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Information Theoretic Global Measures of Dirac Equation With Morse and Trigonometric Rosen–Morse Potentials

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Abstract In this study, the information-theoretic measures of (1+1)-dimensional Dirac equation in both position and momentum spaces are investigated for the trigonometric Rosen–Morse and the Morse potentials. The solutions of the corresponding Dirac equation are obtained in an exact analytical manner in the first step. Next, using the Fourier transformation, the position and momentum Shannon information entropies are obtained and some features of the probability densities are analyzed. The consistency with Bialynicki-Birula–Mycielski inequality and Heisenberg uncertainty is checked.

1 Introduction

Dirac equation connects the pure physics to many problems of modern technology including graphene systems and information technology. Fortunately, the equation, in many cases, appears in Schrödinger form and has been therefore been solved with the approaches of nonrelativistic quantum mechanics including Nikiforov -Uvarov (NU) technique, perturbation, the polynomial solution, the wave function ansatz method, etc. [10,29,43]. Depending on the application of the equation, it has been considered with various interaction terms such Morse, Kratzer, Harmonic, etc. [3,12,41]. On the other hand, the information entropies have become quite interesting interdisciplinary concepts relating science and technology. In particular, the Shannon information entropy, proposed in 1940s [33], provides us with invaluable source of knowledge on atomic, molecular and nuclear systems and has been therefore investigated from various aspects [2,7,17,22,26,30]. In addition, the entropic uncertainty proposed by Beckner, Bialynicki-Birula, and Mycieslki (named BBM equality after them) [5] considers a stronger version of the Heisenberg uncertainty and appears as

$$S_x + S_p \geqslant D(1 + \ln \pi) \tag{1}$$

where D represents the spatial dimension and the position state (S_x) and momentum (S_p) information entropies are defined [9,18], respectively, by

$$\begin{aligned} S_x &= -\int_{-\infty}^{\infty} |\psi(x)|^2 \ln |\psi(x)|^2 \, \mathrm{d}x \\ S_p &= -\int_{-\infty}^{\infty} |\phi(p)|^2 \ln |\phi(p)|^2 \, \mathrm{d}p \end{aligned} \tag{2}$$

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where $\psi(x)$ is the normalized eigenfunction in is spatial coordinate and $\phi(p)$ denotes its normalized Fourier transform. Actually, the Shannon entropy provides a measure of information about the probability distribution and shows the correlation between particles [1,25,34].

The concept has been studied for various interactions of physics including harmonic [23], Pöschl-Teller (PT) [8,14], Morse [4], Coulomb [13], isospectral PT [21], etc. [15,20,21,31,32]. The Shannon entropy also been studied with generalized non-central potentials such as the trigonometric Rosen–Morse [36,38], the PT-like [35], squared tangent [16] and hyperbolic double-well potential [37]. The study within the framework of position-dependent-mass Schrödinger equation has been considered with null [42] and hyperbolic potentials [40]. The concept has also been studied within Dirac–Fock [19], Klein–Gordon frameworks [6,24] and for Killingbeck and Kratzer potentials [27,28].

In the present work, we first solve Dirac equation for both Rosen–Morse and Morse potentials using the Nikiforov-Uvarov (NU) method. In the next section, using the Fourier transformation, we obtain the momentum space wave functions and as well as standard deviation and thereby report the position and momentum information entropy densities, respectively denoted by $\rho_s(x)$ and $\gamma_s(p)$. The consistency with Bialynicki-Birula-Mycielski inequality and standard deviation is also checked regarding the results.

2 Solution of Dirac Equation

The Dirac equation is written as [11]

$$\left(\gamma^{\mu} p_{\mu} - mcI - \frac{V}{c}\right)\psi = 0 \tag{3}$$

with

$$\begin{pmatrix} 0 & \sigma_\mu \\ \sigma_\mu & 0 \end{pmatrix} \quad ; \quad p_\mu = -i\hbar\delta_\mu \tag{4}$$

where γ^{μ} denotes he Dirac matrices and m represents the mass of the particle. The potential is considered in the form

$$V = \gamma^{\mu} A_{\mu} + I V_s \tag{5}$$

where A_μ and V_s are the vector and scalar potentials, respectively. By choosing $\mu = 0, 1$; the Hamiltonian in $(1+1)$-dimensions is written as

$$H = \gamma^5 c\left(p_1 + \frac{A_1}{c}\right) + IA_0 + \gamma^0(mc^2 + V_s) \tag{6}$$

where $\gamma^5 = \gamma^0\gamma^1$. By using

$$A_1 = 0; \quad \psi = \begin{pmatrix} \Phi^+ \\ \Phi^- \end{pmatrix}; \quad \gamma^0 = \sigma_1; \quad p = -i\hbar\frac{d}{dx}; \quad A_0 = V_s \tag{7}$$

the equation appears in the coupled form

$$-i\hbar c\frac{d}{dx}\Phi^- + (mc^2 + 2V_s - E)\Phi^+ = 0 \tag{8}$$

$$-i\hbar c\frac{d}{dx}\Phi^+ - (mc^2 - E)\Phi^- = 0 \tag{9}$$

which give the relations between the upper and lower components as

$$\Phi^- = -\frac{i\hbar c}{(mc^2 + E)}\frac{d}{dx}\Phi^+ \tag{10}$$

By substitution of Eq. (11) into Eq. (9), we obtain the Schrödinger-like equation

$$-\hbar^2 c^2\frac{d^2}{dx^2}\Phi^+ + (mc^2 + E)(mc^2 + 2V_s - E)\Phi^+ = 0 \tag{11}$$

In fact, our purpose in this article is the solution of Eq. (11) for the Rosen–Morse ans Morse potentials in two position and momentum spaces by which we determine the two universal quantities, i.e. the Shannon entropy and Heisenberg relations as well as related aspects.

Trigonometric Rosen–Morse Potential

Here, we consider the trigonometric Rosen–Morse potential as

$$V_s = d\cot^2\left(\frac{\pi x}{a}\right) - f\csc^2\left(\frac{\pi x}{a}\right); \qquad x \in [0, a] \tag{12}$$

Choosing

$$\psi(x) = \sin\left(\frac{\pi x}{a}\right)^{(-2\lambda)} u(x) \tag{13}$$

the position part of the equation appears as

$$\begin{aligned}&\left\{\frac{d^2}{dx^2} - \frac{4\pi}{a}\lambda\cot\left(\frac{\pi x}{a}\right)\frac{d}{dx} + \frac{2\pi^2(1+2\lambda)\lambda}{a^2}\cot^2\left(\frac{\pi x}{a}\right) + \frac{2\pi^2}{a^2}\lambda - 2d(1+E)\cot^2\left(\frac{\pi x}{a}\right)\right.\\&\left.+(E^2-1) + 2f(1+E)\csc^2\left(\frac{\pi x}{a}\right)\right\}u(x) = 0\end{aligned} \tag{14}$$

Defining the new variables

$$\lambda = \frac{1}{4}\left(\sqrt{1 + \frac{8d(1+E)a^2}{\pi^2}} - 1\right) \tag{15}$$

$$\nu = \sqrt{\frac{a^2}{\pi^2}((E^2-1) + 2d(1+E))} \tag{16}$$

$$\delta = \frac{1}{4}\left(\sqrt{1 - \frac{8f(1+E)\pi^2}{a^2}} + 1\right) \tag{17}$$

Eq. (14) reduces to

$$\left\{\frac{d^2}{dx^2} - \frac{4\pi\lambda}{a}\cot\left(\frac{\pi x}{a}\right)\frac{d}{dx} + \frac{\pi^2}{a^2}(\nu^2 - \lambda^2) + \frac{4\pi^2}{a^2}\delta(\tfrac{1}{2} - \delta)\csc^2\left(\frac{\pi x}{a}\right)\right\}u(x) = 0 \tag{18}$$

Using the new change of variable $y = \cos^2\left(\frac{\pi x}{a}\right)$ Eq. (18) can be rewritten as

$$\left\{\frac{d^2}{dx^2} + \frac{\frac{1}{2} - (1-2\lambda)y}{y(1-y)}\frac{d}{dy} + \frac{\nu^2 - \lambda^2}{4y(1-y)} + \frac{\delta(\frac{1}{2} - \delta)}{y^2(1-y)}\right\}u(y) = 0 \tag{19}$$

