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A New Feature of the Screened Coulomb Potential in Momentum Space

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Abstract A Coulomb equivalent screened Coulomb potential is proposed for solving the Schrödinger equation and/or the Calogero first order differential equation, where some critical range bands are obtained. Phase shifts for "any" two-charged particle system (from electron–electron to heavy ion–heavy ion) are reproduced by using the universal critical range bands and the appropriate Sommerfeld parameter over a very wide energy region. A Coulomb-like off-shell amplitude is introduced using two-potential theory without employing the usual Coulomb renormalization method.

1 Introduction

Obtaining an accurate method for treating charged particle scattering is one of the more important problems in nuclear reactions. Almost all approaches in the past half century use a screened Coulomb potentials. However, precise three-body calculations require a more accurate solution for the Coulomb problem. In the three-body configuration (r-) space calculation, the three-body wave function [$\Psi(\mathbf{x}, \mathbf{y})$ with the Jacobi coordinates $\mathbf{x}$ and $\mathbf{y}$] is approximately expanded in a separable manner $\sum_i \psi_x^i(\mathbf{x})\psi_y^i(\mathbf{y})$. In the p-space calculation, the three-body equations, with the nuclear potential and the screened Coulomb potential, are solved assuming convergence is achieved by increasing the range. However, we still have a concern as to whether those converged values exist or not, or whether the method can be used to predict a physical properties.

Usually, in order to obtain the LS equation from the Schrödinger equation, the boundary condition:

$$\lim_{r\to\infty} rV(r) \to 0 \tag{1}$$

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must be satisfied. The nuclear potential satisfies this condition, but the Coulomb potential does not because the Coulomb potential is

$$V^C(r) = 2k\eta(k)/r \tag{2}$$

so that

$$\lim_{r\to\infty} rV^C(r) \to 2k\eta(k) \neq 0, \tag{3}$$

where $\eta(k) = \nu ZZ'e^2/k$ is the Sommerfeld parameter, with the reduced mass $\nu = m_1m_2/(m_1+m_2)$ and charges Ze and $Z'e$ in units of $\hbar = c = 1$, and $k = \sqrt{2\nu E}$ denotes the on-(energy-)shell momentum.

On the other hand, for the p-space approach, the "usual Fourier transform" of the Coulomb potential is performed by using a screened Coulomb potential such as a Yukawa-type potential with a dimension-less "universal range": $\mathcal{R} = kR$,

$$V^{\mathcal{R}}(r;k) = \frac{2k\eta(k)}{r}e^{-kr/\mathcal{R}}. \tag{4}$$

The Fourier transform of Eq. (4) is given by

$$V^{\mathcal{R}}(\mathbf{p},\mathbf{p}';k) = \frac{8\pi k\eta(k)}{(\mathbf{p}-\mathbf{p}')^2 + (k/\mathcal{R})^2}. \tag{5}$$

Because the Coulomb potential in r-space is obtained in the limit $\mathcal{R}\to\infty$ in Eq. (4), we expect that the Coulomb potential in momentum space is obtained by taking $\mathcal{R}\to\infty$ in Eq. (5),

$$V^C(\mathbf{p},\mathbf{p}';k) = \lim_{\mathcal{R}\to\infty} V^{\mathcal{R}}(\mathbf{p},\mathbf{p}';k) = \frac{8\pi k\eta(k)}{(\mathbf{p}-\mathbf{p}')^2}, \tag{6}$$

where $V^{\mathcal{R}}(\mathbf{p},\mathbf{p}';k)$ and $V^C(\mathbf{p},\mathbf{p}';k)$ are the screened Coulomb and the Coulomb potentials in momentum space, respectively.

The screened Coulomb potential $V^{\mathcal{R}}(\mathbf{p},\mathbf{p}';k)$ satisfies the Lippmann–Schwinger (LS) equation condition as required by Eq. (1),

$$t^{\mathcal{R}}(\mathbf{p},\mathbf{p}';E) = V^{\mathcal{R}}(\mathbf{p},\mathbf{p}';k) + \int V^{\mathcal{R}}(\mathbf{p},\mathbf{p}'';k)G_0(\mathbf{p}'';E)t^{\mathcal{R}}(\mathbf{p}'',\mathbf{p}';E)\frac{d\mathbf{p}''}{(2\pi)^3}. \tag{7}$$

However, by increasing the range $\mathcal{R}$ (having $1/k \ll R$), the kernel suffers an overlapping singularity between the Green's function and the potential of Eq. (6). In other words, Eq. (6) has a serious divergence at $p = p'' = k \equiv \sqrt{2\nu E}$ in the kernel. We conclude that such a long range limit for the screened Coulomb potential does not converge Eq. (7) to the LS equation for the Coulomb problem or to the Coulomb solution. A condition similar to Eq. (1) in p-space is given by

$$\lim_{p'\to p} V(\mathbf{p},\mathbf{p}') < \infty. \tag{8}$$

It should be noted that both of a long range and the short range potentials can not be calculated in the *same bases* in the LS equation.

Recently, we introduced a new method for solving the proton-deuteron three-body problem with a rigorous treatment of the Coulomb interaction [1]. To accomplish this the most important breakthrough was demonstrated in the two-charged particle problem for the proton–proton system (using the LS equation) [2–4].

In order to verify the condition Eq. (8), we introduced a modified Coulomb potential V^{MCP} in [3,4] which is defined by using the Eq. (6)-type Coulomb potential and a screened Coulomb potential $V^{\mathcal{R}}(\mathbf{p},\mathbf{p}';k)$ with a finite universal range $\mathcal{R}$,

$$V^{MCP}(\mathbf{p},\mathbf{p}';k) = V^C(\mathbf{p},\mathbf{p}';k)\overline{\delta}_{p,p'} + V^{\mathcal{R}}(\mathbf{p},\mathbf{p}';k)\delta_{p,p'}, \tag{9}$$

where we define the delta function $\delta_{p,p'} = 1$ for $p = p'$, and 0 for $p \neq p'$, and $\overline{\delta}_{p,p'} = (1-\delta_{p,p'}) = 1$ for $p \neq p'$ and 0 for $p = p'$, respectively. The new potential has no diagonal divergence in p-space; therefore V^{MCP}

satisfies the condition in Eq. (8). Furthermore, it was proved that the Fourier transform of V^{MCP} becomes the pure Coulomb potential in r-space [2], which means that the V^{MCP} potential is equivalent to the pure Coulomb potential when it serves as an integrand in momentum space. Therefore, V^{MCP} is an alternative Coulomb potential in the integral sense. The kernel of the LS equation for V^{MCP} is free from the overlapping singularity in principle, where the screening $range$ of V^{MCP} in Eq. (9) defined by a critical range $R = e^{a\gamma}/2k$ for energy larger than 1 keV in the proton-proton case [1–4] and a is defined in "Appendix A".

