}\Gamma(n_1 + 2k_1 + 1)\,\Gamma(n_2 + 2k_2 + 1)}} \times \int_{-1}^{1} dx \left(1 - x^2\right)^{(k_1 + k_2 - 2)/2} C_{n_1}^{(k_1 + 1/2)}(x)\, C_{n_2}^{(k_2 + 1/2)}(x) \tag{17}$$

and

$$\int_{-1}^{1} dx \left(1 - x^2\right)^{-2} \Theta_{n_1 m_1}(x)\, \Theta_{n_2 m_2}(x) = \frac{\Gamma(2k_1 + 1)\,\Gamma(2k_2 + 1)}{\Gamma(k_1 + 1)\,\Gamma(k_2 + 1)} \sqrt{\frac{(2n_1 + 2k_1 + 1)(2n_2 + 2k_2 + 1)}{2^{2k_1 + k_2 + 2}\Gamma(n_1 + 2k_1 + 1)\,\Gamma(n_2 + 2k_2 + 1)}} \times \int_{-1}^{1} dx \left(1 - x^2\right)^{(k_1 + k_2 - 4)/2} C_{n_1}^{(k_1 + 1/2)}(x)\, C_{n_2}^{(k_2 + 1/2)}(x) \tag{18}$$

We use the following integral of the product of two Gegenbauer polynomials [43] with $\theta > -1/2$, $\mu > -1/2$, and $\lambda > -1/2$:

$$\int_{-1}^{1} C_l^{\theta}(x)\, C_m^{\mu}(x) \left(1 - x^2\right)^{\lambda - 1/2} dx = \frac{\pi 2^{1-2\lambda}}{\Gamma(\mu)\,\Gamma(\theta)} \sum_{k=0}^{[l/2]} \left[\frac{(l - 2k + \lambda)}{(l - 2k)!k!s!} \times \frac{\Gamma(l - \theta - k)}{\Gamma(l + \lambda - k + 1)} \frac{\Gamma(l - 2k + 2\lambda)\,\Gamma(m + \mu - s)}{\Gamma(m + \lambda - s + 1)} (\theta - \lambda)_k (\mu - \lambda)_s \right], \tag{19}$$

where $m = l - 2k + 2s$, with $[m/2] \geq s \in N$, and $(z)_n$ is the Pochhammer symbol [44]:

$$(z)_n = z(z + 1) \ldots (z + n - 1) = \frac{\Gamma(z + n)}{\Gamma(z)}. \tag{20}$$

The integral (19) vanishes for odd values of $l + m$. Thus, the expressions (17) and (18) take the following forms:

$$\int_{-1}^{1} dx \left(1 - x^2\right)^{-1} \Theta_{n_1 m_1}(x)\, \Theta_{n_2 m_2}(x) = \left|\cos \frac{\pi}{2}(n_1 + n_2)\right| \times \frac{\Gamma(2k_1 + 1)}{\Gamma(k_1 + 1)} \frac{\Gamma(2k_1 + 1)}{\Gamma(k_1 + 1)} \left[\frac{(2n_1 + 2k_1 + 1)}{2^{2(k_1 + k_2 + 1)}\Gamma(n_1 + 2k_1 + 1)} \frac{(2n_2 + 2k_2 + 1)}{\Gamma(n_2 + 2k_2 + 1)} \right]^{1/2} \times \frac{\pi 2^{2 - k_1 - k_2}}{\Gamma(k_1 + 1/2)\,\Gamma(k_2 + 1/2)} \sum_{p=0}^{[n_1/2]} \left[\frac{[n_1 - 2p + (k_1 + k_2 - 1)/2]}{(n_1 - 2p)!p!s!} \times \frac{\Gamma(n_1 - 2p + k_1 + k_2 - 1)\,\Gamma(n_2 + k_1 - s + 1/2)}{\Gamma\left(n_2 + \frac{k_1 + k_2 + 1}{2} - s\right)} \times \frac{\Gamma(n_1 - k_1 - p - 1/2)}{\Gamma\left(n_1 + \frac{k_1 + k_2 + 1}{2} - p\right)} \left(\frac{k_1 - k_2}{2} + 1\right)_p \left(\frac{k_2 - k_1}{2} + 1\right)_s \right]. \tag{21}$$