Introducing the gauge transformation $u(y) = y^{-\frac{1}{2}}\phi(y)$ Eq. (19) takes the form

$$\left\{\frac{d^2}{dy^2} + \frac{-\frac{1}{2}+2\lambda y}{y(1-y)}\frac{d}{dy} + \frac{-\left(\frac{\nu^2}{4} - \frac{\lambda^2}{4} - \lambda - \frac{1}{4}\right)y^2 + \left(\frac{\nu^2}{4} - \frac{\lambda^2}{4} - \lambda - \frac{3}{4} - \delta(\frac{1}{2} - \delta)\right)y + \frac{1}{2} + \delta(\frac{1}{2} - \delta)}{(y(1-y))^2}\right\}\phi(y) = 0 \tag{20}$$

The latter can be simply solved using the NU method [39],with the required parameters being

$$\begin{aligned}&\alpha_1 = -\frac{1}{2}, \quad \alpha_2 = -2\lambda, \quad \alpha_3 = 1 \quad \alpha_4 = \frac{3}{4},\\&\alpha_5 = -(\lambda+1), \quad \alpha_6 = \frac{3}{4}\left(\lambda^2 + 1 + \frac{\nu^2}{3}\right) + \lambda,\\&\alpha_7 = \frac{1}{4}(\lambda^2 - 2\lambda - 3 - \nu^2) + \delta\left(\frac{1}{2} - \delta\right),\\&\alpha_8 = \left(\delta - \frac{1}{4}\right)^2, \quad \alpha_9 = \left(\lambda + \frac{1}{4}\right)^2, \quad \alpha_{10} = 2\delta + \frac{1}{2},\\&\alpha_{11} = 2(\delta + \lambda + 1), \quad \alpha_{12} = \delta + \frac{1}{2}, \quad \alpha_{13} = -(\delta + 2\lambda + 1),\end{aligned} \tag{21}$$

With

$$\xi_1 = \frac{1}{4}\left(\nu^2 - \lambda^2 - 4\lambda - 1\right)$$
$$\xi_2 = \frac{1}{4}\left(\nu^2 - \lambda^2 - 4\lambda - 3\right) - \delta\left(\frac{1}{2} - \delta\right) \tag{22}$$
$$\xi_3 = -\frac{1}{2} - -\delta\left(\frac{1}{2} - \delta\right)$$

and possesses the solutions in terms of Jacobi polynomials as

$$\Phi^+(x) = \cos\left(\frac{\pi x}{a}\right)^{2\delta} \sin\left(\frac{\pi x}{a}\right)^{2\lambda+1} P_n^{\left(2\delta-\frac{1}{2}, 2\lambda+\frac{1}{2}\right)}\left(1 - 2\cos^2\left(\frac{\pi x}{a}\right)\right) \tag{23}$$

In the special case of $f = 0$

$$\Phi^+(x) = \cos\left(\frac{\pi x}{a}\right) \sin\left(\frac{\pi x}{a}\right)^{2\lambda+1} P_n^{\left(\frac{1}{2}, 2\lambda+\frac{1}{2}\right)}\left(1 - 2\cos^2\left(\frac{\pi x}{a}\right)\right) \tag{24}$$

for $n = 0$, the wave functions become

$$\Phi^+(x) = \cos\left(\frac{\pi x}{a}\right) \sin\left(\frac{\pi x}{a}\right)^{2\lambda+1} \tag{25}$$

$$\Phi^-(x) = \frac{i\pi\left[\lambda + (1+\lambda)\cos\left(\frac{2\pi x}{a}\right)\right]\sin\left(\frac{\pi x}{a}\right)^{2\lambda}}{a(1+E)} \tag{26}$$

we obtain a relation for the energy spectrum in the special case as

$$n^2 + 2n(\lambda + 1) + \frac{\lambda^2}{4} - \frac{\nu^2}{4} + 2\lambda + \frac{11}{8} = 0 \tag{27}$$

where the two variables λ and ν , as indicated in Eqs. (15) and (16), include the energy and therefore the energy corresponding to each state can be simply determined.

Hence, the total wave function for this system is obtained by having the two spinors and the normalization constant determined from $\int_0^a \psi\psi^* \, dx = 1$.

Morse Potential

The Morse potential possesses the form

$$V_s = V_0(e^{-2\alpha x} - e^{-\alpha x}) \tag{28}$$

where x is the distance between the particles, V_0 represents the well depth and α controls the 'width' of the potential (the smaller α is, the larger the well we have).Using the transformation $y = e^{-\alpha x}$, we have

$$\left\{\frac{d^2}{dy^2} + \frac{1}{y}\frac{d}{dy} - \frac{(1-E^2)}{\alpha^2 y^2} + \frac{2V_0(1+E)}{\alpha^2 y} - \frac{2V_0(1+E)}{\alpha^2}\right\}\Phi^+(y) = 0 \tag{29}$$

Introducing

$$\beta^2 = \frac{(1-E^2)}{\alpha^2} \quad ; \quad \gamma^2 = \frac{2V_0(1+E)}{\alpha^2} \tag{30}$$

Eq. (29) is more neatly written as

$$\left\{\frac{d^2}{dy^2} + \frac{1}{y}\frac{d}{dy} + \frac{-\gamma^2 y^2 + \gamma^2 y - \beta^2}{y^2}\right\}\Phi^+(y) = 0 \tag{31}$$

By using the NU method [39], the required parameters being

$$\begin{aligned}
&\alpha_1 = 1, \quad \alpha_2 = \alpha_3 = \alpha_4 = \alpha_5 = 0, \\
&\alpha_6 = \gamma^2, \quad \alpha_7 = -\gamma^2, \quad \alpha_8 = \beta^2, \\
&\alpha_9 = \gamma^2, \quad \alpha_{10} = 1 + 2\beta, \quad \alpha_{11} = 2\gamma, \\
&\alpha_{12} = \beta, \quad \alpha_{13} = -\gamma,
\end{aligned} \tag{32}$$

With

$$\begin{aligned}
&\xi_1 = -\xi_2 = \gamma^2 \\
&\xi_3 = \beta^2
\end{aligned} \tag{33}$$

Therefore in the special case of $\alpha_3 = 0$, the upper spinor is obtained as

$$\Phi_n^+(x) = \exp\left[-\alpha\beta x - \gamma e^{-\alpha x}\right] L_n^{2\beta}\left(2\gamma e^{-\alpha x}\right) \tag{34}$$

which simply gives

$$\Phi_n^-(x) = -i\alpha \exp\left[-\alpha\beta x - \gamma e^{-\alpha x}\right]\left(2\gamma L_{n-1}^{2\beta+1}\left(2\gamma e^{-\alpha x}\right) - \left(e^{\alpha x}\beta - \gamma\right) L_n^{2\beta}\left(2\gamma e^{-\alpha x}\right)\right)/(1 + E_n) \tag{35}$$

In this case, we obtain a relation for the energy spectrum as

$$(2n + 1) + \gamma - 2\beta = 0 \tag{36}$$

Therefore, the total form of the wave function is obtained from Eq. (7).

The analytical calculation of Shannon information is quite difficult for higher states. Therefore, we concentrate on the ground-state $n = 0$ and discuss the two higher states numerically.

3 Information Entropy

In this section we are going to obtain the position and momentum information entropies for the two potentials. In the previous section we got the wave function in position space and now we use the Fourier transform to calculate the corresponding momentum space wave functions.