In order to solve the problem, we required an auxiliary potential: $V^{\phi} \equiv V^{C} - V^{\mathcal{R}}$ and a *Lemma* in ref. [5,6]. Using the *Lemma* is the only way to obtain a "critical range (or decisive range)" [2–4], and the LS equation for the auxiliary potential was numerically solved by two methods, the first one adopted "special Gaussian points" with a critical range [7], and the second one solved by a contour deformation method with the critical range [2,8].

The above mentioned critical range should be generalized to the wider energy region: $0 < E < \infty$, especially for the very low energy region of less than 1 keV for the proton-proton case. In this paper for convenience, we obtain a critical range which represents the auxiliary phase shift becoming zero (or the *Lemma* in terms of the phase shift): $\phi(k) = \sigma_l(k) - \delta_l^{\mathcal{R}}(k) \to 0$ to high accuracy using the r-space calculation.

On the other hand, we can choose a screening range for the Coulomb potential which can directly determine the Coulomb phase shift by solving a differential equation for the phase shift [9–11] or also by solving the Schrödinger equation. We obtain a very high precision with more than six digits. In momentum space, the LS equation gives the same accurate phase shift.

In our past several papers, we demonstrated that two potentials: $V^{C} = V^{\mathcal{R}} + V^{\phi}$ can be summarized by selecting V^{ϕ} to obtain $T^{C} = T^{\mathcal{R}\phi} + T^{\phi}$ where T^{ϕ} is the LS amplitude, and $T^{\mathcal{R}\phi}$ is a modified amplitude with respect to the $V^{\mathcal{R}}$ by T^{ϕ}. Let us call it the *Lemma* method where we require a *Lemma*: $T^{\phi}(k,k) = T^{\phi}(k,p') = T^{\phi}(p,k) = 0$ for the on- and half on-shell amplitudes which is proved in our papers. In this paper, we present another method which is given in "Appendix B" by using the GSE method [12–17]. For this purpose, we have to introduce an innocuous range parameter $\mathcal{R}'$, which is very close to the critical range $\mathcal{R}' \approx \mathcal{R}$. The details are shown in "Appendix B".

As a summary, we assert that the Coulomb problem in momentum space is solved by using the Coulomb phase shift "renormalization" method which was introduced firstly by W. Tobocman et al. in the 1950s [18], by A. M. Veselova in early 1970s [19–24] and by E. O. Alt et al in late 1970s [25–28]. Some improvements have been proposed by A. Deltuva, et al with a benchmark test [29–33], and also by ourselves[34]. However, our method is not the same as the renormalization method, but requires "zero renormalization" to obtain a critical range (to solve the LS equation) which is presented in the references [1–5,7,8].

Unfortunately, almost all approaches can not be solved analytically. Our former method [5,6] analytically reaches this goal only by assistance of the *Lemma*. In order to obtain good accuracy, the renormalization method requires a numerical calculation with many digit precision to avoid a loss of trailing digits (or a cancellation of significant digits). Such a loss of information in the trailing term could propagate to the leading term through the renormalization, and finally affect the higher wave interactions in the three-body problem. We have emphasized, in three-body problems, that the reproduction of the two-body Coulomb phase shifts is important, because the Coulomb higher waves do not converge and interfere with the non-Coulomb higher waves. Therefore, we have to take care regarding the propagation of errors in the "renormalization" method in the three-body problem.

In this context, we found a counterpart of the two-potential formalism by selecting $V^{\mathcal{R}}$ instead of V^{ϕ} to make $T^{C} = T^{\phi\mathcal{R}} + T^{\mathcal{R}}$ where $T^{\mathcal{R}}$ is the LS amplitude for $V^{\mathcal{R}}$, and the remainder term is an amplitude for V^{ϕ} modified by the distortion of $T^{\mathcal{R}}$. Let us call this case a $direct$ method.

In this paper, we will investigate the "direct" method in the Sect. 2. The numerical calculation for the LS equation is done by using the GSE method [12–17] in momentum space: the Schrödinger equation and the phase shift differential equation are solved in r-space. The calculated results for the universal critical range bands are shown together with the Coulomb phase shifts for scattering from "electron–electron" to "heavy ion–heavy ions" in Sect. 3. Our conclusion and a discussion will be given in Sect. 4.

2 A New Direct Method to Obtain a Coulomb Equivalent Phase Shift

In this paper, we would like to present a new method which is practical and applicable to few-body problems and in general.

Let us start from Eq. (9), that is,

$$V_l^{MCP}(p,p') = V_l^{\mathcal{R}}(p,p') + \overline{V}_l^{\phi}(p,p') = \begin{cases} V_l^{\mathcal{R}}(p,p') & (p=p') \\ V_l^{C}(p,p') & (p\neq p'). \end{cases} \tag{10}$$

Contrary to two-potential theory for the *Lemma* method which is "selecting T_l^{ϕ}", and we adopt the two-potential theory for the *direct* method: "selecting $T_l^{\mathcal{R}}$", we obtain the off-shell Coulomb amplitude which is given in "Appendix C",

$$T_l^C(p,p';E) = \int_0^\infty\int_0^\infty \frac{p''^2 p'''^2 dp''dp'''}{(2\pi^2)^2}\overline{\omega}_l^{\mathcal{R}}(p,p'';E) \times t_l^{\phi\mathcal{R}}(p'',p''';E)\omega_l^{\mathcal{R}}(p''',p';E) + T_l^{\mathcal{R}}(p,p';E) \tag{11}$$

with the core amplitude,

$$t_l^{\phi\mathcal{R}}(p,p';E) = V_l^{\phi}(p,p') + \int_0^\infty\int_0^\infty \frac{p''^2 p'''^2 dp''dp'''}{(2\pi^2)^2} \times V_l^{\phi}(p,p'')G_l^{\mathcal{R}}(p'',p''';E)t_l^{\phi\mathcal{R}}(p''',p';E). \tag{12}$$