and

$$\int_{-1}^{1} dx \left(1 - x^2\right)^{-2} \Theta_{n_1 m_1}(x)\, \Theta_{n_2 m_2}(x) = \left|\cos \frac{\pi}{2}(n_1 + n_2)\right|$$

$$\times \frac{\Gamma(2k_1+1)}{\Gamma(k_1+1)}\frac{\Gamma(2k_1+1)}{\Gamma(k_1+1)}\left[\frac{(2n_1+2k_1+1)}{2^{2(k_1+k_2+1)}\Gamma(n_1+2k_1+1)}\frac{(2n_2+2k_2+1)}{\Gamma(n_2+2k_2+1)}\right]^{1/2}$$
$$\times \frac{\pi 2^{4-k_1-k_2}}{\Gamma(k_1+1/2)\Gamma(k_2+1/2)}\sum_{p=0}^{[n_1/2]}\left[\frac{[n_1-2p+(k_1+k_2-3)/2]}{(n_1-2p)!p!s!}\right.$$
$$\times \frac{\Gamma(n_1-2p+k_1+k_2-3)\Gamma(n_2+k_1-s+1/2)}{\Gamma\left(n_2+\frac{k_1+k_2-1}{2}-s\right)}$$
$$\left.\times \frac{\Gamma(n_1-k_1-p-1/2)}{\Gamma\left(n_1+\frac{k_1+k_2-1}{2}-p\right)}\left(\frac{k_1-k_2}{2}+2\right)_p\left(\frac{k_2-k_1}{2}+2\right)_s\right]. \tag{22}$$

Substuting (21) and (22) in (13), the general form of the hamiltonian matrix elements in (13) is given by the expression:

$$\frac{\beta}{4\mu}\langle N_1n_1m_1|\,p^4\,|N_2n_2m_2\rangle = \frac{\mu^2\left(\eta\sigma^2\right)^4 e^8}{4\hbar^4}\left(n_2'\right)^{-4}\delta_{N_1N_2}\delta_{n_1n_2}\delta_{m_1m_2}$$
$$+\frac{\mu\left(\eta\sigma^2\right)^4 e^6}{\hbar^2}\left(n_2'\right)^{-2}\langle N_1n_1m_1|\,r^{-1}\,|N_2n_2m_2\rangle$$
$$+\left(\eta\sigma^2\right)^2 e^4\langle Nn_1m_1|\,r^{-2}\,|N_2n_2m_2\rangle$$
$$+\left[\frac{\left(\eta\sigma^2\right)^4 e^6 q}{2}\left(n_2'\right)^{-2}\langle N_1n_1m_1|\,r^{-2}\,|N_2n_2m_2\rangle\right.$$
$$\left.+\frac{\left(\eta\sigma^2\right)e^2q\hbar^2}{\mu}\langle N_1n_1m_1|\,r^{-3}\,|N_2n_2m_2\rangle\right]$$
$$\times \frac{\Gamma(2k_1+1)}{\Gamma(k_1+1)}\frac{\Gamma(2k_1+1)}{\Gamma(k_1+1)}\left[\frac{(2n_1+2k_1+1)}{2^{2(k_1+k_2+1)}\Gamma(n_1+2k_1+1)}\frac{(2n_2+2k_2+1)}{\Gamma(n_2+2k_2+1)}\right]^{1/2}$$
$$\times \frac{\pi 2^{2-k_1-k_2}}{\Gamma(k_1+1/2)\Gamma(k_2+1/2)}\left|\cos\frac{\pi}{2}(n_1+n_2)\right|$$
$$\times \sum_{p=0}^{[n_1/2]}\left[\frac{[n_1-2p+(k_1+k_2-1)/2]}{(n_1-2p)!p!s!}\right.$$
$$\times \frac{\Gamma(n_1-2p+k_1+k_2-1)\Gamma(n_2+k_1-s+1/2)}{\Gamma\left(n_2+\frac{k_1+k_2+1}{2}-s\right)}$$
$$\left.\times \frac{\Gamma(n_1-k_1-p-1/2)}{\Gamma\left(n_1+\frac{k_1+k_2+1}{2}-p\right)}\left(\frac{k_1-k_2}{2}+1\right)_p\left(\frac{k_2-k_1}{2}+1\right)_s\right]$$
$$+\left|\cos\frac{\pi}{2}(n_1+n_2)\right|\left(\eta\sigma^2\right)^2\frac{q^2\hbar^4}{4\mu^2}\frac{\Gamma(2k_1+1)}{\Gamma(k_1+1)}\frac{\Gamma(2k_1+1)}{\Gamma(k_1+1)}$$
$$\times\left[\frac{(2n_1+2k_1+1)}{2^{2(k_1+k_2+1)}\Gamma(n_1+2k_1+1)}\frac{(2n_2+2k_2+1)}{\Gamma(n_2+2k_2+1)}\right]^{1/2}$$
$$\times \frac{\pi 2^{4-k_1-k_2}}{\Gamma(k_1+1/2)\Gamma(k_2+1/2)}\sum_{p=0}^{[n_1/2]}\left[\frac{[n_1-2p+(k_1+k_2-3)/2]}{(n_1-2p)!p!s!}\right.$$
$$\times \frac{\Gamma(n_1-2p+k_1+k_2-3)\Gamma(n_2+k_1-s+1/2)}{\Gamma\left(n_2+\frac{k_1+k_2-1}{2}-s\right)}$$

