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J. H. Alvarenga Nogueira · T. Frederico · O. Lourenço

$B^+ \to K^-\pi^+\pi^+$: Three-Body Final State Interactions and $K\pi$ Isospin States

Abstract In this exploratory study, final state interactions are considered to formulate the B meson decay amplitude for the $K\pi\pi$ channel. The Faddeev decomposition of the Bethe–Salpeter equation is used in order to build a relativistic three-body model within the light-front framework. The S-wave scattering amplitude for the $K\pi$ system is considered in the 1/2 and 3/2 isospin channels with the set of inhomogeneous integral equations solved perturbatively. In comparison with previous results for the D meson decay in the same channel, one has to consider the different partonic processes, which build the source amplitudes, and the larger absorption to other decay channels appears, that are important features to be addressed. As in the D decay case, the convergence of the rescattering perturbative series is also achieved with two-loop contributions.

1 Introduction

Heavy quark decays are largely explored in the literature. Due to the large B meson mass (m_b), there are several approaches for B decays based on QCD effective field theories within heavy quark expansions [1–4]. They are based on factorization of the hadronic matrix elements and mainly consider short-distance physics. The weak effective Hamiltonian is constructed based on tools from quantum field theory, such as the operator product expansion to separate the problem in the long-distance and short-distance physics. The perturbative treatment is justified by the fact that the strong coupling constant α_s is small in high energy short-distance processes. The long-distance physics and its non-perturbative nature leads to divergent amplitudes that are complicated to deal with and requires care. The called soft final state interactions (FSI) shows to be essential in studies involving B meson decays, since it does not disappear for large m_b [5]. However, within the QCD factorization approach it was shown that type of effects are suppressed in the heavy quark limit in the case of two-body decays [1,2]. Contributions coming from long-distance inelastic rescattering is expected to be the main source of soft FSI and can be substantial in charge-parity (CP) violation distributions [6,7]. Rescattering effects can also explain the appearance of events in very suppressed decay channels. A recent experimental study of the charmless B_c decay to the $KK\pi$ channel, which within the Standard Model can only occur by weak annihilation diagrams, shows some events in the phase space of this channel [8]. This can be related with hadronic rescattering inelastic transitions to that final decay channel. QCD factorization calculations of two-body B_c decays, also suppressed, can explain that small branching ratios [9]. Contributions coming from final state interaction for the $B^+ \to J/\psi\pi^+$ decay within the QCD factorization approach was further considered in [10].

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J. H. A. Nogueira · T. Frederico (✉)
Instituto Tecnológico de Aeronáutica, São José dos Campos, SP 12228-900, Brazil
E-mail: tobfrederico@yahoo.com.br

O. Lourenço
Universidade Federal do Rio de Janeiro, Macaé, RJ 27930-560, Brazil

FSI play an important role in heavy meson weak decays. This interactions usually appears as suppressed non-factorizable effects in QCD factorization, but even within this approach it is shown that in the center of the Dalitz plot physical values of m_b seem not to be large enough to suppress significantly that power-corrections [11]. As a test of *CP* violation, FSI are essential to guarantee *CPT* invariance [6,7]. A practical theoretical approach was used to study these three-body charmless $B^{\pm}$ decays in [6]. A more general formulation, including resonances and its interferences, applied for four *B* decay channels is found in Ref. [7]. *CP* violation in the low invariant mass of the $\pi\pi$ system of the $B \rightarrow \pi\pi\pi$ decay channel is also studied in Ref. [12,13], where contributions from scalar and vector resonances are considered. The S-wave $\pi\pi$ elastic scattering in the region below the ρ mass has also the important contribution from the $f_0(600)$ resonance, as showed in Ref. [14] for a four-body semileptonic decay.

Our goal in the present work is to address the issue of three-body FSI in the specific $B^+ \rightarrow K^-\pi^+\pi^+$ decay, with emphasis in the S-wave $K^-\pi^+$ amplitude, as an exploratory first approach study. In order to proceed in such direction, we closely follow the formalism developed for the *D* decay in Ref. [15].

Our study is based in a relativistic model for the three-body FSI that was applied to the $D^+ \rightarrow K^-\pi^+\pi^+$ decay [15–17]. In Ref. [15], the isospin projection of the decay amplitude was performed to study different isospin state contributions to the $K^-\pi^+$ rescattering. In that model, by starting from a Bethe–Salpeter like equation and using the Faddeev decomposition, the decay amplitude was separated into a smooth term and a three-body fully interacting contribution. Moreover, the amplitude was factorized in the standard two-meson resonant amplitude times a reduced complex amplitude for the bachelor meson, that carries the effect of the three-body rescattering mechanism. The off-shell bachelor amplitude is a solution of an inhomogeneous Faddeev type integral equation, that has as input the S-wave isospin 1/2 and 3/2 $K^-\pi^+$ transition matrix. In the Faddeev formulation, the integral equation has a connected kernel, which is written in terms of the two-body amplitude. The light-front (LF) projection of the equations [18] was performed to simplify the numerical calculations, and interactions between identical charged pions were neglected. A different coupled-channel framework, considering both $\pi\pi$ and $K\pi$ empirical scattering amplitudes, was used in Ref. [19] to study the $D^+ \rightarrow K^-\pi^+\pi^+$ Dalitz plot.

Here we discuss the perturbative solutions of the LF integral equations for the bachelor amplitude in the *B* meson decay. To check the convergence of the series expansion, we go up to terms of third order in the two-body transition matrix. The numerical results for the $B^+ \rightarrow K^-\pi^+\pi^+$ decay with three-body FSI and $K\pi$ interactions in $I = 1/2$ and 3/2 states are presented. The S-wave $K\pi$ amplitude depends on the isospin of the system. There are two isospin states possible for this system, namely, $I = 1/2$ and $I = 3/2$. The LASS experimental data [20] shows resonances and the corresponding scattering amplitude poles only in the isospin 1/2 channel. This feature is used here to model the $K\pi$ S-matrix used in the *B* decay amplitude.