The Møller functions for the screened Coulomb potential,

$$\overline{\omega}_l^{\mathcal{R}}(p,p'';E) = \frac{2\pi^2}{p^2}\delta(p-p'') + T_l^{\mathcal{R}}(p,p'';E)G_0(p'';E), \tag{13}$$

$$\omega_l^{\mathcal{R}}(p''',p';E) = \frac{2\pi^2}{p'^2}\delta(p'''-p') + G_0(p''';E)T_l^{\mathcal{R}}(p''',p';E), \tag{14}$$

the Green's function for the screened Coulomb potential,

$$G_l^{\mathcal{R}}(p'',p''';E) = \frac{2\pi^2}{p''^2}\delta(p''-p''')G_0(p'';E) + G_0(p'';E)T_l^{\mathcal{R}}(p'',p''';E)G_0(p''';E), \tag{15}$$

and the screened Coulomb amplitude,

$$T_l^{\mathcal{R}}(p,p';E) = V_l^{\mathcal{R}}(p,p') + \frac{1}{2\pi^2}\int_0^\infty p''^2 dp'' \times V_l^{\mathcal{R}}(p,p'')G_0(p'';E)T_l^{\mathcal{R}}(p'',p';E). \tag{16}$$

$T_l^{\mathcal{R}}$ is the dominant term in Eq. (11), and the first term of the r.h.s. in Eq. (11) is a secondary term with a small auxiliary potential, although Eq. (12) may produce a numerical difficulty in the kernel $V^{\phi}(p,p'')G_0(p'';E) \equiv \overline{V}^{\phi}(p,p'')G_0(p'';E)$. However, the kernel is regular at $p=p'=k$ by Eq. (10). Since we can obtain the on-shell $T_l^{\mathcal{R}}$ precisely by using the universal range $\mathcal{R}$ in Eq. (50), then the amplitude $t_l^{\phi\mathcal{R}}$ could be minimized. The universal range is obtained through the r space calculation with the Schrödinger equation in "Appendix A". Therefore $T_l^{\mathcal{R}}$ can represent the on- and half-off Coulomb amplitude; however, the off-shell part of T_l^C could be compensated by the first term of the r.h.s. in Eq. (11).

Therefore, we confirm that if the secondary term $T_l^{\phi\mathcal{R}}$ is obtained, then the off-shell amplitude $T_l^C(p',p'';E)$ can be calculated together with $T_l^{\mathcal{R}}$. This result guarantees the existence of the off-shell Coulomb-like t-matrix by Eq. (11) [35], which was not defined in Ref. [36].

Table 1 The parameters a_m and b_m (with $m = 0, 1, 2, \ldots$) in Eq. (17) for five bands: $\mathcal{R} = \mathcal{R}_n(\eta)$ (with n=0, 1, 2,…) are shown as calculated by the direct method

$\mathcal{R}_n$	m	0	1	2	3	4
$\mathcal{R}_0$	a_m	−0.122527	67.0542	33.4284	−33.6583	
$0.00353 \le \eta \le 1.58$	b_m	−0.14194	90.8834	15.8711	12.7285	
$\mathcal{R}_1$	a_m	5.57504	9.02951	12.9823	−3.20748	
$1.58 \le \eta \le 4.22$	b_m	−0.731152	−0.925015	1.87304	0.68854	
$\mathcal{R}_2$	a_m	1.04150	2.62713	5.58478	−0.64538	
$3.95 \le \eta \le 5.60$	b_m	2.11619	3.39619	3.91962	−2.57886	0.397671
$\mathcal{R}_3$	a_m	3.26102	12.2893	43.3765	−5.55534	
$5.60 \le \eta \le 7.90$	b_m	0.419669	−0.514822	−1.45874	0.924641	0.00556185
$\mathcal{R}_4$	a_m	1.1429	1.8394	3.89062	−0.245209	−0.0191509
$7.07 \le \eta \le 9.13$	b_m	2.12168	3.59564	−1.4752	0.186987	

3 Numerical Result for Screening Ranges

3.1 Universal Critical Range Bands

To obtain the range $\mathcal{R}$ by the direct-method, the Schrödinger equation and/or the Calogero differential equation [9–11] can be used. We obtain a very accurate on-shell phase shift by Calogero's differential equation with a finite range potential; therefore, using the same range, the phase shift is reproduced by the LS equation with high accuracy. It seems clear that a proper screening range is obtained by increasing the range in the potential: $\lim_{R\to\infty} e^2 e^{-r/R}/r \to e^2/r$; in the two- and three-body problems, however, that idea is wrong. Because the long range singularity can not be covered with a closed neighborhood in the potential theory, the limiting procedure fails. It was confirmed that a critical screening range exists [2–4].

We found that five discrete screening range bands are necessary to reproduce the Coulomb phase shift for $\sigma_0 \le 4\pi$.

1. The first (lowest) band (black circle in Fig. 1) can represent the higher energy region better than the energy which satisfies $\sigma_0(k) = 0$, and the range should be $\mathcal{R} = \mathcal{R}_0(k) = 0$ fm at $\sigma_0(k) = 0$.
2. The second lower band (black nabla symbol) can be applied in the phase shift region: $0 \le \sigma_0(k) \le \pi$ and the range must be $\mathcal{R} = \mathcal{R}_1(k) = 0$ fm at $\sigma_0(k) = \pi$.
3. The next band (black square symbol) covers the phase shift region: $\pi \le \sigma_0(k) \le 2\pi$, and the range should be $\mathcal{R} = \mathcal{R}_2(k) = 0$ fm at $\sigma_0(k) = 2\pi$.
4. The next band (white circle symbol) applies for: $2\pi \le \sigma_0(k) \le 3\pi$, and the range $\mathcal{R} = \mathcal{R}_3(k) = 0$ fm at $\sigma_0(k) = 3\pi$.

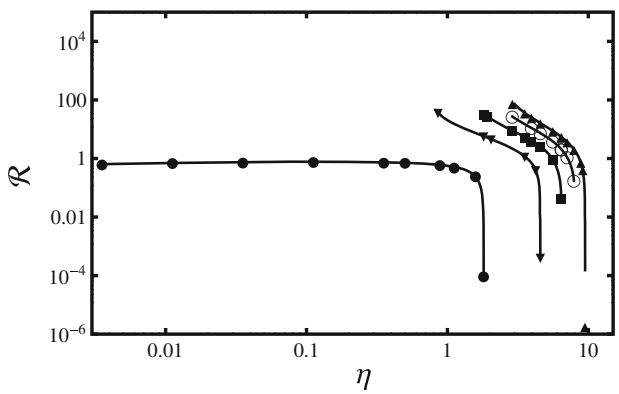

Fig. 1 The range bands by the *direct* fitting method. The universal ranges $\mathcal{R}(k)$ are denoted by five different symbols: *black circle*, *black nabla*, *black square*, *white circle*, and *black delta*. The universal ranges $\mathcal{R} = \mathcal{R}(k) = \mathcal{R}_n(k)$ ($n = 0, 1, 2, 3, 4$) are fitted as functions of $\eta = \eta(k)$. 1) $\mathcal{R}_0 = 0$ at $\sigma_0(k) = 0$, 2) $\mathcal{R}_1 = 0$ at $\sigma_1(k) = \pi$, 3) $\mathcal{R}_2 = 0$ at $\sigma_2(k) = 2\pi$, 4) $\mathcal{R}_3 = 0$ at $\sigma_3(k) = 3\pi$, 5) $\mathcal{R}_4 = 0$ at $\sigma_0(k) = 4\pi$. All bands become zero at $\sigma_0(k) = n\pi$ ($n = 0, 1, 2, \ldots$); however, one could replace them by the finite range of the upper band. Each band makes it possible to derive the Coulomb phase shifts for the corresponding energy region