$$\times \frac{\Gamma\left(n_1-k_1-p-1/2\right)}{\Gamma\left(n_1+\frac{k_1+k_2-1}{2}-p\right)}\left(\frac{k_1-k_2}{2}+2\right)_p\left(\frac{k_2-k_1}{2}+2\right)_s\Bigg]$$
$$\times \langle N_1 n_1 m_1 | \, r^{-4} \, | N_2 n_2 m_2 \rangle \ , \tag{23}$$

where $\langle N_1 n_1 m_1 | \, r^s \, | N_2 n_2 m_2 \rangle$ are given by replacing $s = \{-1, -2, -3, -4\}$ in (16).

Example: The States **|010⟩** and **|100⟩**

In the ordinary case (i.e. $\beta = 0$), $|010\rangle$ and $|100\rangle$ are two degenerate states. It is clear that the matrix (13) is actually diagonal ($\langle 100| \, p^4 \, |010\rangle = 0$) which can be seen by using the expressions:

$$L_0^{(\alpha)}(x) = 1, \qquad L_1^{(\alpha)}(x) = -x + \alpha + 1 \tag{24}$$
$$C_0^{(\alpha)}(x) = 1, \qquad C_1^{(\alpha)}(x) = 2\alpha x \tag{25}$$

From Eq. (13), a straightforward calculation gives the energy corrections at first order in the parameter β as follows:

$$\begin{aligned}
\Delta E_{010} = 4\mu^2 \Bigg[& \left(E_{010}^{(0)}\right)^2 - \frac{abAE_{010}^{(0)}(2k_0+1)^2}{k_0+2}\left(\frac{2^{2k_0+1}[\Gamma(k_0+1)]^2}{\Gamma(2k_0+2)} - \frac{2^{2k+3}[\Gamma(k_0+2)]^2}{\Gamma(2k_0+4)}\right) \\
& - \frac{a^2bBE_{010}^{(0)}(2k_0+1)^2}{(k_0+2)(2k_0+3)}\left(\frac{2^{2k_0-1}[\Gamma(k_0)]^2}{\Gamma(2k)} - \frac{2^{2k_0+1}[\Gamma(k_0+1)]^2}{\Gamma(2k_0+2)}\right) \\
& + \frac{a^2bA^2(2k_0+1)^2}{2(k_0+2)(2k_0+3)}\left(\frac{2^{2k_0+1}[\Gamma(k_0+1)]^2}{\Gamma(2k_0+2)} - \frac{2^{2k_0+3}[\Gamma(k_0+2)]^2}{\Gamma(2k_0+4)}\right) \\
& + \frac{a^4bB^2(2k_0+1)}{2(k_0+2)(2k_0+3)(2k_0+2)}\left(\frac{2^{2k_0-1}[\Gamma(k_0-1)]^2}{\Gamma(2k_0-2)} - \frac{2^{2k_0+1}[\Gamma(k_0)]^2}{\Gamma(2k_0)}\right) \\
& + \frac{a^3bAB(2k_0+1)^2}{(k_0+2)(2k_0+3)(2k_0+2)}\left(\frac{2^{2k_0-1}[\Gamma(k_0)]^2}{\Gamma(2k_0)} - \frac{2^{2k_0+1}[\Gamma(k_0+1)]^2}{\Gamma(2k_0+2)}\right)\Bigg] \\
\Delta E_{100} = 4\mu^2 \Bigg[& \left(E_{100}^{(0)}\right)^2 - \frac{abAE_{100}^{(0)}}{(2+k_0)}\frac{2^{2k_0+1}[\Gamma(k_0+1)]^2}{\Gamma(2k_0+2)} \\
& - \frac{a^2bBE_{100}^{(0)}2^{2k_0-1}[\Gamma(k_0)]^2}{(2+k_0)(2k_0+1)\Gamma(2k_0)} + \frac{a^2bA^2 2^{2k_0}[\Gamma(k_0+1)]^2}{(2+k_0)(2k_0+1)\Gamma(2k_0+2)} \\
& + \frac{a^4bB^2 2^{2k_0-3}(2k_0-2)(4+k_0)[\Gamma(k_0-1)]^2}{(2+k_0)(2k_0+3)} + \frac{a^3bAB2^{2k_0}[\Gamma(k_0)]^2}{\Gamma(2k_0+3)}\Bigg]
\end{aligned} \tag{26}$$

