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# Bound and Scattering State of Position Dependent Mass Klein–Gordon Equation with Hulthen Plus Deformed-Type Hyperbolic Potential

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**Abstract** In this article we use supersymmetry quantum mechanics and factorization methods to study the bound and scattering state of Klein–Gordon equation with deformed Hulthen plus deformed hyperbolic potential for arbitrary state in D-dimensions. The analytic relativistic bound state eigenvalues and the scattering phase factor are found in closed form. We report on the numerical results for the bound state energy in D-dimensions.

**Keywords** Klein–Gordon equation · Supersymmetric quantum mechanics · Factorization method

## 1 Introduction

In recent times there has been a considerable effort by different authors to investigate the analytical solutions of the Klein–Gordon equation within the framework of higher spatial dimensions or D-dimensions with motivated physical potential models [1–5]. The famous Klein–Gordon equation in relativistic quantum mechanics is one of the most frequently used equations that describes spin zero particles such as mesons [6–8]. The exact or approximate solutions of the Klein–Gordon equation are very important in physics and chemistry because their solution contain all the necessary information needed for the complete description of the quantum state of the system such as probability density and entropy [9–11] and this quantum system is exactly solvable if all the eigenvalues and eigenfunctions calculate analytically [12]. In order to obtain the exact and approximate solutions of the Klein–Gordon equation, various quantum mechanical techniques have been employed such as Nikiforov–Uvarov (NU) [13], Supersymmetric quantum mechanics (SUSYQM) [14], asymptotic iteration method (AIM) [15], exact quantization rule [16] and others [17]. Jia et al. [18] studied the Klein–Gordon equation with improved Manning–Rosen potential using SUSYQM. Xie and Jia [19] investigated the Klein–Gordon equation in higher dimensions with Morse potential. The KG equation in D-dimensions has been discussed for some while, the early work by Nieto in 1979 [20] and later was developed and extended [21–24]. Ikot et al. [25] studied the Klein–Gordon equation in D-dimensions for multiparameter exponential typed potential. Many other researchers have investigated the solutions of the wave equation with different potential models such as Hylleraas potential [26], Mobius square potential [27], Kratzer potential [28], Rosen–Morse

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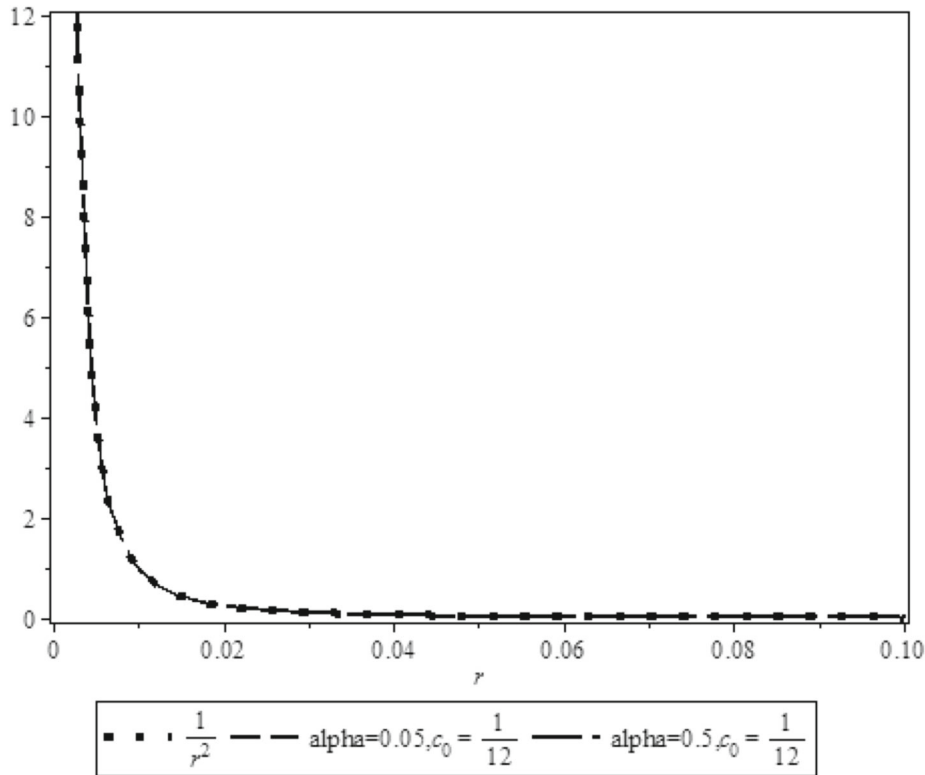