Trigonometric Rosen–Morse Potential

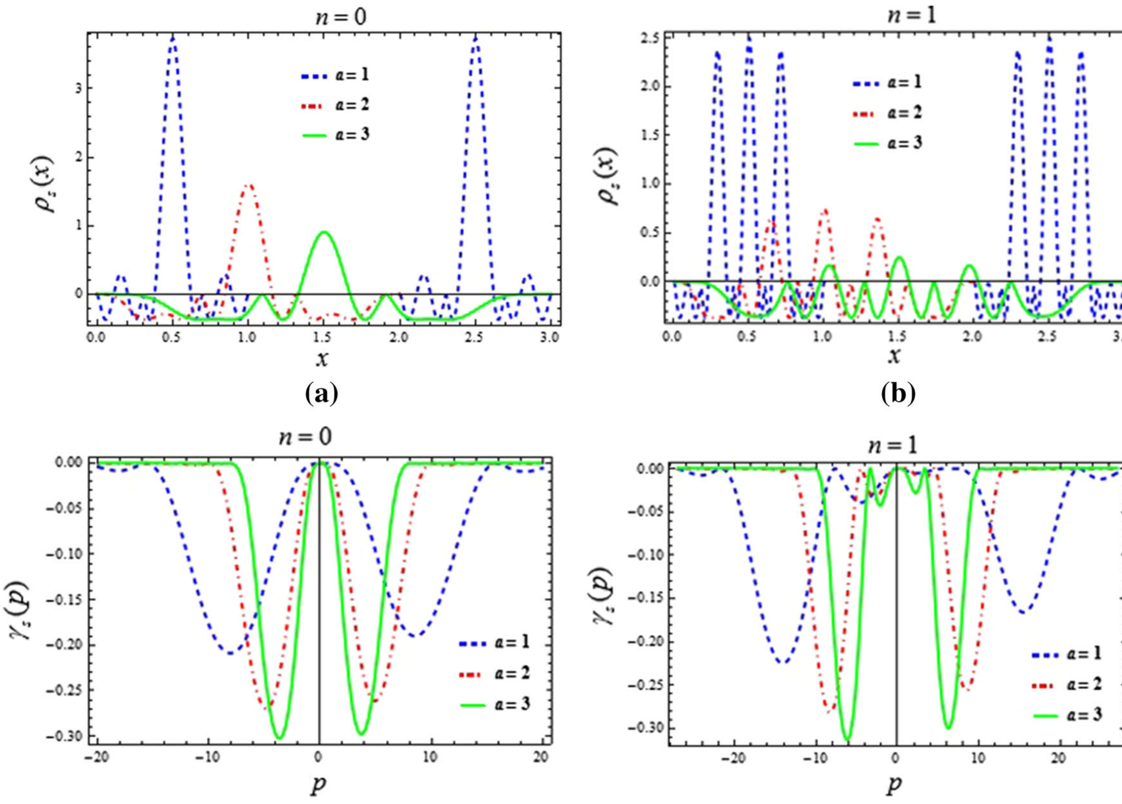
We consider the lower states of $n = 0, 1$ due to the difficulty of the calculations and only report the numerical results corresponding to $n = 2$ state. Using the Fourier transform as

$$\phi^{\pm}(p) = \frac{1}{\sqrt{2\pi}} \int_0^a \Phi^{\pm}(x) e^{-ipx}\, \mathrm{d}x \tag{37}$$

we find for the ground-state $n = 0$

$$\phi^+(p) = \frac{-ap(1+i)2^{-\frac{5}{2}-2\lambda} e^{-i\left(ap+\pi\lambda-\frac{\pi}{2}\right)}\left(1 - i e^{i(ap+2\pi\lambda)}\right)\Gamma\left(-\frac{1}{4} - \frac{ap}{2\pi} - \lambda\right)}{\sqrt{\pi}\cos(2\pi\lambda)\Gamma\left(\frac{1}{2} - 2\lambda\right)\Gamma\left(\frac{5}{4} - \frac{ap}{2\pi} + \lambda\right)} \tag{38}$$

$$\phi^-(p) = \frac{-ap^2 2^{-\frac{7}{2}-2\lambda} e^{-i(ap+\pi\lambda)}\left(-1 + e^{i(ap+2\pi\lambda)}\right)\Gamma\left(-1 - \frac{ap}{2\pi} - \lambda\right)}{\sqrt{\pi}(1+E)\sin(2\pi\lambda)\Gamma(-1-2\lambda)\Gamma\left(2 - \frac{ap}{2\pi} + \lambda\right)} \tag{39}$$



(a) **(b)** **(c)** **(d)**

Fig. 1 Position and momentum space entropy densities for $n = 0, 1$ and $d = 10$

Thus, the position and momentum probability densities are

$$\rho_{n=0}(x) = N_x^2 \sin\left(\frac{\pi x}{a}\right)^{4\lambda}\left[\frac{\pi^2\left(\lambda + (1+\lambda)\cos\left(\frac{\pi x}{a}\right)\right)^2}{a^2(1+E)^2} + \frac{1}{4}\sin^2\left(\frac{\pi x}{a}\right)\right] \tag{40}$$

$$\gamma_{n=0}(p) = \frac{2^{-4\lambda}a^2p^2N_p^2}{\pi}\left[\frac{2^{-\frac{3}{2}}\left(1 - ie^{i(ap+2\pi\lambda)}\right)\Gamma\left(-\frac{1}{4} - \frac{ap}{2\pi} - \lambda\right)}{\cos(2\pi\lambda)\Gamma\left(\frac{1}{2} - 2\lambda\right)\Gamma\left(\frac{5}{4} - \frac{ap}{2\pi} + \lambda\right)}\right.$$
$$\left. + \frac{2^{-3}p\left(-1 + e^{i(ap+2\pi\lambda)}\right)\Gamma\left(-1 - \frac{ap}{2\pi} - \lambda\right)}{\sin(2\pi\lambda)(1+E)\Gamma(-1-2\lambda)\Gamma(2 - \frac{ap}{2\pi} + \lambda)}\right]^2 \tag{41}$$

The position and momentum space information entropies for the one-dimensional Morse potential can be calculated using Eq. (2). As already stated, in general, explicit derivations of the information entropy is not simple. In some cases, even, the derivation of analytical expression is quite cumbersome, better say, impossible, as shown in instructive works [36,38]. We represent the position and momentum information entropy densities respectively by $\rho_s(x) = |\psi(x)|^2 \ln|\psi(x)|^2$ and $\gamma_s(p) = |\phi(p)|^2 \ln|\phi(p)|^2$ since they play a similar role to the probability density $\rho(x) = |\psi(x)|^2$ in quantum mechanics.

The characteristic features of the position and momentum information entropies $\rho_s(x)$ and $\gamma_s(p)$ are shown in Figs. 1a–d for lowest states $n = 0, 1$ vs. the parameter a for constant d . The probability densities $\rho(x)$ and $\gamma(p)$ are illustrated in Figs. 2a–d. We find that the density amplitudes of $n = 0$ are greater than those of $n = 1$. It is seen that the quantity first decreases and then increases to zero vs. the momentum which implies the local behavior of the momentum. In addition, the maxima of $\rho_s(x)$ and $\gamma_s(p)$ decreases with the parameter d . However, the behavior of the maximum amplitude $\rho_s(x)$ is reverse to $|\gamma_s(p)|$.

As seen in Figs. 1a, b, 2a, b, the wave function is symmetric around point $x = a/2$ and more generally in points where $x = ra/2$ with $r = 1, 2, \ldots$. However, in Figs. 1a, b, 2c, d the wave functions in momentum