where

$$E_{100}^{(0)} = E_{010}^{(0)} = -\frac{\mu\left(\eta\sigma^2\right)^2 e^4}{2\hbar^2}\left[2+\sqrt{\frac{q\eta\sigma^2\hbar^2}{2\mu}}\right]^{-2}, \ A = \eta\sigma^2 e^2, \ B = \frac{\eta q\sigma^2\hbar^2}{2\mu}, \tag{27}$$

$$a = \frac{2\mu\eta\sigma^2 e^2}{\hbar^2 n'}, \ b = \left[\frac{\Gamma(2k+1)}{\Gamma(k+1)}\right]^2 \frac{(2n+2k+1)}{2^{2k+1}\Gamma(n+2k+1)}, \ k_0 = \sqrt{\frac{q\eta\hbar^2\sigma^2}{2\mu}}. \tag{28}$$

Thus, at first order in the parameter β, the degeneracy of the two levels $|010\rangle$ and $|100\rangle$ is completely lifted.

3 Matrix Elements for $\Delta N = 0$, $\Delta n = 0$ and $\Delta m = 0$

Taking $N_1 = N_2 = N, n_1 = n_2 = n$ and $m_1 = m_2 = m$, we derive the explicit form of the energy corrections given in Eq. (13). Using the relation between the confluent hypergeometric function $F(-n; l+1; x)$ and the associated Laguerre polynomials $L_n^{(l)}(x)$, namely:

$$L_n^{(l)}(z) = \frac{\Gamma(n+l+1)}{\Gamma(n+1)\,\Gamma(l+1)} F(-n; l+1; z), \tag{29}$$

where $z = ar$ and $a = \frac{2\mu\eta\sigma^2 e^2}{\hbar^2 n'}$, and using the integral:

$$\int_0^\infty z^{l-1} e^{-z} \left[F(-n;\gamma;z)\right]^2 dx = \frac{n!\Gamma(l)}{\gamma(\gamma+1)\cdots(\gamma+n-1)} \Bigg\{1+ \frac{n(\gamma-l-1)(\gamma-l)}{1^2\gamma} + \frac{n(n-1)(\gamma-l-2)(\gamma-l-1)(\gamma-l)(\gamma-l+1)}{1^2 2^2 \gamma(\gamma+1)} + \cdots \cdots + \frac{n(n-1)\cdots 1(\gamma-l-n)\cdots(\gamma-l+n-1)}{1^2 2^2\cdots n^2 \gamma(\gamma+1)\cdots(\gamma+n-1)}\Bigg\}, \tag{30}$$

we obtain the following average values for the radial part:

$$\langle Nnm \mid r^{-1} \mid Nnm\rangle = \int_0^\infty [R(r)]^2 r^{-1} dr = \frac{a}{2n'}, \tag{31}$$