In the recent paper of Nakamura [19], it is discussed that the effect of the $\pi^+\pi^0$ p-wave ($I = 1$) interaction can contribute to the $D^+ \rightarrow K^-\pi^+\pi^+$ decay only through the rescattering, with this contribution being a pure coupled-channel effect. Our model is a single channel model for $B^+ \rightarrow K^-\pi^+\pi^+$. It is well known that the effect of coupled channels in single channel model, like the present one, is represented by effective absorptive interaction. In our single channel model, this will correspond to a three-body absorptive interaction. Indeed, the coupling to other channels is introduced by the ε parameter in the three-body propagator, and, as we will show, this parameter is important for the resulting three-body decay amplitude. From this point of view, other channels can have important consequences on the form of *B*-meson decay amplitude. The loss of flux in our three-body model corresponds to the presence of other channels. As one introduce explicitly all the coupled channels, three-body unitarity has to be satisfied. A single channel representation of the rescattering process has to include the loss of probability flux given in our case by the finite width given in the three-body propagator. Physically, this finite width has to be associated with the transition to different channels and the channel formed by the neutral kaon, neutral pion and charged pion, is particularly important, since it allows the $\pi^+\pi^0$ p-wave ($I = 1$) interaction that generates $\rho(770)$, and plays a major role in the FSI in the $D^+ \rightarrow K^-\pi^+\pi^+$ decay. The same effect is possibly present also in the $B^+ \rightarrow K^-\pi^+\pi^+$ decay. However, effectively the coupling to other channels in our approach is taken into account qualitatively by allowing to a finite width (ε) to the three-body propagator.

2 Decay Amplitude for $B^+ \rightarrow K^-\pi^+\pi^+$ Decay with FSI

2.1 S-wave $K\pi$ Scattering Amplitude

The three-body rescattering model used here to study the decay amplitude with FSI, requires a two-body transition matrix as input. In the same way we have done in the $D^+ \rightarrow K^-\pi^+\pi^+$ decay in Ref. [15], the $K\pi$

S-wave elastic scattering amplitude is introduced in the resonant $I_{K\pi} = 1/2$ and non-resonant $I_{K\pi} = 3/2$ isospin states. We use the same parametrization fitted to the LASS data [20] including two resonances above $K_0^*(1430)$, namely $K^*(1630)$ and $K_0^*(1950)$. The main reason to use the additional $K_0^*(1630)$ and $K_0^*(1950)$ resonances is the LASS data, where the whole kinematical range up to 1.89 GeV is fitted. We choose here to not introduce new resonances in the $K^- \pi^+$ $I = 1/2$ channels as it seems that no higher mass resonances are present in this channels, according to the PDG. In addition, the $I = 3/2$ channels seem to be a simple S-wave scattering parametrized by the first two terms in the effective range expansion. In order to not introduce more assumptions, we prefer to be conservative and keep what was used in the previous paper [15]. In our analysis, we also neglect the $\pi\pi$ interaction. The same approximation was also considered in the D decay case of Ref. [15].

The parametrized S-matrix ($S_{K\pi}^{1/2}$) is written as:

$$S_{K\pi}^{1/2} = \frac{k \cot \delta + i \, k}{k \cot \delta - i \, k} \prod_{r=1}^{3} \frac{M_r^2 - M_{K\pi}^2 + i z_r \bar{\Gamma}_r}{M_r^2 - M_{K\pi}^2 - i z_r \Gamma_r} \tag{1}$$

where $z_r = k \, M_r^2 / (k_r \, M_{K\pi})$ and k is the c. m. momentum of each meson of the $K\pi$ pair. Following this S-matrix, the scattering amplitude reads

$$\tau_{I_{K\pi}} \left(M_{K\pi}^2 \right) = 4\pi \frac{M_{K\pi}}{k} \left(S_{K\pi}^{I_{K\pi}} - 1 \right). \tag{2}$$

The parameters associated to the $K_0^*(1430)$, $K_0^*(1630)$ and $K_0^*(1950)$ resonances are (M_r, Γ_r, $\bar{\Gamma}_r$) given by (1.48, 0.25, 0.25), (1.67, 0.1, 0.1) and (1.9, 0.2, 0.14), respectively [16].

The non-resonant part of the scattering amplitude is parameterized by an effective range expansion as $k \cot \delta = \frac{1}{a} + \frac{1}{2} r_0 k^2$ using $a = 1.6$ GeV^{-1} and $r_0 = 3.32$ GeV^{-1}. By using such a model, the S-wave $K\pi$ amplitude in the $I = 3/2$ state is given by $S_{K\pi}^{3/2} = \frac{k \cot \delta + i \, k}{k \cot \delta - i \, k}$, with the effective range expansion parameters $a = -1.00$ GeV^{-1} and $r_0 = -1.76$ GeV^{-1} taken from Ref. [21].

The parametrization from the three-resonance model and the $I_{K\pi} = 1/2$ S-wave phase-shift compared to the LASS data shows good agreement. The results of the parametrization for $|S_{K\pi}^{1/2} - 1|/2$ are shown and discussed in more details in Ref. [15].

2.2 Three-Body Rescattering Bethe–Salpeter Model

The full decay amplitude including the rescattering series and the $3 \to 3$ transition matrix is written as [15]:

$$\begin{aligned} \mathcal{A}(k_\pi, k_{\pi'}) = &\, B_0(k_\pi, k_{\pi'}) \\ &+ \int \frac{d^4 q_\pi d^4 q_{\pi'}}{(2\pi)^8} T(k_\pi, k_{\pi'}; q_\pi, q_{\pi'}) S_\pi(q_\pi) \, S_\pi(q_{\pi'}) S_K(K - q_{\pi'} - q_\pi) B_0(q_\pi, q_{\pi'}), \end{aligned} \tag{3}$$

where the momentum of the pions are k_π and $k_{\pi'}$ and K is the total momentum of the system, which is used to write the two-body invariant mass of the $K\pi$ system as $M_{K\pi}^2 = (K - k_{\pi'})^2$.

The short-distance physics resides in the $B_0(k_\pi, k_{\pi'})$ amplitude, which represents the quark level amplitude. The sum of rescattering diagrams, considered in the ladder approximation, is in the second term of Eq. (3) and composes the long range physics. This term is composed by the $3 \to 3$ transition matrix $T(k_\pi, k_{\pi'}; q_\pi, q_{\pi'})$ with the source term and the meson propagators $S_i(q_i) = i(q_i^2 - m_i^2 + i\epsilon)^{-1}$, where self-energies are neglected. The $K\pi$ transition matrix sum all $2 \to 2$ collision terms. The full transition matrix with the FSI is a solution of the Bethe–Salpeter equation, used with its Faddeev decomposition.