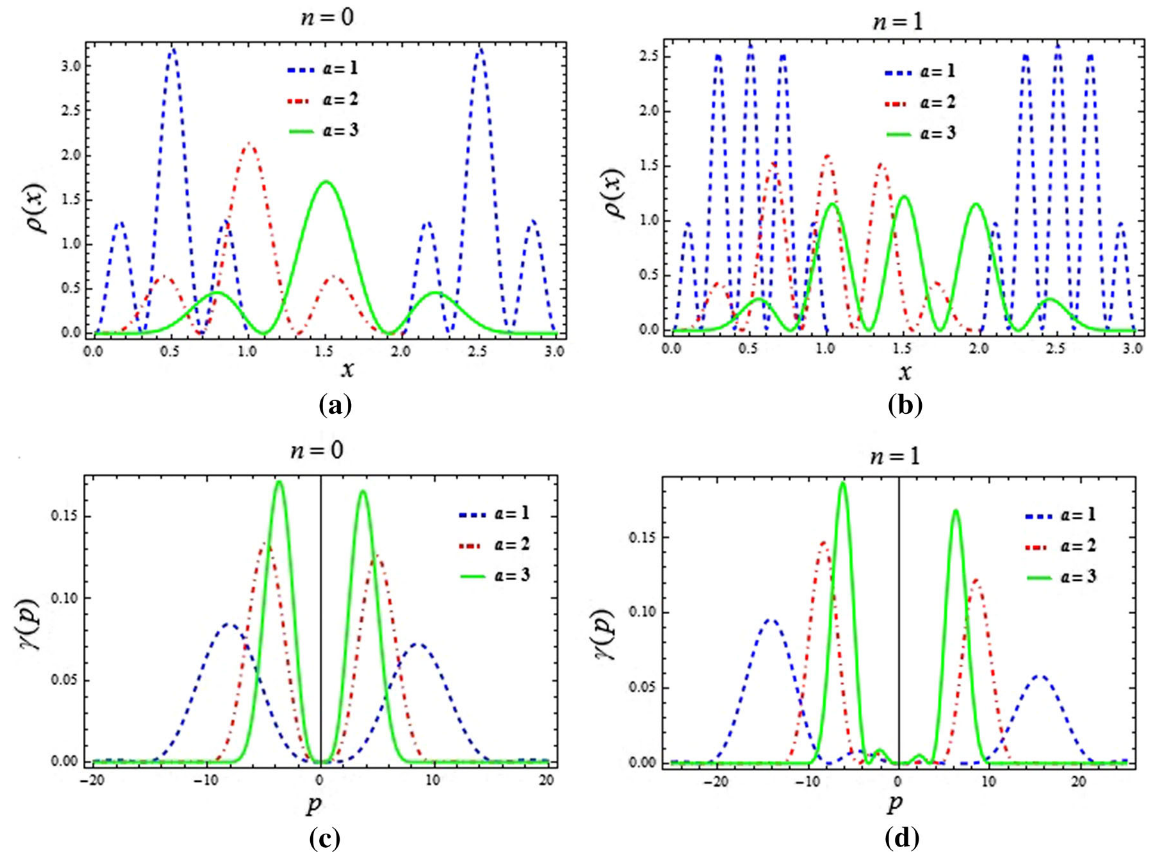


Fig. 2 Position and momentum space probability densities for $n = 0$, 1 and $d = 10$

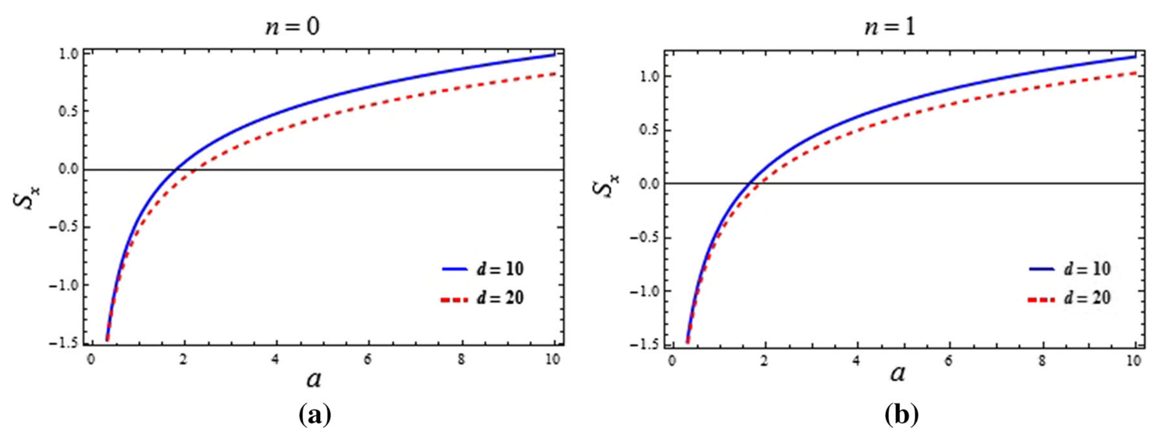


Fig. 3 The Shannon entropy in position space for $n = 0$, 1 and $d = 10$, 20

space represent a symmetric behavior around point zero. The position probability density has a symmetric shape and the position probability density has three peaks for $n = 0$, five peaks in the exited state $n = 1$ and 2n+3 peaks in the nth state as can be inferred from Fig. 2a, b. It is also noted that with increasing parameter a, for constant d, S_x increases. For increasing potential depth, S_x decreases which implies the stability of the system. And by reducing the amount of information entropy increases (information) or the accuracy in predicting the localization of the particle. Also it should be noted behavior of S_p is contrary S_x. In the Fig. 3a, b we see that S_x is negative for some given parameter values a. It might be understood that the moving particle becomes condensed so that it does not move at all. The system becomes more stable at this moment.So we see that negative S_x is physically meaningless.

Table 1 Information entropy and the uncertainty relation of the eigenstates $n = 0, 1, 2$ for different a and d

n	d	a	E	$\langle x\rangle$	$\langle x^2\rangle$	$\langle p\rangle$	$\langle p^2\rangle$	Δx	Δp	$\Delta x\Delta p$	S_x	S_p	S_x+S_p
0	10	1	−0.7452	0.5	0.3002	−0.4916	81.7407	0.2241	9.0277	2.0231	−0.4119	3.0306	2.6187
		2	−0.9361	1	1.1251	−0.1363	26.8827	0.3537	5.1831	1.8333	0.3193	2.2348	2.5596
		3	−0.9716	1.5	2.4543	−0.0602	15.6649	0.4519	3.9574	1.7883	0.0692	2.2348	2.5541
	20	1	−0.8744	0.5	0.2901	−0.2639	89.6232	0.2004	9.4633	1.8964	−0.5084	3.0873	2.5789
		2	−0.9686	1	1.0953	−0.6667	33.7881	0.3086	5.8124	1.7937	−0.0626	2.6169	2.5543
		3	−0.9860	1.5	2.4024	−0.0293	20.4187	0.3904	4.5186	1.7641	0.1742	2.3813	2.5555
	30	1	−0.9167	0.5	0.2848	−0.1772	99.0354	0.1865	9.9500	1.8557	−0.5741	3.1396	2.5655
		2	−0.9792	1	1.0805	−0.0439	39.2214	0.2838	6.2625	1.7773	−0.1452	2.6997	2.5545
		3	−0.9907	1.5	2.3776	−0.0193	24.0997	0.3571	4.9091	1.7530	0.0855	2.4715	2.5570
1	10	1	0.0773	0.5	0.3054	−2.7938	243.261	0.2353	15.3446	3.6106	−0.3789	3.1124	2.7335
		2	−0.7272	1	1.1560	−0.7137	72.7527	0.3950	8.4996	3.3573	0.1556	2.5073	2.6629
		3	−0.8785	1.5	2.5222	−0.2999	40.3795	0.5217	6.3473	3.3114	0.4405	2.2373	2.6778
	20	1	−0.4629	0.5	0.2971	−1.4649	251.753	0.2170	15.7989	3.4284	0.0532	2.6212	2.6701
		2	−0.8653	1	2.6615	0.2152	2.6924	1.3440	1.6267	2.1863	1.2537	1.6921	2.9457
		3	−0.7483	1.5	2.4631	−0.1410	50.5900	0.4617	7.1113	3.2833	0.3222	2.3775	2.6997
	30	1	−0.6430	0.5	0.2923	−0.9559	272.984	0.2057	16.4945	3.3929	−0.5019	3.1642	2.6623
		2	−0.9107	1	1.1096	−0.2162	99.4210	0.3310	9.9686	3.2996	−0.0128	2.6992	2.6864
		3	−0.9603	1.5	2.4328	−0.2162	99.4210	0.4276	9.9687	4.2626	0.2474	2.6992	2.9466
2	10	1	1.5873	0.5	0.3110	−3.9738	291.251	0.2469	16.5969	4.0978	−0.3421	3.5421	3.2000
		2	−0.3336	1	1.1851	−0.0682	21.9574	0.4302	4.6854	2.0156	0.2153	2.8877	3.1341
		3	−0.7021	1.5	2.5235	−0.0345	20.8033	0.5808	5.7174	3.3207	0.5207	2.9188	3.4084
	20	1	0.3123	0.5	0.3037	−1.0519	313.9150	0.7331	17.6864	12.9659	−0.4123	3.7929	3.3806
		2	−0.6694	1	1.1549	−0.0065	20.2889	0.3935	4.5043	1.7724	0.1293	2.8785	3.0078
		3	−0.8528	1.5	2.5235	−0.0345	20.8033	0.5230	4.5609	2.3853	0.4201	2.7999	3.2200
	30	1	−0.1248	0.5	0.2993	−0.4049	286.8700	0.2221	16.9324	3.7607	−0.4527	3.8668	3.4141
		2	−0.7804	1	1.1378	−0.0016	20.6915	0.3713	4.5488	2.6889	0.0740	2.9013	3.9753
		3	−0.9023	1.5	2.4891	0.0091	18.5847	0.4889	4.3109	2.1067	0.3555	2.7462	3.1017

In Table (1), we present the numerical results of the information entropies S_x and S_p and their sum for the lowest states $n = 0, 1, 2$ with the parameters a and d . It should be noted that the sum of the entropies, in consistency with the BBM inequality, possesses the stipulated lower bound $1 + \ln \pi$.