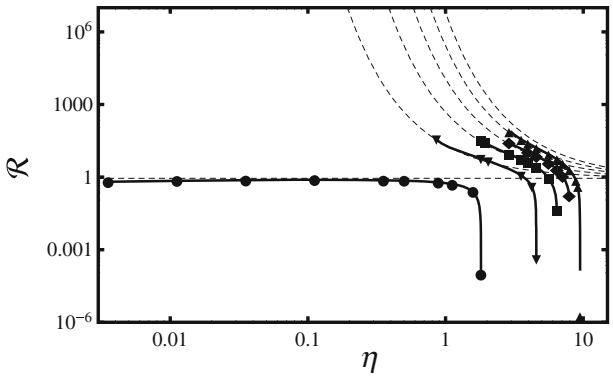

Fig. 2 The direct fitting range bands and extension by our model range. The legends are the same as in Fig. 1. *Dashed lines* are given by Eq. (50) in the case: $m = 1$ of the Yukawa type from the lowest line $n = 0$ to the highest line $n = 5$

5. The highest band (black delta symbol) applies for: $3\pi \leq \sigma_0(k) \leq 4\pi$ and the range $\mathcal{R} = \mathcal{R}_4(k) = 0$ fm at $\sigma_0(k) = 4\pi$.

We find that $\sigma_0(k) = n\pi$ $(n = 0, 1, 2, \ldots)$ leads to $\mathcal{R}_n(k) = 0$ which denotes $V^{\mathcal{R}}(r) = 0$. However, the zero range screened Coulomb property seems to be unusual; then we adopt the finite range of the upper band instead. These bands are fitted by the N/D form

$$\mathcal{R} = \frac{a_0 + a_1\eta + a_2\eta^2 + a_3\eta^3 + \cdots}{b_0 + b_1\eta + b_2\eta^2 + b_3\eta^3 + \cdots} \tag{17}$$

with the universal ranges $\mathcal{R} \equiv kR$, and the Sommerfeld parameter $\eta = \eta(k)$. Parameters $a_0, a_1, \ldots$ and $b_0, b_1, \ldots$ are shown in the Table 1. Therefore, the Coulomb phase shifts of the each system are obtained by taking the individual range $R = \mathcal{R}/k$ and incorporating the individual values of the Sommerfeld parameters.

Especially for the cases: $\sigma_l(k) = n\pi$ $(n = 0, 1, 2, \ldots)$, the range converges to a zero value. The given ranges provide the nine-digit results with the Calogero equation.

Obviously, the phase shifts for the higher partial waves require the individual range-bands for the higher waves, because the screened Coulomb potential plus the centrifugal term $l(l+1)/r^2$ are used to obtain the range. Therefore, Fig. 1 is available only for the S-wave. However, the longer range $\mathcal{R}$ which is given by Eq. (50) is the partial wave independent, because it is satisfied in the asymptotic region where the centrifugal potential is negligible.

Let us call the critical range bands which are defined by Eq. (50) "asymptotic" bands, while the former critical range bands which are obtained numerically, "numerical" bands. Therefore, the universal critical range bands are given by the smooth continuation between both range bands in Fig. 2. It is seen that universal critical ranges of the asymptotic bands are generally very large except for $n = 0$, therefore it is rather difficult to obtain the numerical solution by the LS equation because of cancellation of significant digits. On the other hand, universal critical ranges within numerical bands are smaller, therefore the LS equation is successfully solved.

3.2 Coulomb Phase Shifts

By using our universal critical range bands, we can easily obtain the Coulomb phase shifts by using the LS equation in momentum space. We found that phase shifts for "any" two-charged particle systems (from electron-electron to heavy ion-heavy ion) are automatically reproduced by using the universal critical range bands and the appropriate Sommerfeld parameter over a very wide energy region. Figures 3 and 4 are the Coulomb phase shifts for many different two-charged particle systems for S-wave. The higher partial waves are shown for reference in Fig. 5. The Coulomb-like off-shell amplitude could be introduced using the two-potential theory. The numerical results will be shown on another occasion.

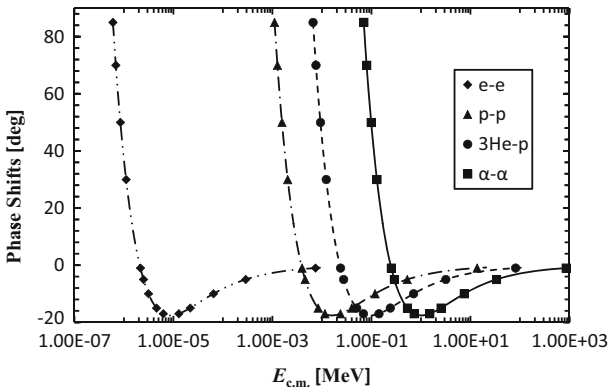

Fig. 3 The Coulomb phase shifts in the S-wave are illustrated for several systems from e^--e^- to $\alpha - \alpha$ by the *diamond*, *triangle*, *circle*, and *square symbols*, respectively. The results are compared with the analytic solutions by the *double-dashed*, *dotted-dashed*, *dashed*, and *solid lines*, respectively. These are calculated using the universal ranges which are given in Fig. 1. The present range is obtained for the Yukawa-type screened Coulomb potential. The results are illustrated only for $0 \leq \sigma_0 \leq \pi$; however, our results are fitted with the analytic solution in an energy region greater than that