2.3 Decay Amplitude

The full three-body T-matrix gives the final state interactions between the mesons in the decay channel and it is a solution of the Bethe–Salpeter equation. Here we follow the formalism developed in Ref. [15], where the Faddeev decomposition including only two-body irreducible diagrams for spinless particles without self-energies is considered. Only two body interactions are considered, involving all three-particles except between the equal charged pions.

The two-body transition matrix written with a four-conservation delta factorized out reads

$$T_i\left(k'_j, k'_k; k_j, k_k\right) = (2\pi)^4 \tau_i(s_i) S_i^{-1}(k_i)\, \delta\left(k'_i - k_i\right) , \tag{4}$$

where the Mandelstam variable $s_i = (k_j + k_k)^2$ is the only dependence considered and $\tau_i(s_i)$ is the unitary S-wave scattering amplitude of particles j and k. Using the separable form of Eq. (4) the problem is reduced to a four-dimensional integral equation in one momentum variable for the Faddeev components of the vertex function.

The full decay amplitude considering interactions between all the final states mesons reduces to

$$\mathcal{A}_0(k_i, k_j) = B_0(k_i, k_j) + \sum_\alpha \tau(s_\alpha)\xi^\alpha(k_\alpha) , \tag{5}$$

where the subindex in $\mathcal{A}_0$ denotes the S-wave two-meson scattering and the bachelor amplitude $\xi(k_i)$ carries the three-body rescattering effect and is represented by the connected Faddeev-like equations

$$\xi^i(k_i) = \xi_0^i(k_i)$$
$$+ \int \frac{d^4 q_j}{(2\pi)^4} S_j(q_j)S_k(K - k_i - q_k)\tau_j(s_j)\xi^j(q_j) + \int \frac{d^4 q_k}{(2\pi)^4} S_j(K - k_i - q_k)S_k(q_k)\tau_k(s_k)\xi^k(q_k). \tag{6}$$

with $q_k = K - k_i - q_j$. In Eq. (6), both, amplitude and phase, depending on the bachelor meson on-mass-shell momentum and $\tau(s_i)$, can take into account two-meson resonances. The parameterized $K\pi$ scattering amplitude $\tau_i(M^2_{K\pi})$ reproduces the LASS experimental [20] S-wave phase-shift in the isospin 1/2 and 3/2 channels.

By taking into account all the model assumptions, the decay amplitude for the $B^+ \to K^-\pi^+\pi^+$ process is given by

$$\mathcal{A}_0(k_\pi, k_{\pi'}) = B_0(k_\pi, k_{\pi'}) + \tau(M^2_{K\pi})\xi(k_{\pi'}) + \tau(M^2_{K\pi'})\xi(k_\pi) , \tag{7}$$

where $M^2_{K\pi} = (K - k_{\pi'})^2$, $M^2_{K\pi'} = (K - k_\pi)^2$ and the bachelor pion on-mass-shell momentum is given by

$$|\mathbf{k}_\pi| = \left[\left(\frac{M^2_B + m^2_\pi - M^2_{K\pi'}}{2\, M_B} \right)^2 - m^2_\pi \right]^{\frac{1}{2}} . \tag{8}$$

The rescattering series comes from the solution of Eq. (9), where the second and third terms in Eq. (6) correspond to higher order loop diagrams.

The inhomogeneous integral equation for the spectator amplitude in the three-body collision process is a function only of the bachelor momentum (see [15]),

$$\xi(k) = \xi_0(k) + \int \frac{d^4 q}{(2\pi)^4} \tau\left((K - q)^2\right) S_K(K - k - q)\, S_\pi(q)\, \xi(q), \tag{9}$$

where the first term is

$$\xi_0(k) = \int \frac{d^4 q}{(2\pi)^4} S_\pi(q)S_K(K - k - q)B_0(k, q), \tag{10}$$

with the partonic decay amplitude described by $B_0(k, q)$.

The two basic contributions for the decay amplitude are the well behaved function $B_0(k_\pi, k_{\pi'})$ and three-body rescattering term $\tau\left(M^2_{K\pi'}\right)\xi(k_\pi)$. The operator τ acts on the isospin states 1/2 and 3/2. The complex decay amplitude can be decomposed in terms of phase and amplitude as

$$A\left(M^2_{K\pi'}\right) = \frac{1}{2}\langle K\pi\pi|B_0\rangle + \langle K\pi\pi|\tau\left(M^2_{K\pi'}\right)|\xi(k_\pi)\rangle = a_0\left(M^2_{K\pi'}\right) e^{i\Phi_0\left(M^2_{K\pi'}\right)}, \tag{11}$$

which is a function of only $M^2_{K\pi'}$ and $|K\pi\pi\rangle$ represents the state in isospin space.

3 FSI Light-Front Equations

The equations presented for the decay processes considering FSI effects are simplified when treated in light-front dynamics. The light-front (LF) projection of the four-dimensional coupled equations presents a three-dimensional form. Such a technique was successfully applied for the heavy meson decays presented in Ref. [15] and will also be used here to treat the $B \to K\pi\pi$ decay problem.

The light-front projection performed in the field-theoretical inhomogeneous three-body BS equation to build the integral equations used in our work, corresponds to the truncation of the light-front Fock-space to the three-meson valence component in the intermediate. The advantages of performing the LF projection is that Z-diagrams are in general suppressed [22]. In addition, the integral equations with only the valence three-meson state are covariant under seven LF kinematical transformation, namely, the ones that keep the null-plane invariant, which includes three translations, rotation around the z-direction, two other kinematical boosts, and the boost along the z-direction. The truncation of the LF Fock-space is stable under the kinematical boosts [23]. In contrast, the Fock-space truncation in the instant form has only three translations and three rotations and no-boosts.