Morse Potential

For simplicity, we study the $n = 0, 1$ cases in more details. We can derive the corresponding momentum representation in analytical form for $n = 0$ as

$$\phi^{+}(p) = \frac{\gamma^{-\frac{ip}{\alpha}-\beta}\Gamma\left(\frac{ip}{\alpha}+\beta\right)}{\sqrt{2\pi}\alpha} \tag{42}$$

$$\phi^{-}(p) = \frac{p\gamma^{-\frac{ip}{\alpha}-\beta}\Gamma\left(\frac{ip}{\alpha}+\beta\right)}{\sqrt{2\pi}\alpha(1+E)} \tag{43}$$

For the first exited state $n = 1$

$$\phi^{+}(p) = \frac{(-2ip+\alpha)\gamma^{-\frac{ip}{\alpha}-\beta}\Gamma\left(\frac{ip}{\alpha}+\beta\right)}{\sqrt{2\pi}\alpha^2} \tag{44}$$

$$\phi^{-}(p) = \frac{p(-2ip+\alpha)\gamma^{-\frac{ip}{\alpha}-\beta}\Gamma\left(\frac{ip}{\alpha}+\beta\right)}{\sqrt{2\pi}\alpha^2(1+E)} \tag{45}$$

Hence, the position and momentum probability densities are respectively written for two lowing states $n = 0, 1$ as

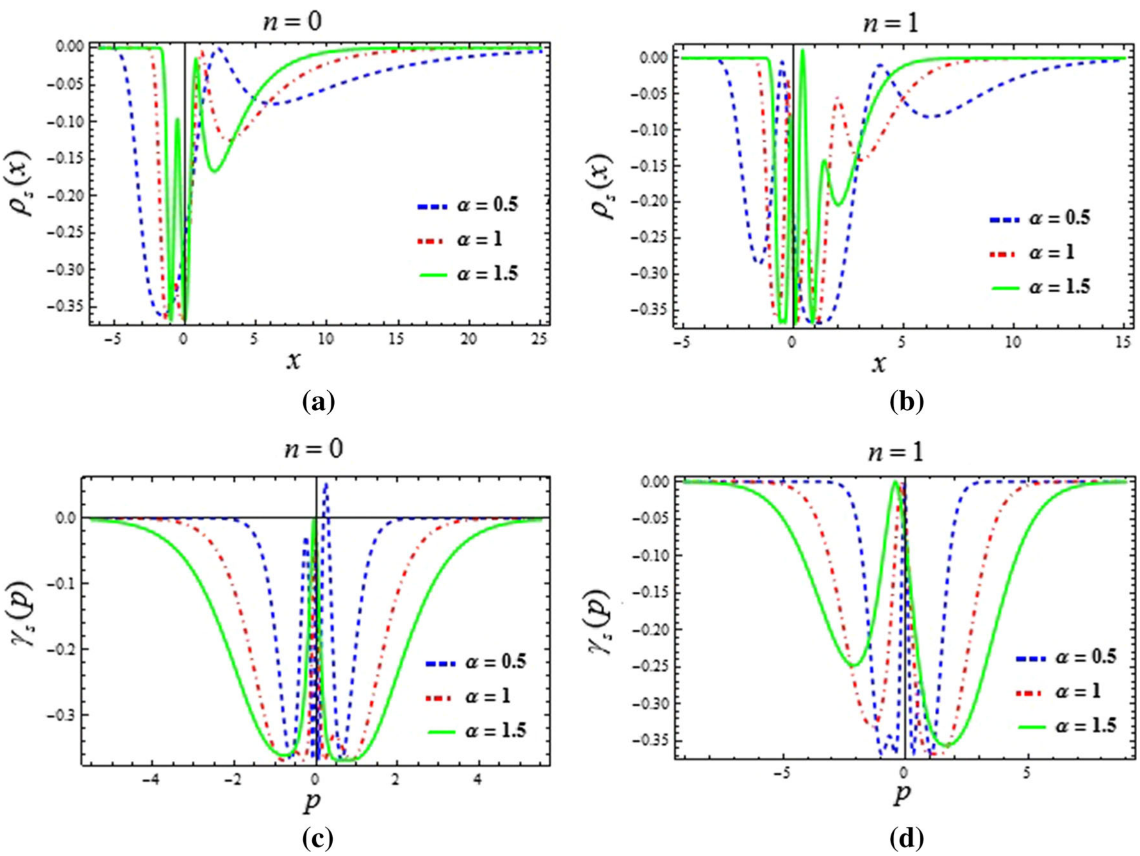


Fig. 4 The position and momentum space entropy densities for $n = 0, 1$ and $V_0 = 10$

$$\rho_{n=0}(x) = N_x^2 \exp\left[-2\gamma e^{-\alpha x}\right]\left(e^{-2\alpha\beta x} + \frac{\alpha^2(-\beta e^{\alpha x} + \gamma)^2 e^{-2\alpha(1+\beta)x}}{(1+E)^2}\right) \tag{46}$$

$$\gamma_{n=0}(p) = \frac{N_p^2 \gamma^{-2\beta}\left|\Gamma\left(\frac{ip}{\alpha} + \beta\right)\right|^2}{2\pi\alpha^2}\left(1 + \frac{p}{1+E}\right)^2 \tag{47}$$

and

$$\rho_{n=1}(x) = N_x^2 \exp\left[-2\gamma e^{-\alpha x}\right] \times \left(e^{-2\alpha\beta x}\left(1 + 2\beta - 2\gamma e^{-\alpha x}\right)^2 + \frac{\alpha^2(\beta(1+2\beta)e^{2\alpha x} - e^{\alpha x}(3+4\beta) + 2\gamma^2)^2 e^{-2\alpha(2+\beta)x}}{(1+E)^2}\right) \tag{48}$$

$$\gamma_{n=1}(p) = \frac{N_p^2 \gamma^{-2\beta}\left(4p^2 + \alpha^2\right)\left|\Gamma\left(\frac{ip}{\alpha} + \beta\right)\right|^2}{2\pi\alpha^4}\left(1 + \frac{p}{1+E}\right)^2 \tag{49}$$

Due to analytical limitations, we will use numerical method to show correct results.

Characteristic features of the position and momentum information entropies, $\rho_s(x)$ and $\gamma_s(p)$, are shown in Figs. 4a, b, 5a, b for the lowest states $n = 0, 1$ vs. the parameter α . The probability densities $\rho(x)$ and $\gamma(p)$ are illustrated in Figs. 4c,, d, 5c, d. We find that the density amplitudes of the ground state $n = 0$ are greater than those of the exited states $n = 1, 2, \ldots$. The forms mentioned show that there is no symmetry in such cases. From Fig. 5a, b it can be inferred that the position probability density has only one peak for $n = 0$, two peaks in the exited state $n = 1$ and $n + 1$ peaks in the nth state.