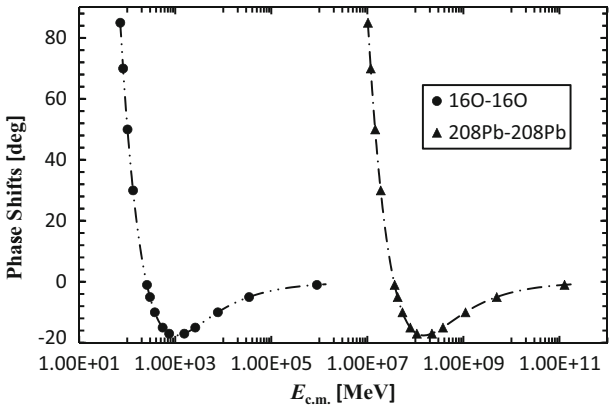

Fig. 4 The Coulomb phase shifts in the S-wave are illustrated for ^{16}O-^{16}O and ^{208}Pb-^{208}Pb. Our calculated results are shown by the *solid-circle* and the *triangle*. These are compared with the analytic results by the *double-dashed*, and the *dashed-dotted lines*, respectively. The other aspects are the same as in Fig. 3

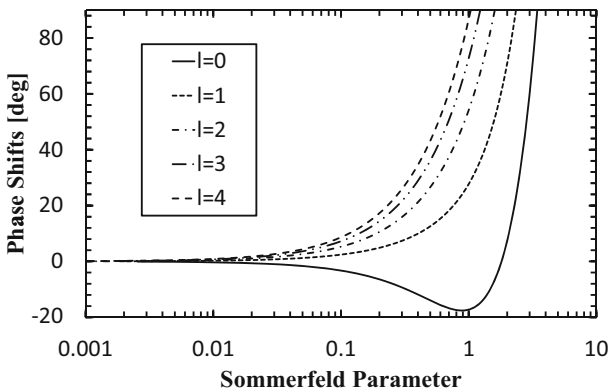

Fig. 5 Our screened Coulomb potential in Eq. (4) which satisfies the *Lemma* is applicable for the higher partial wave cases. The calculated results are fitted to the analytic partial wave phase shifts: $\sigma_l(k) = \arg \Gamma(l + 1 + i\eta(k))$

4 Conclusion and Discussion

In this paper, we derived a critical screening range which is used to calculate the Coulomb phase shift in the energy region: $0 < E < \infty$ with high accuracy without using the Coulomb phase shift renormalization

method but instead the direct method. The direct method gives not only the on- and half off-shell amplitudes very accurately, but also the off-shell part is analytically generated where the same asymptotic behavior as the Coulomb wave function is in effect.

In this paper, the quantized critical range in the V^{MCP}-potential is introduced with the aid of the r-space calculation for convenience in the relatively short range region, but in the longer range region with an analytic form shown in "Appendix A". The quantized critical range includes the long range Coulomb information as shown in "Appendix A". By using the critical range, we proposed a new direct method instead of the historical renormalization method.

Our Coulomb treatment by the dimensionless universal range and appropriate Sommerfeld parameter is general for any two-charged particle system from "electron"–"electron" to the "heavy ion"–"heavy ion". Because the Sommerfeld parameter, as a universal variable is determined by the charges and the reduced mass, then our method may be applicable to systems with non-integer charges such as quarks.

Finally, we conclude that the Coulomb potential can be treated on the same basis of the usual scattering theory for the short range potential by the aid of the critical range bands, where the pinching singularity, by the dispersion theoretical terminology, does not occur in our theory. We believe that the two-body Coulomb problems in momentum space can be solved not only theoretically but also numerically with high precision in this paper. A partial wave-independent asymptotic critical range calculation is ready for performing with very high precision. A practical three-body calculation is now in progress.

A An Auxiliary Phase Shift $\phi(k)$ and a Critical Range $\mathcal{R}$

The auxiliary phase shift $\phi(k)$ was introduced in [5,6]. In order to obtain the phase shift, let us start from the Schrödinder equation for the screened Coulomb potential and the pure Coulomb potential.

$$\left[\frac{d^2}{dr^2} - \frac{L(L+1)}{r^2} - V^C(r) + k^2\right] w_L^{(\pm)}(r) = 0 \tag{18}$$

$$\left[\frac{d^2}{dr^2} - \frac{L(L+1)}{r^2} - V^{\mathcal{R}}(r) + k^2\right] h_L^{(\pm)}(r) = 0 \tag{19}$$

Therefore in Eq. (18), using the auxiliary potential: $V^{\phi}(r) = \{V^C(r) - V^{\mathcal{R}}(r)\}$, we have

$$\left[\frac{d^2}{dr^2} - \frac{L(L+1)}{r^2} - V^{\mathcal{R}}(r) - V^{\phi}(r) + k^2\right] w_L^{(\pm)}(r) = 0 \tag{20}$$

The potentials are defined by using the two-body reduced mass and the charges with Ze and $Z'e$,

$$V^C(r) = 2\nu\frac{ZZ'e^2}{r} = 2k\frac{\eta(k)}{r} \tag{21}$$

$$V^{\mathcal{R}}(r) = 2\nu\frac{ZZ'e^2}{r}e^{-(r/R)^m} = 2k\frac{\eta(k)}{r}e^{-(r/R)^m} \tag{22}$$

with the Sommerfeld parameter $\eta(k) = ZZ'e^2\nu/k$. The ratio between the Coulomb wave function $w_L^{(\pm)}(r)$ and the screened Coulomb wave function $h_L^{(\pm)}(r)$ is given by a new function $y_L^{(\pm)}(r)$,

$$h_L^{(\pm)}(r) \equiv y_L^{(\pm)}(r) \cdot w_L^{(\pm)}(r). \tag{23}$$

Substituting Eq. (23) into Eq. (19), we have the following relation by using Eq. (18),

$$\frac{y_L''(r)}{y_L(r)} + 2\frac{w_L'(r)}{w_L(r)} \cdot \frac{y_L'(r)}{y_L(r)} = V^{\mathcal{R}}(r) - V^C(r) \equiv -V^{\phi}(r); \tag{24}$$

equivalently we have a differential equation for the auxiliary potential,

$$y_L''(r) + 2\frac{w_L'(r)}{w_L(r)} \cdot y_L'(r) + V^{\phi}(r)y_L(r) = 0. \tag{25}$$

The asymptotic wave functions based on these potentials are given by

$$w_L^{(\pm)}(r) \sim \exp\left[\pm i\left(kr - \frac{\pi L}{2} - \eta(k)\ln 2kr + \sigma_L(k)\right)\right] \tag{26}$$

$$h_L^{(\pm)}(r) \sim \exp\left[\pm i\left(kr - \frac{\pi L}{2} + \delta_L^R(k)\right)\right] \tag{27}$$

where $\sigma_L(k)$ and $\delta_L^R(k)$ are the Coulomb and the screened Coulomb phase shifts, respectively. It is obvious that the main difference in the asymptotic wave functions between the long range and the short range interactions appears in the r dependent phase: $\ln 2kr$ in the Coulomb wave function Eq. (26).