After all manipulations, discussed in details in [15], the integral equation in terms of the LF variables reads

$$
\xi^i(y, \mathbf{k}_\perp) = \xi_0^i(y, \mathbf{k}_\perp)
$$
$$
+ \frac{i}{2(2\pi)^3} \int_0^{1-y} \frac{dx}{x(1-x-y)} \int d^2 q_\perp \left[\frac{\tau_j\left(M_{ik}^2(x, q_\perp)\right) \xi^j(x, \mathbf{q}_\perp)}{M^2 - M_0^2(x, \mathbf{q}_\perp; y, \mathbf{k}_\perp) + i\varepsilon} + (j \leftrightarrow k) \right], \quad (12)
$$

where $M^2 = K^\mu K_\mu$, $y = k_i^+/K^+$, $x = q_j^+/K^+$ or $x = q_k^+/K^+$ in the first or second integral in the right-hand side of the equation. The free three-body squared mass is

$$
M_0^2(x, \mathbf{q}_\perp; y, \mathbf{k}_\perp) = \frac{k_\perp^2 + m_i^2}{y} + \frac{q_\perp^2 + m_j^2}{x} + \frac{(\mathbf{k}_\perp + \mathbf{q}_\perp)^2 + m_k^2}{1 - x - y}. \quad (13)
$$

The argument of the two-body amplitude $\tau_j\left(M_{ik}^2(x, q_\perp)\right)$ should be understood as

$$
M_{ik}^2(x, q_\perp) = (1 - x)\left(M^2 - \frac{q_\perp^2 + m_j^2}{x}\right) - q_\perp^2. \quad (14)
$$

The driven term in Eq. (12) is rewritten as

$$
\xi_0^i(y, \mathbf{k}_\perp) = \frac{i}{2(2\pi)^3} \int_0^{1-y} \frac{dx}{x(1-y-x)} \int d^2 q_\perp \frac{B_0(x, \mathbf{q}_\perp; y, \mathbf{k}_\perp)}{M^2 - M_0^2(x, \mathbf{q}_\perp; y, \mathbf{k}_\perp) + i\varepsilon} = B_0 \, \xi_0(y, k_\perp) \quad (15)
$$

where $B_0(x, \mathbf{q}_\perp; y, \mathbf{k}_\perp) = B_0$ is the short-distance amplitude, taken as a constant in this work.

Since the integral over $q_\perp$ is divergent, a regularization procedure is needed. Here we use a finite subtraction constant $\lambda(\mu^2)$, and a subtraction point within the integration kernel of Eq. (15). This method leads to the following driven term

$$
\xi_0(y, k_\perp) = \lambda(\mu^2)
$$
$$
+ \frac{i}{2} \int_0^1 \frac{dx}{x(1-x)} \int_0^{2\pi} d\theta \int_0^\infty \frac{dq_\perp q_\perp}{(2\pi)^3} \left[\frac{1}{M_{K\pi}^2(y, k_\perp) - M_{0,K\pi}^2(x, q_\perp) + i\varepsilon} - \frac{1}{\mu^2 - M_{0,K\pi}^2(x, q_\perp)} \right]
$$
$$(16)$$

with the $K\pi$ system free squared-mass given by $M_{0,K\pi}^2(x, q_\perp) = \frac{q_\perp^2 + m_\pi^2}{x} + \frac{q_\perp^2 + m_K^2}{1-x}$. After integration over θ and $q_\perp$, Eq. (16) is finally written as

$$
\xi_0(y, k_\perp) = \lambda(\mu^2) + \frac{i}{4} \int_0^1 \frac{dx}{(2\pi)^2} \ln \frac{(1-x)\left(x M_{K\pi}^2(y, k_\perp) - m_\pi^2 + i x\varepsilon\right) - x m_K^2}{(1-x)\left(x\mu^2 - m_\pi^2\right) - x m_K^2}. \quad (17)
$$

4 Application in the $B^+ \to K^- \pi^+ \pi^+$ Decay

The model for the $B^+ \to K^- \pi^+ \pi^+$ decay with FSI is based on an inhomogeneous integral equation for the spectator meson, with the meson-meson scattering amplitude as input. Isospin states of the $\pi\pi$ interaction are disregarded here, unlike the $I_{K\pi} = 1/2$ and $I_{K\pi} = 3/2$ states for the $K^\mp \pi^\pm$ channel, consider in our calculations. Our parametrization for the $K\pi$ amplitude follows the experimental results of [20], where the resonant $I_{K\pi} = 1/2$ channel below $K_0^*(1430)$ dominates and the $I_{K\pi} = 3/2$ amplitude is comparable. This model is the same used in Ref. [15] to study the $D^+ \to K^- \pi^+ \pi^+$ decay. A calculation up to two loops for this same decay was performed in Ref. [17] bellow $K_0^*(1430)$.

Here the LF model is applied to the B decay and the calculations are performed up to three-loops in order to check the numerical convergence of the integrals. There are two possible total isospin states, namely, $I_T = 5/2$ and $3/2$. In our notation, the bachelor amplitude has the total isospin index and the one related with the interacting pair $\xi^{I_T^z}_{I_T, I_{K\pi}}(y, k_\perp)$, where we also consider the isospin projection index. The source amplitude written in terms of the $K\pi$ isospin state reads

$$|B_0\rangle = \sum_{I_T, I_{K\pi}} \alpha^{I_T^z}_{I_T, I_{K\pi}} |I_T, I_{K\pi}, I_T^z\rangle + \sum_{I_T, I_{K\pi'}} \alpha^{I_T^z}_{I_T, I_{K\pi'}} |I_T, I_{K\pi'}, I_T^z\rangle, \tag{18}$$

which has no dependence on the momentum variables and has an arbitrary normalization, since we are not considering explicitly short-distance processes in our calculations. For sake of simplicity we define the recoupling coefficients as $R^{I_T^z}_{I_T, I_{K\pi}, I_{K\pi'}} = \langle I_T, I_{K\pi}, I_T^z | I_T, I_{K\pi'}, I_T^z \rangle$. This allows us to write the set of isospin coupled integral equations as

$$\xi^{I_T^z}_{I_T, I_{K\pi}}(y, k_\perp) = \langle I_T, I_{K\pi}, I_T^z | B \rangle \xi_0(y, k_\perp) + \frac{i}{2} \sum_{I_{K\pi'}} R^{I_T^z}_{I_T, I_{K\pi}, I_{K\pi'}} \int_0^{1-y} \frac{dx}{x(1-y-x)} \int_0^\infty \frac{dq_\perp}{(2\pi)^3}$$
$$\times K_{I_{K\pi'}}(y, k_\perp; x, q_\perp) \xi^{I_T^z}_{I_T, I_{K\pi'}}(x, q_\perp), \tag{19}$$

where the free squared mass of the $K\pi\pi$ system is

$$M^2_{0, K\pi\pi}(x, q_\perp, y, k_\perp) = \frac{k_\perp^2 + m_\pi^2}{y} + \frac{q_\perp^2 + m_\pi^2}{x} + \frac{q_\perp^2 + k_\perp^2 + 2q_\perp k_\perp \cos\theta + m_K^2}{1 - x - y}, \tag{20}$$

with the squared-mass of the virtual $K\pi$ system $M^2_{K\pi}(z, p_\perp) = (1-z)\left(M_B^2 - \frac{p_\perp^2 + m_\pi^2}{z}\right) - p_\perp^2$. The kernel carrying the $K\pi$ scattering amplitude is

$$K_{I_{K\pi'}}(y, k_\perp; x, q_\perp) = \int_0^{2\pi} d\theta \, \frac{q_\perp \tau_{I_{K\pi'}}\left(M^2_{K\pi'}(x, q_\perp)\right)}{M_B^2 - M^2_{0, K\pi\pi}(x, q_\perp, y, k_\perp) + i\varepsilon}. \tag{21}$$