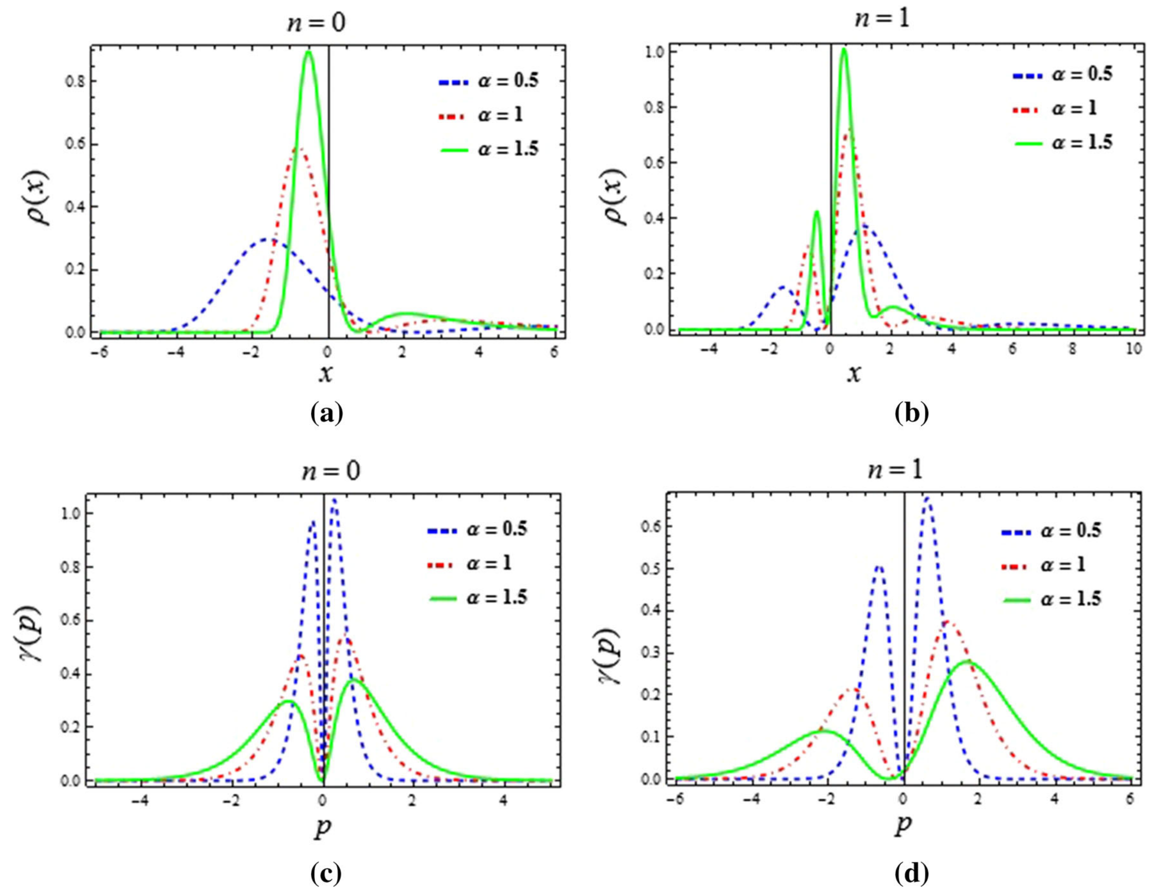


Fig. 5 The position and momentum space probability densities for $n = 0$, 1 and $V_0 = 10$

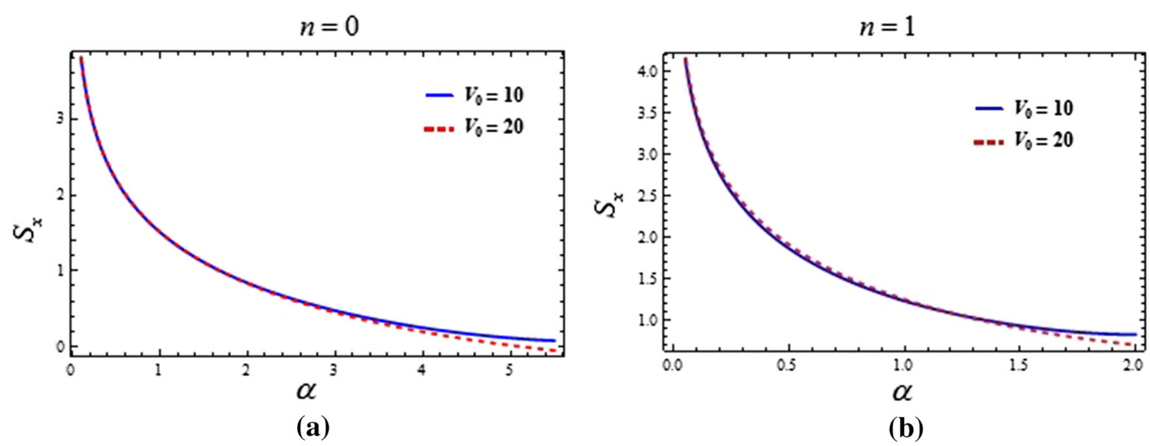


Fig. 6 The Shannon entropy in position space for $n = 0$, 1 and $V_0 = 10$, 20

In Fig. 6a, b, we observe that the position entropy decreases with increasing α and with two different values for parameter depth potential $V_0 = 10, 20$ in the ground state at limit to zero and in the exited state $n = 1$ is reduced to less than one. With a little vision can see in Fig. 6a, b in $n = 1$ state of entropy, the greater the potential depth $V_0 = 10, 20$, the less energy and entropy is proportional to the it and decreases. Thus, here the control system is stable and will be more accurately in predict the location of particle in front in momentum entropy S_p acts quite the reverse.

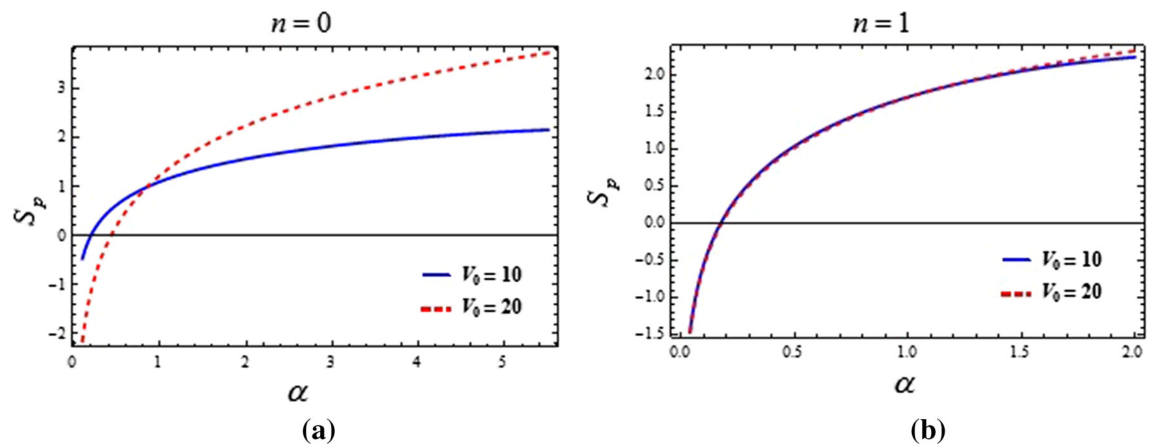


Fig. 7 The Shannon entropy in momentum space for $n = 0, 1$ and $V_0 = 10, 20$

Table 2 Information entropy and the uncertainty relation of the eigenstates $n = 0, 1, 2, 3$ for different α and V_0