By using the asymptotic forms of Eqs. (26), (27), the first and the second terms of Eq. (24) are given by where the asymptotic form of the second term of Eq. (24) is given by

$$\frac{w_L'(r)}{w_L(r)} = \frac{d}{dr}[\ln w_L(r)] = \left(k - \frac{\eta(k)}{r}\right) i \approx ik \tag{28}$$

$$\frac{y_L'(r)}{y_L(r)} = \frac{d}{dr}[\ln y_L(r)] \equiv \frac{d\mathcal{Y}_L(r)}{dr} \tag{29}$$

with a new function $\mathcal{Y}_L(r)$,

$$\mathcal{Y}_L(r) \equiv \ln y_L(r), \tag{30}$$

and with the asymptotic phase

$$y_L^{(\pm)}(r) \sim \exp\left[\pm i\left(\eta(k)\ln 2kr - \sigma_L(k) + \delta_L^R(k)\right)\right]. \tag{31}$$

The first term of Eq. (24) is rewritten using Eq. (29) as

$$\frac{y_L''(r)}{y_L(r)} = \frac{d}{dr}\left[\frac{y_L'(r)}{y_L(r)}\right] + \left[\frac{y_L'(r)}{y_L(r)}\right]^2 = \frac{d^2\mathcal{Y}_L(r)}{dr^2} + \left[\frac{d\mathcal{Y}_L(r)}{dr}\right]^2 \tag{32}$$

where the terms are assumed to be very small or vanishing in the asymptotic region. Because, $y_L(r) \to 1$, or $\mathcal{Y}_L(r) \to 0$ is verified and are expected to be smooth function due to Eqs. (23) and (30). Consequently, Eq. (24) is rewritten, by using (28) and (29), as a simple differential equation in the asymptotic region,

$$2ik\mathcal{Y}_L'(r) = V^{\mathcal{R}}(r) - V^C(r) = -V^{\phi}(r). \tag{33}$$

Because the integral of Eq. (33) should be performed in the region between r and ∞, which is the Coulomb boundary, then the solution is given by using an arbitrary constant b,

$$\mathcal{Y}_L(r) = \int_b^r \frac{\{V^{\mathcal{R}}(r) - V^C(r)\}}{2ik}dr = i\eta(k)\int_b^r \frac{1 - e^{-(r/R)^m}}{r}dr. \tag{34}$$

Here, $\mathcal{Y}_L(r)$ could be separated by using the screening range R

$$\begin{aligned}
\mathcal{Y}_L(r) &= i\eta(k)\left[\int_b^0 \frac{1 - e^{-(r/R)^m}}{r}dr + \int_0^R \frac{1 - e^{-(r/R)^m}}{r}dr + \int_R^r \frac{1 - e^{-(r/R)^m}}{r}dr\right] \\
&= i\eta(k)\left[C + \left\{\int_0^R \frac{1 - e^{-(r/R))^m}}{r}dr + \int_R^r \frac{1}{r}dr - \int_R^r \frac{e^{-(r/R)^m}}{r}dr\right\}\right] \\
&= i\eta(k)\left[C + \left\{\int_0^R \frac{1 - e^{-(r/R)r)^m}}{r}dr - \int_R^\infty \frac{e^{-(r/R)^m}}{r}dr\right\}\right. \\
&\quad \left. + \int_R^r \frac{1}{r}dr + \int_r^\infty \frac{e^{-(r/R)^m}}{r}dr\right]
\end{aligned} \tag{35}$$

with

$$C = \int_b^0 \frac{1 - e^{-(r/R)^m}}{r} dr. \tag{36}$$

The first part {} in Eq. (35) is the Euler constant $\gamma = 0.57721\ldots$ which is defined by,

$$\gamma \equiv \int_0^1 \frac{1 - e^{-t}}{t} dr - \int_1^\infty \frac{e^{-t}}{t} dt. \tag{37}$$

That is, by putting $(r/R)^m = t$ and $dt = m(r/R)^{m-1} dr/R = mtdr/R$, and also $t = 1$ at $r = R$, then we obtain

$$\left\{ \int_0^R \frac{1 - e^{-(r/R)^m}}{r} dr - \int_R^\infty \frac{e^{-(r/R)^m}}{r} dr \right\}$$
$$= \int_0^1 \frac{1 - e^{-t}}{t} \frac{dt}{m} - \int_R^1 \frac{e^{-t}}{t} \frac{dt}{m} = \left\{ \frac{\gamma}{m} \right\}. \tag{38}$$

Therefore, Eq. (35) becomes

$$\mathcal{Y}_L(r) = i\eta(k) \left[C + \left\{ \frac{\gamma}{m} \right\} + (\ln 2kr - \ln 2kR) + \int_r^\infty \frac{e^{-(r/R)^m}}{r} dr \right]$$
$$= i\eta(k) \left[a\gamma + (\ln 2kr - \ln 2kR) + \int_r^\infty \frac{e^{-(r/R)^m}}{r} dr \right] \tag{39}$$

with $a = (C/\gamma + 1/m)$, and $\mathcal{Y}_L(r)$ is an r-dependent function.

The asymptotic phase in Eq. (31) can be compared with Eq. (39) by neglecting the last term of Eq. (39), and also by adopting $b = 0$ or $C = 0$. We obtain

$$\eta(k) \left[\left\{ \frac{\gamma}{m} \right\} + (\ln 2kr - \ln 2kR) \right] = \left(\eta(k) \ln 2kr - \sigma_L(k) + \delta_L^R(k) \right) \tag{40}$$

Therefore, we have

$$\eta(k) \left[\left\{ \frac{\gamma}{m} \right\} - \ln 2kR \right] = -\sigma_L(k) + \delta_L^R(k), \tag{41}$$

and

$$\sigma_L(k) = \delta_L^R(k) + \eta(k) \left(\ln 2kR - \left\{ \frac{\gamma}{m} \right\} \right) \tag{42}$$
$$\equiv \delta_L^R(k) + \phi(k, R). \tag{43}$$

If we can choose

$$\phi(k, R) \equiv \eta(k) \left(\ln 2kR - \left\{ \frac{\gamma}{m} \right\} \right) = 2n\pi \tag{44}$$

with $n = 0, 1, 2, 3, \ldots$, then we have

$$\ln 2kR = \frac{2n\pi}{\eta(k)} + \left\{ \frac{\gamma}{m} \right\}. \tag{45}$$