Isospin 2 states of pion-pion interactions are not considered in the model, which will be explored as a single channel model, with the $K\pi$ S-wave interaction in the resonant $I = 1/2$, and as a coupled channel model with both $I = 1/2$ and $3/2$ $K\pi$ S-wave interactions.

The symmetrized decay amplitude with respect to the identical pions is written as

$$\mathcal{A}_0 = A_0\left(M^2_{K\pi'}\right) + A_0\left(M^2_{K\pi}\right). \tag{22}$$

The isospin projection on each term leads to

$$A_0\left(M^2_{K\pi'}\right) = \sum_{I_T, I_{K\pi'}, I_T^z} \langle K^- \pi^+ \pi^+ | I_T, I_{K\pi'}, I_T^z \rangle \left[\frac{1}{2} \langle I_T, I_{K\pi'}, I_T^z | B_0 \rangle + \tau_{I_{K\pi}}\left(M^2_{K\pi'}\right) \xi^{I_T^z}_{I_T, I_{K\pi'}}(k_\pi)\right]$$
$$= a_0\left(M^2_{K\pi'}\right) e^{i\Phi_0\left(M^2_{K\pi'}\right)}. \tag{23}$$

5 Numerical Perturbative Solutions

The problem is solved by integrating the terms starting from the driving term and iterating as a perturbative series. The integrations are done up to three loops in order to check the convergence. In the coupled-channel calculations, the total isospin states $I = 3/2$ are performed coupling $I_{K\pi} = 1/2$ or $I_{K\pi} = 3/2$ states. We also consider the $I_T = 5/2$ with its single contribution in the $K\pi$ interaction for the isospin $3/2$ states.

For the single channel case we consider only $K\pi$ interaction in the resonant isospin $1/2$ states and the perturbative solution of the equation up to three-loops reduces to

$$
\begin{aligned}
\xi^{3/2}_{3/2,1/2}(y, k_\perp) = {} & \frac{1}{6}\sqrt{\frac{2}{3}}\xi_0(y, k_\perp) - \frac{i}{3}\left(\frac{1}{6}\sqrt{\frac{2}{3}}\right)\int_0^\infty \frac{dq_\perp}{(2\pi)^3}\int_0^{1-y} dx\, K_{1/2}(y, k_\perp; x, q_\perp)\,\xi_0(x, q_\perp) \\
& -\frac{1}{9}\left(\frac{1}{6}\sqrt{\frac{2}{3}}\right)\int_0^\infty \frac{dq_\perp}{(2\pi)^3} \\
& \times \int_0^{1-y} dx\, K_{1/2}(y, k_\perp; x, q_\perp)\int_0^\infty \frac{dq'_\perp}{(2\pi)^3}\int_0^{1-x} dx'\, K_{1/2}(x, q_\perp; x', q'_\perp)\,\xi_0(x', q'_\perp) \\
& + \cdots
\end{aligned}
\tag{24}
$$

where we compute driving term considering $\alpha^{3/2}_{3/2,1/2} = 1$ and the kernel $K_{1/2}$ is defined by Eq. (21). This equation has an arbitrary normalization factor, coming from the source partonic amplitude, which we assume to be constant.

The numerical integration over the radial variable is computed introducing a momentum cut-off $\Lambda = 0.8$ GeV. This is smaller than in the D decay case, but in that case the change of the cutt-off parameter from 2.0 to 0.8 GeV practically does not alter the results. In the B case, the use of $\Lambda = 2.0$ GeV is very expensive numerically, and probably this is related to the large non-physical region accessed.

The finite value of the momentum cut-off represents roughly the two-meson interaction range, that is somewhat related to the size of the mesons itself. If the $K\pi$ model interaction had a finite range, naturally a cut-off in the hadronic loop would appear and, in our case, it is brought by the momentum cut-off.

Concerning the ε parameter, the value used here was $\varepsilon = 0.5$ GeV^2, which is larger than the one used in the D decay case. In fact, since the B phase space is very large, we know that the absorption is higher comparing with the D decay. Here we mimic this effect by using a larger value for the ε parameter. We have tested different values of ε (close to $\varepsilon = 0.5$ GeV^2), obtaining a small difference in the results. The subtraction constant in the driving term is chosen to be zero.

Regarding the convergence of the loop expansion, we have studied it up to three-loops. The results concerning phase and modulus of the bachelor function is depicted in Fig. 1.
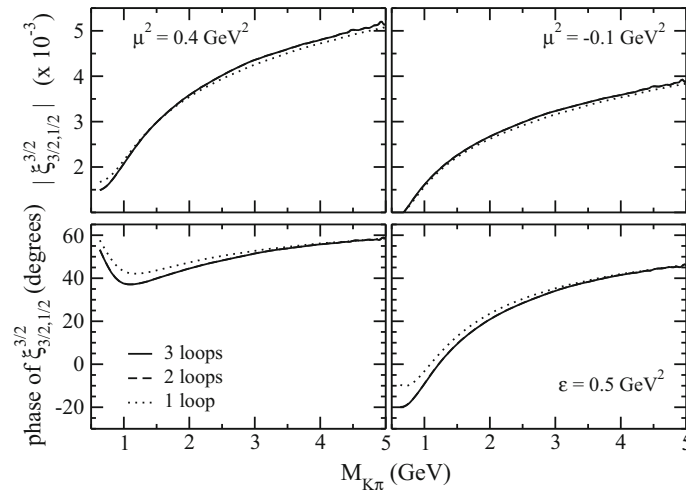
Fig. 1 Modulus and phase of $\xi^{3/2}_{3/2,1/2}$ for $\varepsilon = 0.5$ GeV^2, $\mu^2 = 0.4$ GeV^2 (*left*) and $\mu^2 = -0.1$ GeV^2 (*right*)

It was used $\mu^2 = (0.4, -0.1)$ GeV^2, in order to verify the effect of the subtraction point in the calculations. We have also used $\varepsilon = 0.5$ GeV^2. For a fixed value of μ^2 it is clear that the two-loop solution already present convergence and is enough for practical applications. This finding is similar to that observed in the D decay case, but now the results are even better concerning the convergence. The phase is always positive for $\mu^2 = 0.4$ GeV^2, but can be either positive or negative for $\mu^2 = -0.1$ GeV^2. The phase variation is large for $\mu^2 = -0.1$ GeV^2 and presents a minimum increasing again for $\mu^2 = 0.4$ GeV^2. In both cases the modulus increases for larger two-body invariant masses.