$$\langle Nnm \mid r^{-2} \mid Nnm\rangle = \int_0^\infty [R(r)]^2 r^{-2} dr = \frac{1}{2l+1}\frac{a^2}{2n'}, \tag{32}$$

$$\langle Nnm \mid r^{-3} \mid Nnm\rangle = \int_0^\infty [R(r)]^2 r^{-3} dr = \frac{a^3}{2l(2l+1)(2l+2)} = \frac{a^3\Gamma(2l)}{\Gamma(2l+3)}, \tag{33}$$

$$\langle Nnm \mid r^{-4} \mid Nnm'\rangle = \int_0^\infty [R(r)]^2 r^{-4} dr = \frac{a^4}{n'}\left(\frac{3(n')^2 - l(l+1)}{(2l-1)\,2l\,(2l+1)(2l+2)(2l+3)}\right) = \frac{a^4\left[3(n')^2 - l(l+1)\right]}{n'}\frac{\Gamma(2l-1)}{\Gamma(2l+4)} \tag{34}$$

Now, we evaluate the following integral of the angular functions:

$$\int_{-1}^{1} dx \left(1-x^2\right)^{-t} [\Theta(x)]^2 = \frac{(2n+2k+1)}{2^{2k+1}\Gamma(n+2k+1)} \times \left[\frac{\Gamma(2k+1)}{\Gamma(k+1)}\right]^2 \int_{-1}^{1} dx \left(1-x^2\right)^{k-t} \left[C_n^{(k+1/2)}(x)\right]^2 \tag{35}$$

We use the following expression given in Ref. [45]:

$$\int_0^\pi d\theta \sin^\nu\theta \sin(\gamma\theta)\left[C_n^{(\lambda)}\left(\sqrt{1+\rho\sin^2\theta}\right)\right]^2 = \frac{2^{-\nu}\pi\Gamma(\nu+1)(2\lambda)_n^2}{(n!)^2\,\Gamma\left(\frac{\nu-\gamma}{2}+1\right)\Gamma\left(\frac{\nu+\gamma}{2}+1\right)} \times \sin\left(\frac{\gamma\pi}{2}\right) {}_5F_4\left(\begin{matrix}-n, \lambda, 2\lambda+n, \frac{\nu+1}{2}, 1+\frac{\nu}{2}; -\rho \\ \lambda+\frac{1}{2}, 2\lambda, \frac{\nu-\gamma}{2}+1, \frac{\nu+\gamma}{2}+1\end{matrix}\right), \qquad (\mathrm{Re}\,\nu > -1) \tag{36}$$

with ${}_pF_q\left({a_1,a_2,\ldots a_p;x \atop b_1,b_2,\ldots,b_q}\right)$ being the hypergeometric function defined as:

$${}_pF_q\left(\begin{matrix}a_1, a_2, \ldots, a_p; x \\ b_1, b_2, \ldots, b_q\end{matrix}\right) = {}_pF_q\left(a_1, a_2, \ldots, a_p; b_1, b_2, \ldots, b_q; x\right)$$

$$= \sum_{k=0}^{\infty} \frac{(a_1)_k (a_2)_k \ldots (a_p)_k}{(b_1)_k (b_2)_k \ldots (b_q)_k} \frac{x^k}{k!} \tag{37}$$

We obtain:

$$\int_{-1}^{1} dx \left(1-x^2\right)^{-1} [\Theta(x)]^2 = \frac{\pi}{2^{4k-3}(n!)^2} \frac{(2n+2k+1)}{[\Gamma(k+1)]^2} \frac{\Gamma(n+2k+1)\Gamma(2k-3)}{\Gamma(k-3/2)\Gamma(k-1/2)} \times {}_5F_4\left(\begin{matrix} -n, k+1/2, n+2k+1, k-1/2, k; 1 \\ k+1, 2k+1, k-1/2, k+1/2 \end{matrix}\right), \tag{38}$$

$$\int_{-1}^{1} dx \left(1-x^2\right)^{-2} [\Theta(x)]^2 = \frac{\pi}{2^{4k}(n!)^2} \frac{(2n+2k+1)}{[\Gamma(k+1)]^2} \frac{\Gamma(n+2k+1)\Gamma(2k-1)}{\Gamma(k-1/2)\Gamma(k+1/2)} \times {}_5F_4\left(\begin{matrix} -n, k+\frac{1}{2}, n+2k+1, k-3/2, k-1; 1 \\ k+1, 2k+1, k-3/2, k-1/2 \end{matrix}\right). \tag{39}$$