By substituting Eq. (45) into Eq. (39), $\mathcal{Y}_L(r)$ becomes

$$\begin{aligned}\mathcal{Y}_L(r) &= i\eta(k)\left[\left\{\frac{\gamma}{m}\right\} + (\ln 2kr - \ln 2kR)\right] \\ &= i\eta(k)\left[\left\{\frac{\gamma}{m}\right\} + \left(\ln 2kr - \frac{2n\pi}{\eta(k)} - \left\{\frac{\gamma}{m}\right\}\right)\right] \\ &= i\left[\eta(k)\ln 2kr - 2n\pi\right]. \end{aligned} \tag{46}$$

Therefore, Eq. (31) is given by

$$\begin{aligned} y_L^{(\pm)}(r) &\sim \exp\left[\pm i\left(\eta(k)\ln 2kr - \sigma_L(k) + \delta_L^R(k)\right)\right] \\ &= \exp\left[\pm i\left(\eta(k)\ln 2kr - 2n\pi\right)\right]. \end{aligned} \tag{47}$$

We can confirm that the long range property of the Coulomb wave function can be compensated by this phase of Eq. (47) in the screened Coulomb wave function. Hence, by using Eqs. (47) and (26), Eq. (23) becomes

$$\begin{aligned} h_L^{(\pm)}(r) &= y_L^{(\pm)}(r)\cdot w_L^{(\pm)}(r) \\ &= \exp\left[\pm i\left(\eta(k)\ln 2kr - 2n\pi\right)\right] \\ &\times \exp\left[\pm i\left(kr - \frac{\pi L}{2} - \eta(k)\ln 2kr + \sigma_L(k)\right)\right] \\ &= \exp\left[\pm i\left(kr - \frac{\pi L}{2} + \sigma_L(k) - 2n\pi\right)\right] \\ &= \exp\left[\pm i\left(kr - \frac{\pi L}{2} + \delta_L^R(k)\right)\right]. \end{aligned} \tag{48}$$

This result is equivalent to Eq. (27).

Finally, we can conclude that the energy dependent "critical range" parameter in the asymptotic region is given by using Eq. (45),

$$R = \frac{1}{2k}\exp\left[\frac{2n\pi}{\eta(k)} + \left\{\frac{\gamma}{m}\right\}\right], \tag{49}$$

and the universal range is

$$\mathcal{R} = \frac{1}{2}\exp\left[\frac{2n\pi}{\eta(k)} + \left\{\frac{\gamma}{m}\right\}\right]. \tag{50}$$

B Proof of the *Lemma* for the Modified Coulomb Potential

The *Lemma* for the modified Coulomb potential will be proved. In order to satisfy the *Lemma*: $\overline{T}_l^{\phi}(k,k;E) = \overline{T}_l^{\phi}(p,k;E) = \overline{T}_l^{\phi}(k,p';E) = 0$ for $\overline{V}_l^{\phi}(p,p')$, we adopt the GSE method [12–17] where $\overline{V}_l^{\phi}(k,k)$ should not be zero, for this purpose, we use Eq. (9) by taking $\mathcal{R}' \sim \mathcal{R}$. Therefore Eq. (10) is rewritten by omitting $(;k)$ in the potentials for simplicity,

$$\begin{aligned} \overline{V}_l^{\phi}(p,p') &\equiv \begin{cases} V_l^{\mathcal{R}'}(p,p') - V_l^{\mathcal{R}}(p,p') \sim \varepsilon & (p = p') \\ V_l^{C}(p,p') - V_l^{\mathcal{R}}(p,p') & (p \neq p') \end{cases} \\ &= V_l^{MCP}(p,p') - V_l^{\mathcal{R}}(p,p'), \end{aligned} \tag{51}$$

where we can choose a very small number ε which verifies Eq. (51) for any partial waves. Therefore, the LS equation for the potential $\overline{V}_l^{\phi}$ becomes,

$$\overline{T}_l^{\phi}(p, p'; E) = \overline{V}_l^{\phi}(p, p') + \frac{1}{2\pi^2}\int_0^{\infty} p''^2 dp'' \times \overline{V}_l^{\phi}(p, p'')G_0(p''; E)\overline{T}_l^{\phi}(p'', p'; E). \tag{52}$$

The GSE method gives the solutions of $\overline{T}_l^{\phi}(k, k; E)$, $\overline{T}_l^{\phi}(p, k; E)$ and $\overline{T}_l^{\phi}(k, p'; E)$, [12–16]

$$\overline{T}_l^{\phi}(k, k; E) = \frac{[\overline{V}_l^{\phi}(k, k)]^2}{A_l^{\phi}(k, k; E)}, \tag{53}$$

$$\overline{T}_l^{\phi}(k, p'; E) = \frac{\overline{V}_l^{\phi}(k, k)\overline{\varphi}_l^{\phi}(k, p'; E)}{A_l^{\phi}(k, k; E)}, \tag{54}$$

$$\overline{T}_l^{\phi}(p, k; E) = \frac{\overline{\varphi}_l^{\phi}(p, k; E)\overline{V}_l^{\phi}(k, k)}{A_l^{\phi}(k, k; E)}, \tag{55}$$

with

$$\overline{\varphi}_l^{\phi}(p, k; E) = \overline{V}_l^{\phi}(p, k) + \int_0^{\infty} \overline{V}_l^{\phi(2)}(p, p')G_0(p'; E)\overline{\varphi}_l^{\phi}(p', k; E)\frac{p'^2 dp'}{2\pi^2} \tag{56}$$

$$\overline{A}_l^{\phi}(k, k; E) = \overline{V}_l^{\phi}(k, k) + \int_0^{\infty} \overline{V}_l^{\phi}(k, p')G_0(p'; E)\overline{\varphi}_l^{\phi}(p', k; E)\frac{p'^2 dp'}{2\pi^2} \tag{57}$$

and,

$$\overline{V}_l^{\phi(2)}(p, p') = \frac{\begin{vmatrix} \overline{V}_l^{\phi}(k, k) & \overline{V}_l^{\phi}(k, p') \\ \overline{V}_l^{\phi}(p, k) & \overline{V}_l^{\phi}(p, p') \end{vmatrix}}{\overline{V}_l^{\phi}(k, k)}. \tag{58}$$