5.1 Interaction for Coupled Channels in $I_{K\pi} = 1/2$ and 3/2 States

In the coupled channels case the set of integral equations obtained from Eq. (19) reads

$$
\xi_{3/2,1/2}^{3/2}(y, k_\perp) = A_w \, \xi_0(y, k_\perp)
$$
$$
+ \frac{i \, R_{3/2,1/2,1/2}^{3/2}}{2} \int_0^{1-y} \frac{dx}{x(1-y-x)} \int_0^\infty \frac{dq_\perp}{(2\pi)^3} K_{1/2}(y, k_\perp; x, q_\perp) \, \xi_{3/2,1/2}^{3/2}(x, q_\perp)
$$
$$
+ \frac{i \, R_{3/2,1/2,3/2}^{3/2}}{2(2\pi)^3} \int_0^{1-y} \frac{dx}{x(1-y-x)} \int_0^\infty \frac{dq_\perp}{(2\pi)^3} K_{3/2}(y, k_\perp; x, q_\perp) \, \xi_{3/2,3/2}^{3/2}(x, q_\perp), \qquad (25)
$$
$$
\xi_{3/2,3/2}^{3/2}(y, k_\perp) = B_w \, \xi_0(y, k_\perp)
$$
$$
+ \frac{i \, R_{3/2,3/2,1/2}^{3/2}}{2} \int_0^{1-y} \frac{dx}{x(1-y-x)} \int_0^\infty \frac{dq_\perp}{(2\pi)^3} K_{1/2}(y, k_\perp; x, q_\perp) \, \xi_{3/2,1/2}^{3/2}(x, q_\perp)
$$
$$
+ \frac{i \, R_{3/2,3/2,3/2}^{3/2}}{2} \int_0^{1-y} \frac{dx}{x(1-y-x)} \int_0^\infty \frac{dq_\perp}{(2\pi)^3} K_{3/2}(y, k_\perp; x, q_\perp) \, \xi_{3/2,3/2}^{3/2}(x, q_\perp). \qquad (26)
$$

and for $I_T = 5/2$:

$$
\xi_{5/2,3/2}^{3/2}(y, k_\perp) = C_w \, \xi_0(y, k_\perp)
$$
$$
+ \frac{i \, R_{5/2,3/2,3/2}^{3/2}}{2} \int_0^{1-y} \frac{dx}{x(1-y-x)} \int_0^\infty \frac{dq_\perp}{(2\pi)^3} K_{3/2}(y, k_\perp; x, q_\perp) \, \xi_{5/2,3/2}^{3/2}(x, q_\perp), \qquad (27)
$$

where the isospin states related to the projection of the partonic amplitude (18) brings the weights A_w, B_w and C_w, given by $A_w = \langle I_T = 3/2, I_{K\pi} = 1/2, I_T^z = 3/2| B_0 \rangle$, $B_w = \langle 3/2, 3/2, 3/2| B_0 \rangle$ and $C_w = \langle 5/2, 3/2, 3/2| B_0 \rangle$ where the isospin coefficients are $A_w = \alpha_{3/2,1/2}^{3/2}(1 + R_{3/2,1/2,1/2}^{3/2}) + \alpha_{3/2,3/2}^{3/2} R_{3/2,1/2,3/2}^{3/2}$, $B_w = \alpha_{3/2,3/2}^{3/2}(1 + R_{3/2,3/2,3/2}^{3/2}) + \alpha_{3/2,1/2}^{3/2} R_{3/2,3/2,1/2}^{3/2}$ and $C_w = \alpha_{5/2,3/2}^{3/2}(1 + R_{5/2,3/2,3/2}^{3/2})$. The coefficients α come from the partonic decay amplitude (18) projected onto the isospin space and are defined as $\alpha_{3/2,1/2}^{3/2} = \frac{W_1}{2} C_{1/2\,1\,3/2}^{1/2\,1\,3/2} C_{1\,-1/2\,1/2}^{1\,1/2\,1/2}$, $\alpha_{3/2,3/2}^{3/2} = \frac{W_2}{2} C_{1/2\,1\,3/2}^{3/2\,1\,3/2} C_{1\,-1/2\,1/2}^{1\,1/2\,3/2}$ and $\alpha_{5/2,3/2}^{3/2} = \frac{W_3}{2} C_{1/2\,1\,3/2}^{3/2\,1\,5/2} C_{1\,-1/2\,1/2}^{1\,1/2\,3/2}$ and the Clebsch-Gordan and recoupling coefficients $C_{1/2\,1\,3/2}^{1/2\,1\,3/2} = 1$, $C_{1\,-1/2\,1/2}^{1\,1/2\,1/2} = \sqrt{2/3}$, $C_{1/2\,1\,3/2}^{3/2\,1\,3/2} = -\sqrt{2/5}$, $C_{1\,-1/2\,1/2}^{1\,1/2\,3/2} = 1/\sqrt{3}$, $C_{1/2\,1\,3/2}^{3/2\,1\,5/2} = \sqrt{3/5}$, $R_{3/2,1/2,1/2}^{3/2} = -2/3$, $R_{3/2,1/2,3/2}^{3/2} = \sqrt{5}/3$, $R_{3/2,3/2,3/2}^{3/2} = 2/3$, $R_{3/2,3/2,1/2}^{3/2} = \sqrt{5}/3$, and $R_{5/2,3/2,3/2}^{3/2} = 1$. With all these manipulations the weights A_w, B_w, and C_w reads $A_w = \sqrt{\frac{1}{54}}(W_1 - W_2)$, $B_w = \sqrt{\frac{5}{54}}(W_1 - W_2)$ and $C_w = \frac{W_3}{\sqrt{5}}$.