Finally, the diagonal matrix elements up to first order in β take the form:

$$\begin{aligned} \frac{\beta}{4\mu} \langle Nnm| p^4 |Nnm\rangle &= \beta \frac{a^4}{(n')^8} \left\{ \frac{\hbar^2}{8\mu} \left(\frac{\hbar^2}{2} + \frac{n'}{2l+1} \right) + \right. \\ &+ \frac{\eta q \sigma^2 \hbar^4}{\mu} \left(\frac{1}{2n'(2l+1)} + \frac{4n'\Gamma(2l)}{\Gamma(2l+3)} \right) \frac{\pi}{2^{4k-2}(n!)^2} \frac{(2n+2k+1)}{[\Gamma(k+1)]^2} \\ &\times \frac{\Gamma(n+2k+1)\Gamma(2k-3)}{\Gamma(k-3/2)\Gamma(k-1/2)} {}_5F_4\left(\begin{matrix} -n, k+1/2, n+2k+1, k-1/2, k; 1 \\ k+1, 2k+1, k-1/2, k+1/2 \end{matrix}\right) \\ &+ \frac{\eta^2 q^2 \sigma^4 \hbar^4}{\mu} \frac{\left[3\left(n'\right)^2 - l(l+1)\right]\Gamma(2l-1)}{\Gamma(2l+4)n'} \frac{\pi}{2^{4k+2}(n!)^2} \frac{(2n+2k+1)}{[\Gamma(k+1)]^2} \\ &\times \frac{\Gamma(n+2k+1)\Gamma(2k-1)}{\Gamma(k-1/2)\Gamma(k+1/2)} {}_5F_4\left(\begin{matrix} -n, k+\frac{1}{2}, n+2k+1, k-3/2, k-1; 1 \\ k+1, 2k+1, k-3/2, k-1/2 \end{matrix}\right) \end{aligned} \tag{40}$$

This last expression depends on $l(l = n+k)$, which lifts the degeneracy.

4 Generalized Recurrence Relations

For theoretical studies of atoms and molecules, it is useful to know the average values of the powers of radial and angular coordinates. In order to determine all powers, we need a recurrence relation and first few values. In our case and for the diagonal matrix elements, the radial part verifies the following recurrence relations given in Ref. [31](restoring μ, e and $\hbar$):

$$\frac{\hbar^4 a^2}{4\mu^2 e^4} (s+1) \left\langle r^s \right\rangle = \frac{\hbar n' a}{2\mu e^2} \left\langle r^{s-1} \right\rangle - \frac{s\left[(2l+1)^2 - s^2\right]}{4} \left\langle r^{s-2} \right\rangle, \tag{41}$$

where the first average elements of r^s were evaluated in Ref. [31]. Next, we evaluate the recurrence formula for the angular part. For this we denote:

$$\begin{aligned} \left\langle \sin^{2t}\theta \right\rangle_{n,k} &= \langle Nnm \mid \sin^{2t}\theta \mid Nnm\rangle = \int_{-1}^{1} dx \left(1-x^2\right)^t [\Theta(x)]^2 \\ &= \frac{(2n+2k+1)}{2^{2k+1}\Gamma(n+2k+1)} \left[\frac{\Gamma(2k+1)}{\Gamma(k+1)}\right]^2 \int_{-1}^{1} dx \left(1-x^2\right)^{k+t} \left[C_n^{(k+1/2)}(x)\right]^2. \end{aligned} \tag{42}$$

We can write:

$$\int_{-1}^{1} dx \left(1-x^2\right)^{k+t+1} \left[C_n^{(k+1/2)}(x)\right]^2$$

$$= \int_{-1}^{1} dx \left(1-x^2\right)^{k+t} \left(\left[C_n^{(k+1/2)}(x)\right]^2 - \left[x C_n^{(k+1/2)}(x)\right]^2\right). \tag{43}$$