In Eq. (51), because of the relation,

$$|\overline{V}_l^{\phi}(p, k)| \sim |\overline{V}_l^{\phi}(k, p')| \sim |\overline{V}_l^{\phi}(p, p')| > |\overline{V}_l^{\phi}(k, k)|, \tag{59}$$

and $|\overline{V}_l^{\phi}(k, k)| \sim |\varepsilon|$, we have

$$\overline{V}_l^{\phi(2)}(p, p') \sim |\varepsilon|^{-1}, \tag{60}$$

$$|\overline{\varphi}_l^{\phi}(p, k; E)| \sim |\varepsilon|^{-1}, \tag{61}$$

$$|\overline{A}_l^{\phi}(k, k; E)| \sim |\varepsilon|^{-1}. \tag{62}$$

Therefore, we obtain from Eqs. (53), (54) and (55),

$$\overline{T}_l^{\phi}(k, k; E) \sim |\varepsilon|^3, \tag{63}$$

$$\overline{T}_l^{\phi}(k, p'; E) \sim |\varepsilon|, \tag{64}$$

$$\overline{T}_l^{\phi}(p, k; E) \sim |\varepsilon|. \tag{65}$$

Consequently, the *Lemma* is proven by taking the limit $\mathcal{R}' \to \mathcal{R}$ which satisfies $\varepsilon \to 0$ independent of the partial wave.

C Two-Potential Theory Regarding $T^{\mathcal{R}}$

The two-potential theory in operator form is introduced for the LS equation which is given by the potential V^{MCP} in Eq. (9); hereafter, we adopt simply the notation V^C instead of V^{MCP},

$$T^C = (V^{\mathcal{R}} + V^{\phi}) + (V^{\mathcal{R}} + V^{\phi})G_0T^C \tag{66}$$

$$\begin{aligned} T^{\mathcal{R}} &= V^{\mathcal{R}} + V^{\mathcal{R}}G_0T^{\mathcal{R}} \equiv V^{\mathcal{R}}\omega^{\mathcal{R}} \\ &= V^{\mathcal{R}} + T^{\mathcal{R}}G_0V^{\mathcal{R}} \equiv \overline{\omega}^{\mathcal{R}}V^{\mathcal{R}} \end{aligned} \tag{67}$$

$$\omega^{\mathcal{R}} = 1 + G_0T^{\mathcal{R}}$$

$$\overline{\omega}^{\mathcal{R}} = 1 + T^{\mathcal{R}}G_0. \tag{68}$$

Let us define

$$T^C \equiv T^{\phi\mathcal{R}} + T^{\mathcal{R}}. \tag{69}$$

By using Eqs (66),(69)

$$\begin{aligned} T^{\phi\mathcal{R}} + T^{\mathcal{R}} &= (V^{\mathcal{R}} + V^{\phi}) \\ &\quad + (V^{\mathcal{R}} + V^{\phi})G_0(T^{\phi\mathcal{R}} + T^{\mathcal{R}}) \\ &= (V^{\mathcal{R}} + V^{\phi}) \\ &\quad + V^{\phi}G_0T^C + V^{\mathcal{R}}G_0(T^{\phi\mathcal{R}} + T^{\mathcal{R}}). \end{aligned} \tag{70}$$

Multiply $\overline{\omega}^{\mathcal{R}}$ from the left of Eq. (70),

$$\begin{aligned} (1 + T^{\mathcal{R}}G_0)T^{\phi\mathcal{R}} &= \overline{\omega}^{\mathcal{R}}V^{\phi}\omega^{\mathcal{R}} \\ &\quad + (1 + T^{\mathcal{R}}G_0)V^{\phi}G_0T^{\phi\mathcal{R}} \\ &\quad + (1 + T^{\mathcal{R}}G_0)V^{\mathcal{R}}G_0T^{\phi\mathcal{R}}, \end{aligned} \tag{71}$$

$$T^{\phi\mathcal{R}} = \overline{\omega}^{\mathcal{R}}V^{\phi}\omega^{\mathcal{R}} + \overline{\omega}^{\mathcal{R}}V^{\phi}G_0T^{\phi\mathcal{R}}. \tag{72}$$

Let us define

$$T^{\phi\mathcal{R}} \equiv \overline{\omega}^{\mathcal{R}}t^{\phi\mathcal{R}}\omega^{\mathcal{R}}, \tag{73}$$

substituting Eq. (73) into Eq. (72), we obtain

$$\begin{aligned} t^{\phi\mathcal{R}} &= V^{\phi} + V^{\phi}(G_0 + G_0T^{\mathcal{R}}G_0)t^{\phi\mathcal{R}} \\ &= V^{\phi} + V^{\phi}G^{\mathcal{R}}t^{\phi\mathcal{R}} \end{aligned} \tag{74}$$

$$G^{\mathcal{R}} = G_0\overline{\omega}^{\mathcal{R}} = G_0 + G_0T^{\mathcal{R}}G_0. \tag{75}$$

Therefore we have

$$T^C = \overline{\omega}^{\mathcal{R}}t^{\phi\mathcal{R}}\omega^{\mathcal{R}} + T^{\mathcal{R}}. \tag{76}$$

The momentum representations of these operator relations are given by the partial wave formula of Eq. (76) (see also [35]),

$$\begin{aligned} T_l^C(p, p'; E) = &\int_0^{\infty}\int_0^{\infty} \frac{p''^2dp''}{2\pi^2}\frac{p'''^2dp'''}{2\pi^2} \\ &\times\overline{\omega}_l^{\mathcal{R}}(p, p''; E)t_l^{\phi\mathcal{R}}(p'', p'''; E)\omega_l^{\mathcal{R}}(p''', p'; E) + T_l^{\mathcal{R}}(p, p'; E). \end{aligned} \tag{77}$$

In order to verify the relation between the double integral and single integral, the following relations should be defined,

$$G_l^{\mathcal{R}}(p, p'; E) = \frac{2\pi^2}{p^2}\delta(p - p')G_0(p; E) + G_0(p; E)T_l^{\mathcal{R}}(p, p'; E)G_0(p'; E) \tag{78}$$

$$\overline{\omega}_l^{\mathcal{R}}(p, p'; E) = \frac{2\pi^2}{p^2}\delta(p - p') + T_l^{\mathcal{R}}(p, p'; E)G_0(p'; E) \tag{79}$$

$$\omega_l^{\mathcal{R}}(p, p'; E) = \frac{2\pi^2}{p^2}\delta(p - p') + G_0(p; E)T_l^{\mathcal{R}}(p, p'; E). \tag{80}$$

It should be noted that the volume factors $2\pi^2/p^2$ and $2\pi^2/p'^2$ are missing in our former articles [5,6] in G_l^{ϕ}, $\overline{\omega}_l^{\phi}$, ω_l^{ϕ} regarding T^{ϕ}, because of the different definition of the free Green's function. However, the final results are not changed.

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