Also in this coupled channels case, the bachelor amplitude is computed to check the convergence. The coupled equations of Eq. (25) appear in the case $I_T = 3/2$. For $I_T = 5/2$, it is a single channel equation Eq. (27). The results are shown in Fig. 2, using $\varepsilon = 0.5$ GeV^2 and $\mu^2 = -0.1$ GeV^2, with the parameters from the expansion of the source term given by $W_1 = 1$, $W_2 = 2$ and $W_3 = 0.2$.

Again, the convergence is clear and the two-loop result is already enough for practical applications. Both, phase and modulus of the bachelor amplitudes increases with $M_{K\pi}$ and changes considerably along the large phase space available. In the channel $I_T = 3/2$, both components have similar magnitudes for the phase and are larger than that from the $I_T = 5/2$ case, same pattern observed in the D decay.

6 Results for the Phase and Amplitude in the $B^+ \to K^-\pi^+\pi^+$ Decay

Since the two-loop result presents already a good convergence for the bachelor amplitudes, we restrict our calculations hereafter to decay amplitude up to two-loops in Eq. (24). For the moment, there is no experimental
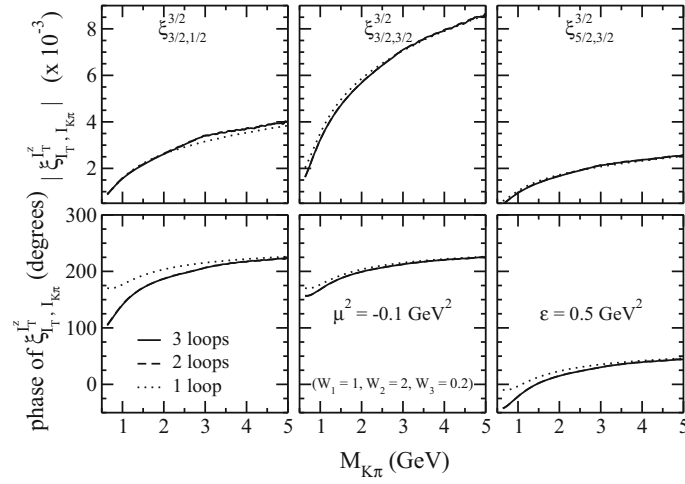
Fig. 2 Modulus and phase of $\xi^{I_T^z}_{I_T, I_{K\pi}}$ for $\varepsilon = 0.5\,\mathrm{GeV}^2$ and $\mu^2 = -0.1\,\mathrm{GeV}^2$ and the parameters $W_1 = 1$, $W_2 = 2$ and $W_3 = 0.2$

data available to perform a comparative analysis as done for the D meson decay in [15]. For the single channel calculations we consider only the S-wave $K\pi$ scattering amplitude in the isospin $1/2$ state, which is fitted to the LASS data [20]. The reduced form of the decay amplitude, that will give us both phase and modulus by means of Eq. (23), reads

$$A_0(M_{K\pi}^2) = \sqrt{\frac{2}{3}} \left[\frac{1}{12}\sqrt{\frac{2}{3}} + \tau_{1/2}\left(M_{K\pi}^2\right) \xi^{3/2}_{3/2,1/2}\left(k_{\pi'}\right) \right]. \tag{28}$$

The iteration of the coupled equations (25)–(26) gives the results for the channel $I_T = 3/2$. For the the $I_T = 5/2$ state, the amplitude is given by the single expression in Eq. (27). We also consider for these calculations the results up to two loops, since the convergence is verified. The S-wave decay amplitude is

$$A_0\left(M_{K\pi}^2\right) = C_1 \left[\frac{A_w}{2} + \tau_{1/2}\left(M_{K\pi}^2\right) \xi^{3/2}_{3/2,1/2}\left(k_{\pi'}\right) \right] + C_2 \left[\frac{B_w}{2} + \tau_{3/2}\left(M_{K\pi}^2\right) \xi^{3/2}_{3/2,3/2}\left(k_{\pi'}\right) \right] +$$
$$+ C_3 \left[\frac{C_w}{2} + \tau_{3/2}\left(M_{K\pi}^2\right) \xi^{3/2}_{5/2,3/2}\left(k_{\pi'}\right) \right] \tag{29}$$

where the constants C_i come from the isospin projection onto the state $K\pi\pi$, Eq. (23). There are two free parameters related with the projected partonic amplitude, namely, $W_1 - W_2$ and W_3. If the first is zero and the second nonzero, only total isospin $5/2$ appears and there is no structure in the decay amplitude, as shown in Ref. [15]. This shows that it is not a good physical solution, since the isospin state contributions are not being taken into account in a reasonable way. A more detailed study of the correct weights using the LF model would be guided by experimental data, as done for the $D^+ \to K^-\pi^+\pi^+$ decay in [15]. Here we just follow that study, where the authors found a small mixture of the total isospin $5/2$ state.