n	V_0	α	E	$\langle x\rangle$	$\langle x^2\rangle$	$\langle p\rangle$	$\langle p^2\rangle$	Δx	Δp	$\Delta x\Delta p$	S_x	S_p	$S_x + S_p$
0	10	0.5	−0.9953	0.5610	25.1614	0.0094	0.2179	4.9846	0.4667	2.3263	2.4927	0.5528	3.0455
		1	−0.9812	0.2899	6.3304	0.0376	0.8694	2.4993	0.4667	2.3286	1.5075	1.2434	2.7509
		1.5	−0.9574	0.2038	2.8439	0.0846	1.9482	1.6740	1.3932	2.3322	1.1125	1.6445	2.7570
		2	−0.9238	0.1643	1.6248	0.1507	3.4429	1.2640	1.8494	2.3376	0.8393	1.9260	2.7653
	20	0.5	−0.9970	0.9688	31.3622	0.0059	0.2039	5.5157	0.4515	2.4903	2.2002	0.5289	2.7291
		1	−0.9880	0.4902	7.8770	0.0239	0.8144	2.7635	0.9021	2.4929	1.5106	1.2208	2.7314
		1.5	−0.9730	0.3332	3.5284	0.0538	1.8286	1.8486	1.3512	2.4978	1.1110	1.66242	2.7352
		2	−0.9519	0.2568	2.0068	0.0957	3.2411	1.3932	1.7977	2.7405	0.8315	1.9089	2.8404
	30	0.5	−0.9978	1.1800	36.0368	0.0045	0.1965	5.8859	0.4432	2.6086	2.1997	0.5159	2.7238
		1	−0.9910	0.5942	9.0424	0.0179	0.7852	2.9477	0.8860	2.6117	1.5090	1.2082	2.7172
		1.5	−0.9798	0.4009	4.0436	0.0403	1.7644	1.9705	1.3277	2.6162	1.1076	1.6124	2.7200
		2	−0.9640	0.3057	2.2945	0.0717	3.1305	1.4836	1.7679	2.6228	0.8256	1.8982	2.7238
1	10	0.5	−0.9574	1.2909	7.5619	0.0845	0.7529	2.4280	0.8635	2.0965	1.8661	1.0366	2.9027
		1	−0.8251	0.7119	2.0802	0.3386	2.9304	1.2543	1.6780	2.1047	1.2352	1.6997	2.9349
		1.5	−0.5856	0.5640	1.1076	0.7633	6.2419	0.8885	2.3789	2.1136	0.9369	2.0491	2.9860
		2	−0.1859	0.5628	0.8703	1.3540	9.9116	0.7440	2.8422	2.1146	0.8294	2.2363	3.0657
	20	0.5	−0.9730	1.7732	10.0797	0.0537	0.6829	2.6335	0.8244	2.1710	1.9122	1.0132	2.9254
		1	−0.8907	0.9246	2.6615	0.2152	2.6924	1.3440	1.6267	2.1863	1.2537	1.6921	2.9457
		1.5	−0.7483	0.6633	1.3036	0.4852	5.8979	0.9293	2.3796	2.9787	0.9064	2.0723	2.9787
		2	−0.5354	0.5564	0.8545	0.8642	10.0314	0.7382	3.0470	2.2492	0.7057	2.3204	3.0261
	30	0.5	−0.9798	2.0348	11.8256	0.0402	0.6455	2.7722	0.8024	2.2244	1.9403	0.9994	2.9397
		1	−0.9186	1.0442	3.0715	0.1612	2.5567	1.4075	1.5908	2.2390	1.2709	1.6837	2.9546
		1.5	−0.8142	0.7282	1.4592	0.3632	5.6529	0.9638	2.3497	2.2646	0.9050	2.0737	2.9787
		2	−0.6623	0.5841	0.9079	0.6470	9.7804	0.7528	3.0597	2.3033	0.6750	2.3381	3.0131
2	10	0.5	−0.8799	1.4327	6.6352	0.2347	1.7970	2.1406	1.3198	2.8252	1.8195	1.3354	3.1549
		1	−0.4751	0.9383	2.3065	0.9399	6.5329	1.1942	2.3767	2.8382	1.3070	1.9680	3.2750
	20	0.5	−0.9245	1.8971	8.8404	0.1493	1.6289	2.2894	1.2675	2.9018	1.8564	1.3220	3.1784
		1	−0.6859	1.0620	2.6145	0.5984	6.2195	1.2193	2.4210	2.9519	1.2612	1.9892	3.2504
	30	0.5	−0.9437	2.1555	10.3658	0.1119	1.5360	2.3915	1.2343	2.9518	1.8832	1.3133	3.1965
		1	−0.7691	1.1546	2.8980	0.4482	5.9597	1.2509	2.3998	3.0019	1.2568	1.9907	3.2475
3	10	0.5	−0.7586	1.5719	6.9222	0.4602	3.2969	2.1098	1.7564	3.7056	1.8372	1.5597	3.3969
		1	−0.2948	1.2018	4.7192	1.7623	9.1338	1.9179	2.4552	4.7088	1.6081	2.0656	3.6737
	20	0.5	−0.8503	1.9866	8.8739	0.2927	3.0189	2.2197	1.7127	3.7026	1.8372	1.5597	3.3969
		1	−0.3441	1.2657	3.2312	1.1687	10.7686	1.2764	3.0664	3.9139	1.3619	2.1885	3.5504
	30	0.5	−0.8888	2.2278	10.2628	0.2193	2.8558	2.3021	1.6756	3.8574	1.8706	1.5556	3.3762
		1	−0.5309	1.2841	3.2517	0.8778	10.6559	1.2661	3.1441	3.9807	1.3121	2.2109	3.5230

As clearly seen from Table 2, the BBM inequality and Heisenberg uncertainty relation hold for the Morse potential and the sum of entropies $S_x + S_p$ is higher than $1 + \ln\pi$ value. The significance of the BBM inequality is that it presents an irreducible lower bound to the entropy sum so that with increasing α, the entropy sum $S_x + S_p$ tends to be saturated to the boundary value defined by the BBM inequality for the different states n, i.e. their sum stays above the stipulated lower bound of $1 + \ln\pi$. It can also be found in the Shannon entropy as a more powerful version of the Heisenberg uncertainty acts.

The other global measurements is check Heisenberg's uncertainty relation. In fact, uncertainty relations form the basic properties of quantum mechanics. Where The uncertainty principle is the product of the uncertainty position and momentum as $(\Delta x)(\Delta p) \geqslant \frac{1}{2}$ $(\hbar = 1)$. Numerical results are shown in Tables 1 and 2 that observe the behave similarly to entropy but with different accuracy .

4 Conclusion

We presented the information-theoretic measures, namely the Shannon information, of Morse and Rosen–Morse interactions for relativistic fermions. We find that the position information entropy S_x decreases for parameter α in the case of Morse potential, but the behavior of S_x is contrary to that of the S_p and S_x increases with increasing a. The impact of parameters V_0 and d on the stability of the systems were shown in detail. Some interesting features of the information entropy densities, $\rho_s(x)$ and $\gamma_s(p)$, were demonstrated via various figures and tables. The consistency with Bialynicki-BirulaMycielski (BBM) inequality and Heisenberg uncertainty were also verified for a number of states.

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Appendix

The following equation is a general form of second order differential equations written for any potentials

$$\left[\frac{d^2}{ds^2} + \frac{\alpha_1 - \alpha_2 s}{s(1-\alpha_3 s)}\frac{d}{ds} + \frac{-\xi_1 s^2 + \xi_2 s - \xi_3}{[s(1-\alpha_3 s)]^2}\right]\psi = 0 \tag{50}$$

That according to the NU method, the eigenfunctions and eigenenergies respectively are

$$\psi(s) = s^{\alpha_{13}}(1-\alpha_3 s)^{-\alpha_{12}-\frac{\alpha_{13}}{\alpha_3}} P_n^{\left(\alpha_{10}-1,\frac{\alpha_{11}}{\alpha_3}-\alpha_{10}-1\right)}(1-2\alpha_3 s) \tag{51}$$

and

$$\alpha_2 n - (2n+1)\alpha_5 + (2n+1)\left(\sqrt{\alpha_9} + \alpha_3\sqrt{\alpha_8}\right) + n(n-1)\alpha_3 + \alpha_7 + 2\alpha_3\alpha_8 + 2\sqrt{\alpha_8\alpha_9} = 0 \tag{52}$$

where

$$\begin{aligned} &\alpha_4 = \frac{1}{2}(1-\alpha_1), \quad \alpha_5 = \frac{1}{2}(\alpha_2 - 2\alpha_3), \quad \alpha_6 = \alpha_5^2 + \xi_1, \\ &\alpha_7 = 2\alpha_4\alpha_5 - \xi_2, \quad \alpha_8 = \alpha_4^2 + \xi_3, \quad \alpha_9 = \alpha_3\alpha_7 + \alpha_3^2\alpha_8 + \alpha_6, \\ &\alpha_{10} = \alpha_1 + 2\alpha_4 + 2\sqrt{\alpha_8}, \quad \alpha_{11} = \alpha_2 - 2\alpha_5 + 2\left(\sqrt{\alpha_9} + \alpha_3\sqrt{\alpha_8}\right), \\ &\alpha_{12} = \alpha_4 + \sqrt{\alpha_8}, \quad \alpha_{13} = \alpha_5 - \left(\sqrt{\alpha_9} + \alpha_3\sqrt{\alpha_8}\right), \end{aligned} \tag{53}$$

In the special case $\alpha_3 = 0$,

$$\lim_{\alpha_3 \to 0} P_n^{\left(\alpha_{10}-1,\frac{\alpha_{11}}{\alpha_3}-\alpha_{10}-1\right)}(1-2\alpha_3 s) = L_n^{(\alpha_{10}-1)}(\alpha_{11}s) \tag{54}$$

$$\lim_{\alpha_3 \to 0}(1-\alpha_3 s)^{-\alpha_{12}-\frac{\alpha_{13}}{\alpha_3}} = e^{\alpha_{13}s} \tag{55}$$