Using the following recurrence rule of the Gegenbauer polynomials [44]:

$$2\alpha \left(1-x^2\right) C_{n-1}^{(\alpha+1)}(x) = (2\alpha+n+1)\, C_{n-1}^{(\alpha)}(x) - n x C_n^{(\alpha)}(x), \tag{44}$$

we straightforwardly obtain:

$$\begin{aligned}\left\langle \sin^{2(t+1)}\theta \right\rangle_{n,k} &= \left\langle \sin^{2t}\theta \right\rangle_{n,k} - \frac{1}{4}(n+2k+1)\left\langle \sin^{2(t+1)}\theta \right\rangle_{n-1,k+1} \\ &\quad - \frac{(n+2k+2)(n+2k+1)}{n(2n+2k+1)} \left\langle \sin^{2t}\theta \right\rangle_{n-1,k+1} \\ &\quad - \frac{(n+2k+2)}{(2n+2k+1)(2n+2k-1)} \left\langle \sin^{2t}\theta \right\rangle_{n-1,k}. \end{aligned} \tag{45}$$

From Eqs. (41) and (45) we can derive the general formula of the averages values of $r^p \sin^{2s}\theta$. The recurrence formula (45) requires the two initial values $\left\langle \sin^{2t}\theta \right\rangle_{0,k}$ and $\left\langle \sin^{2t}\theta \right\rangle_{1,k}$. Then, taking the special cases (25) and using of the following integral [46]:

$$\int_0^{\pi} d\theta \left(z + \sqrt{z^2-1}\cos\theta\right)^{\mu} \sin^{2\nu-1}\theta = \frac{2^{2\nu-1}\Gamma(\mu+1)\left[\Gamma(\nu)\right]^2}{\Gamma(2\nu+\mu)} C_{\mu}^{(\nu)}(z), \tag{46}$$

with $\mathrm{Re}(\nu) > 0$, we obtain the first matrix elements:

$$\left\langle \sin^{2t}\theta \right\rangle_{0,k} = \frac{2^{2t}\Gamma(2k+2)}{\Gamma\left[2(k+t+1)\right]} \left[\frac{\Gamma(k+t+1)}{\Gamma(k+1)}\right]^2, \tag{47}$$

$$\left\langle \sin^{2t}\theta \right\rangle_{1,k} = \frac{2^{2t}(2k+3)}{2k+2t+3} \frac{\left[\Gamma(2k+1)\right]^4}{\Gamma(2k+2)\Gamma(2k+2t+1)} \left[\frac{\Gamma(k+t+1)}{\Gamma(k+1)}\right]^2. \tag{48}$$

5 Conclusion

In this paper we studied the Schrödinger equation for the Hartmann potential with deformed Heisenberg algebra. Using perturbation theory at the first order in the parameter of deformation β, we obtained the general form of the hamiltonian matrix elements and, as an example, we showed that the degeneracy of the two states $|010\rangle$ and $|100\rangle$ is completely lifted. For the diagonal matrix elements, we derived an explicit analytical expression which depends on l. In this case, some degenerate states split into sub-levels, and new transitions appear. It is worth mentioning that the expressions (23) or (40) of matrix elements shifted up to the first order of β have a behaviour similar to the Stark effect up to the second order in Hydrogen atom ($\beta = ((\Delta X)_{\min}/\hbar)^2$), thus, the parameter of deformation β play the same role of an external electric field. In addition to the recurrence formula for the radial average values given in [31], we derived the one for the angular part which leads to the general formula of the average values of $r^p \sin^{2s}\theta$ for the non-relativistic Hartmann potential. These results are useful in the calculations of the bound-state transitions and, on the experimental side, the energy levels can be measured and an upper bound on the minimal length $(\Delta X)_{\min}$ can be obtained. Appliying our results to diatomic molecules, such as CO, NO, N_2, …. For example, we can make an estimation of the minimal length comparing our results with the experimental data of the ground state energy of the N_2 molecule. In Ref [47], this energy is about $-109,58\,(7)\,a.u.$ From (23) and by taking $\left(q=1,\ \eta\sigma^2 = 7.82\right)$, we find that the upper bound of the minimal length is about $(\Delta X)_{\min} \leqslant 10^{-16} m$. This estimation is identical to the one obtained by [22], and slightly disagrees with the one given in Ref [21]. The presented results in this paper may lead to many interesting applications in ring-shaped molecule studies.

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