In Fig. 3, we observe that the amplitude approaches a constant as $M_{K\pi}$ increases. This behavior appears because the two-body amplitude of Eq. (2) damps fast at large $M_{K\pi}$, and only the constant partonic amplitude [first term in Eq. (3)] remains. This behavior of the two-body amplitude in the dominant $I = 1/2$ channel is traced back to the inclusion of resonances below 2 GeV, as suggested by the known resonances given in PDG. Thus, by considering the current available experimental information from LASS and $D^+ \to K^-\pi^+\pi^+$ decay, we opted to be conservative, however, this may be not realistic but only further experiments can decide. Still regarding Fig. 3, we show a comparison between modulus and phase of decay amplitudes for the B^+ and D^+ mesons, both decaying to the same final state $K^-\pi^+\pi^+$. The subtraction scale is fixed in $\mu^2 = -0.1\,\mathrm{GeV}^2$, the ε parameter was chosen to be $\varepsilon = 0.5\,\mathrm{GeV}^2$, and $W_1 - W_2 = -1$ and $W_3 = 0.2$ were used. All these parameters are kept the same for both cases. In order to test the effect of the constants $W_1 - W_2$ and W_3, we have tried a second set of parameters, namely, $W_1 - W_2 = 1$ and $W_3 = 0.3$, which was used in Ref. [15] to study the experimental data for the $D^+ \to K^-\pi^+\pi^+$ decay amplitude, but the results are very similar and with only a change of sign in the phase.
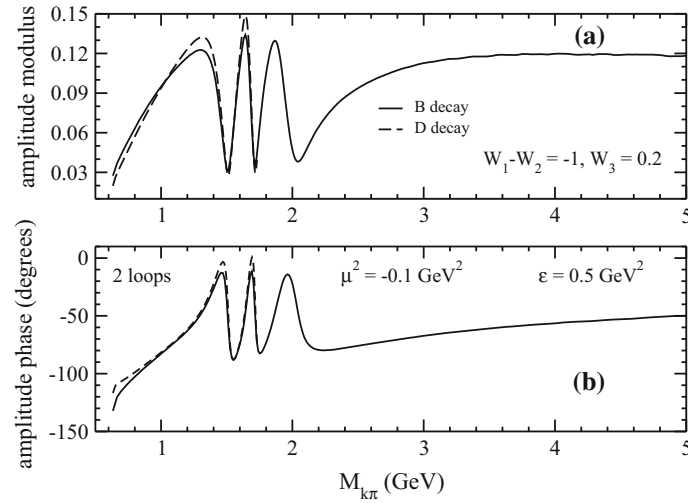
Fig. 3 Comparison of **a** modulus and **b** phase between $D^+ \to K^-\pi^+\pi^+$ and $B^+ \to K^-\pi^+\pi^+$ amplitudes for a initial state in which $W_1 - W_2 = -1$ and $W_3 = 0.2$
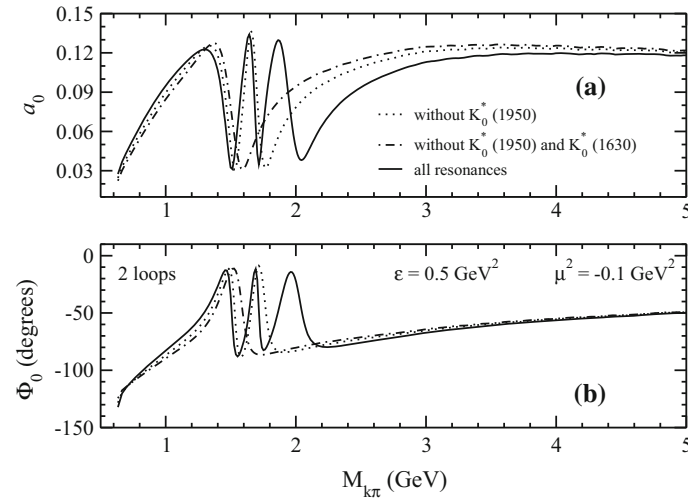
Fig. 4 Modulus (a_0) and phase (Φ_0) of the $B^+ \to K^-\pi^+\pi^+$ amplitude, in the notation of Eq. (23), comparing cases with all resonances, without $K_0^*(1950)$ and without both $K_0^*(1950)$ and $K_0^*(1630)$

In Fig. 4 we compare both modulus and phase of the $B^+ \to K^-\pi^+\pi^+$ decay amplitude with and without the resonant structure, which incorporates $K_0^*(1630)$ and $K_0^*(1950)$.

For this study, we also fix the subtraction point at $\mu^2 = -0.1$ GeV^2 and the other parameters with the same values as before. The figure shows that the inclusion of the resonances produces more bands in both modulus and phase. This is clearly related with the resonances, since the peaks are around its masses and bellow $K_0^*(1430)$ the effect is small. All the cases have the same tail when the two-body invariant mass increases. In amplitude analysis of the three-body B decay to the $K\pi\pi$ channel the $K_0^*(1630)$ resonance is usually included explicitly in the fit, insofar the $K_0^*(1950)$ is more complicated to claim that exists in the channel, therefore it appears indirectly in experimental analysis.

7 Summary and Conclusions

In this exploratory and simplified work, we have used a light-front framework to compute off-shell decay amplitudes starting from the four-dimension Bethe–Salpeter equation decomposed in the Faddeev form. The contribution of final state interactions to the $B^+ \to K^-\pi^+\pi^+$ decay is obtained. This approach can be applied

for charged three-body heavy meson decays, and was used before for the D meson decay, and the calculations were compared to the experimental data expressed in terms of the modulus and phase-shift [15]. Here, we have used the same three-body rescattering model in the final state for the $B \to K\pi\pi$ decay, considering the S-wave $K\pi$ interactions in the resonant 1/2 state, the $K_0^*(1430)$, $K_0^*(1630)$ and $K_0^*(1950)$ resonances and the non-resonant 3/2 isospin states. The scattering matrix was parametrized and fixed with the requirement of fitting the LASS data [20], as done in the D decay case [15].

In the light-front, the inhomogeneous integral equations reduce to three-dimensional ones, solved here with a perturbative series up to three-loops and with the accuracy of the solution checked. The convergence of the series is clear and the two-loop results shows up enough for practical applications, as happened in the D meson decay case of Ref. [15]. In comparison with the decay of the lighter D meson, we needed to use a larger imaginary part for the propagators of the mesons by increasing the ε parameter. Since this parameter mimics the absorption to other decay channels, it is expected that in the B decay, ε increases due to the much larger phase space available. The momentum cut-off was chosen smaller in the B case than in the D decay, in order to have a good convergence. Such a decrease seems reasonable as B is much more massive than D. The heavier particle should have a larger number of decay channels, meaning larger absorption, and therefore the wave function of the particular decay channel at short-distances, where the absorption takes place, is suppressed. The result is that the outgoing state is more concentrated at large distances, which corresponds to the low-momentum region. The smaller cut-off in the B decay with respect to the D one, can be understood as an effective way to parametrize the physics of the larger number of open channels.

The resonant structure above the $K_0^*(1430)$ resonance is also a question that deserves a detailed analysis in face of future experimental data. While the presence of the $K_0^*(1630)$ resonance is expected, and this is in fact used in our amplitude analysis, the $K_0^*(1950)$ influence must be better understood. Other aspect that requires more study are the real weights of the three isospin components of the source amplitudes at the quark level. Three-body rescattering effects are also important because they distribute *CP* violation to different decay channels, since it is one of the mechanisms allowed by the *CPT* constraint [24]. In the near future, this light-front approach will be generalized in order to study *CP* violation in three-body charmless B decays, taking into account the unitarity of the S-matrix, and the *CPT* constraint, exactly as done in Refs. [6,7].

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