So from Eq. (51),we have for the wavefunction

$$\psi(s) = s^{\alpha_{12}} e^{\alpha_{13}s} L_n^{(\alpha_{10}-1)}(\alpha_{11}s) \tag{56}$$

References

1. C. Amovilli, N.H. March, Quantum information: Jaynes and shannon entropies in a two-electron entangled artificial atom. Phys. Rev. A **69**(5), 054302 (2004)
2. J.C. Angulo, J. Antolín, K.D. Sen, Fisher-shannon plane and statistical complexity of atoms. Phys. Lett. A **372**(5), 670–674 (2008)
3. A. Arda, R. Sever, C. Tezcan, Approximate analytical solutions of the Klein-Gordon equation for the hulthén potential with the position-dependent mass. Phys. Scr. **79**(1), 015006 (2008)
4. E. Aydiner, C. Orta, R. Sever, Quantum information entropies of the eigenstates of the morse potential. Int. J. Modern Phys. B **22**(03), 231–237 (2008)
5. W. Beckner, Inequalities in fourier analysis. Ann. Math. **102**(1), 159–182 (1975)
6. V.S. Buyarov, J.S. Dehesa, A. Martinez-Finkelshtein, E.B. Saff, Asymptotics of the information entropy for Jacobi and Laguerre polynomials with varying weights. J. Approx. Theory **99**(1), 153–166 (1999)
7. R.G. Catalán, J. Garay, R. López-Ruiz, Features of the extension of a statistical measure of complexity to continuous systems. Phys. Rev. E **66**(1), 011102 (2002)
8. M.W. Coffey, Semiclassical position and momentum information entropy for sech2 and a family of rational potentials. Can. J. Phys. **85**(7), 733–743 (2007)
9. T.M. Cover, J.A. Thomas et al., Elements of information theory, 2nd edn. (Wiley, New York, 1991)
10. R. De, R. Dutt, U. Sukhatme, Mapping of shape invariant potentials under point canonical transformations. J. Phys. A: Math. Gen. **25**(13), L843 (1992)
11. A.S. de Castro, A. Armat, H. Hassanabadi, Scattering and bound states of fermions in the modified hulthén potential. Eur. Phys. J. Plus **129**(10), 1–7 (2014)
12. A. de Souza Dutra, C.A.S. Almeida, Exact solvability of potentials with spatially dependent effective masses. Phys. Lett. A **275**(1), 25–30 (2000)
13. J.S. Dehesa, W. Van Assche, R.J. Yáñez, Information entropy of classical orthogonal polynomials and their application to the harmonic oscillator and coulomb potentials. Methods Appl. Anal. **4**, 91–110 (1997)
14. J.S. Dehesa, A. Martínez-Finkelshtein, V.N. Sorokin, Information-theoretic measures for morse and Pöschl-Teller potentials. Mol. Phys. **104**(4), 613–622 (2006)
15. J.S. Dehesa, R.J. Yáñez, A.I. Aptekarev, V. Buyarov, Strong asymptotics of laguerre polynomials and information entropies of two-dimensional harmonic oscillator and one-dimensional coulomb potentials. J. Math. Phys. **39**(6), 3050–3060 (1998)
16. S. Dong, G.-H. Sun, S.-H. Dong, J.P. Draayer, *Quantum information entropies for a squared tangent potential well*. Phys. Let. A **378**(3), 124–130 (2014)
17. R. González-Férez, J.S. Dehesa, Shannon entropy as an indicator of atomic avoided crossings in strong parallel magnetic and electric fields. Phys. Rev. Lett. **91**(11), 113001 (2003)
18. H. Hatori, *A note on the entropy of a continuous distribution, in Kodai Mathematical Seminar Reports*, vol. 10 (Tokyo Institute of Technology, Department of Mathematics, 1958), pp. 172–176
19. W. Jian-Jie, Shannon entropy as a measurement of the information in a multiconfiguration diracfock wavefunction. Chin. Phys. Lett. **32**(2), 023102 (2015)
20. J. Katriel, K.D. Sen, Relativistic effects on information measures for hydrogen-like atoms. J. Comput. Appl. Math. **233**(6), 1399–1415 (2010)
21. A. Kumar, Information entropy of isospectral poschl-teller potential. Ind. J. Pure Appl. Phys. **44**(12), 958–963 (2005)
22. R. Lopez-Ruiz, H.L. Mancini, X. Calbet, A statistical measure of complexity. Phy. Lett. A **209**, 321–326 (1995). arXiv:nlin/0205033 [nlin.CD]
23. V. Majernik, T. Opatrnỳ, Entropic uncertainty relations for a quantum oscillator. J. Phys. A: Math. Gen. **29**(9), 2187 (1996)
24. F.J. Marcellán, R.J. Yáñez, A. Zarzo, Special issue dedicated to professor Jesús sánchez-Dehesa on the occasion of his 60th birthday. J. Comput. Appl. Math. **233**(6), 1345–1354 (2010)
25. S.E. Massen, Application of information entropy to nuclei. Phys. Rev. C **67**(1), 014314 (2003)
26. A. Nagy, S. Liu, Local wave-vector, shannon and fisher information. Phys. Lett. A **372**(10), 1654–1656 (2008)
27. S.A. Najafizade, H. Hassanabadi, S. Zarrinkamar, Nonrelativistic shannon information entropy for killingbeck potential. Can. J. Phys. **94**(10), 1085–1092 (2016)
28. S.A. Najafizade, H. Hassanabadi, S. Zarrinkamar, Nonrelativistic shannon information entropy for kratzer potential. Chin. Phys. B **25**(4), 040301 (2016)
29. A.F. Nikiforov, V.B. Uvarov, *Special Functions of Mathematical Physics* (Springer, Berlin, 1988)
30. A. Orłowski, Information entropy and squeezing of quantum fluctuations. Phys. Rev. A **56**(4), 2545 (1997)
31. S.H. Patil, K.D. Sen, Net information measures for modified Yukawa and Hulthen potentials. Int. J. Quant. Chem. **107**(9), 1864–1874 (2007)
32. K.D. Sen, Characteristic features of shannon information entropy of confined atoms. J. Chem. Phys. **123**(7), 074110–074110 (2005)
33. C.E. Shannon, A Mathematical Theory of Communication. Bell syst Tech. J. **27** (379–423), 623–656 (1948)
34. Q. Shi, S. Kais, *Discontinuity of shannon information entropy for two-electron atoms*. Chem. Phys. **309**(2), 127–131 (2005)
35. G.-H. Sun, M.A. Aoki, S.-H. Dong, Quantum information entropies of the eigenstates for the pöschl-Teller-like potential. Chin. Phys. B **22**(5), 050302 (2013)
36. G.-H. Sun, S.-H. Dong, Quantum information entropies of the eigenstates for a symmetrically trigonometric Rosen–Morse potential. Phys. Scr. **87**(4), 045003 (2013)
37. G.-H. Sun, S.-H. Dong, K.D. Launey, T. Dytrych, J.P. Draayer, Shannon information entropy for a hyperbolic double-well potential. Int. J. Quant. Chem. **115**(14), 891–899 (2015)
38. G.-H. Sun, S.-H. Dong, N. Saad, Quantum information entropies for an asymmetric trigonometric Rosen-Morse potential. Annalen der Physik **525**(12), 934–943 (2013)

39. C. Tezcan, R. Sever, A general approach for the exact solution of the Schrödinger equation. Int. J. Theor. Phys. **48**(2), 337–350 (2009)
40. R. Valencia-Torres, G.-H. Sun, S.-H. Dong, Quantum information entropy for a hyperbolical potential function. Phys. Scr. **90**(3), 035205 (2015)
41. O. von Roos, Position-dependent effective masses in semiconductor theory. Phys. Rev. B **27**(12), 7547 (1983)
42. G. Yáñez-Navarro, G.-H. Sun, T. Dytrych, K.D. Launey, S.-H. Dong, J.P. Draayer, Quantum information entropies for position-dependent mass Schrödinger problem. Ann. Phys. **348**, 153–160 (2014)
43. S. Zarrinkamar, A.A. Rajabi, H. Hassanabadi, Dirac equation in the presence of coulomb and linear terms in (1+ 1) dimensions; the supersymmetric approach. Ann. Phys. **325**(8), 1720–1726 (2010)