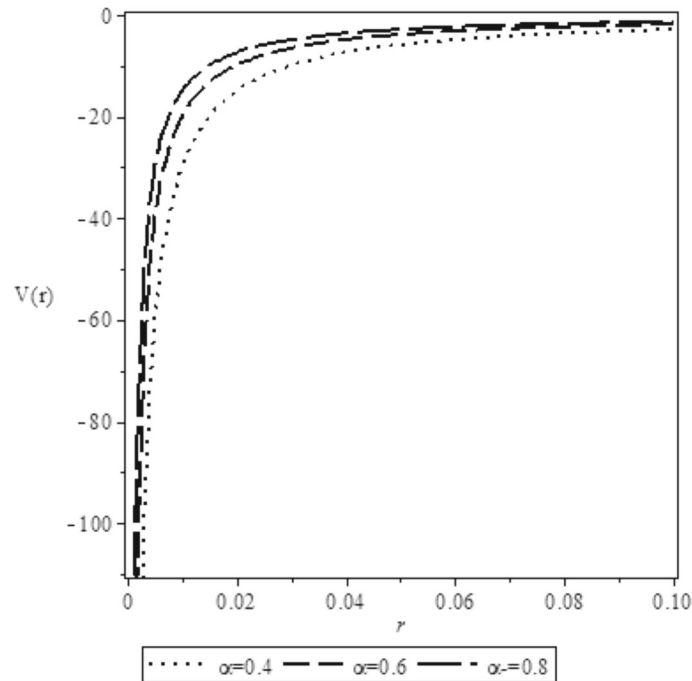
Fig. 1  $\frac{1}{r^2}$  and its approximation of equation (13)

potential [29], new generalized Morse potential [30] and pseudoharmonic potential [31]. Recently, Ikot et al. [32] studied the D-dimensional Klein–Gordon equation with improved Manning–Rosen potential. Hassanabadi and Yazarloo [33] studied the bound and scattering state solutions of Klein–Gordon equation with generalized Poschl–Teller potential. The bound and scattering state solutions of Schrödinger equation with second Poschl–Teller potential have been investigated by You et al. [34]. The scattering state of the coupled Woods–Saxon potential for Duffin Kemmer Petiau (DKP) equation had been examined by Ikot et al. [35]. Arda et al. [36] have investigated the bound and scattering state solutions of the effective mass Klein Gordon equation. Chen et al. [37] reported on the scattering state of Coulomb potential plus ring shaped potential for Klein–Gordon equation. Different approximation scheme has been employed to study the bound state solutions of the wave equation for arbitrary  $l \neq 0$  state [38,39]. Many authors used the approximation schemes [39–42] (see Fig. 1)

$$\frac{1}{r^2} \approx 4\alpha^2 \left[ c_0 + \frac{e^{-2\alpha r}}{1 - e^{-2\alpha r}} + \left( \frac{e^{-2\alpha r}}{1 - e^{-2\alpha r}} \right)^2 \right], \tag{1}$$

for the centrifugal term to obtain the approximate solutions of the bound state of Schrödinger, Klein–Gordon and Dirac equation using various methods of NU, AIM and SUSYQM. As You et al. [34] have noted, this approximation for the centrifugal term cannot be used to obtain the scattering state because of the constant  $c_0$  but can be used effectively to study of the bound state solutions. For the improved Greene–Aldrich approximation Eq. (1), the exact value of  $c_0$  is  $1/12$  [43]. It is well-known that the position dependent mass (PDM) solutions of relativistic and non-relativistic quantum mechanics are of great important in physics and have attracted a considerable attention in recent times [44,45]. Many authors have studied the PDM for Schrödinger, Klein–Gordon and Dirac equations with different potentials [46–51].

The aim of this paper is to obtain the bound and scattering states energy eigenvalues, phase shift and the normalization the eigen function for PDM Klein–Gordon equation with Hulthen plus deformed hyperbolic potential in D-dimensions [52,53]. The Hulthen plus doformed-type hyperbolic potential is defined as [54] (see Fig. 2)



**Fig. 2** Shape of the Hulthen plus deformed hyperbolic potential for  $V_0 = 0.2fm^{-1}$ ,  $V_1 = 0.04fm^{-1}$  and various values of  $\alpha = 0.4, 0.6$  and  $0.8$  respectively

$$V(r) = -V_0 \frac{e^{-2\alpha r}}{1 - qe^{-2\alpha r}} + V_1 \coth_q(\alpha r), \tag{2}$$

where,  $V_0$  and  $V_1$  are the depth of potential wells,  $q$  is the deformation parameter and  $\alpha$  is inverse of screening parameter (see Fig. 2). The mass function is defined as [55]

$$m(r) = m_0 + \frac{m_1}{1 - qe^{-2\alpha r}}, \tag{3}$$

where  $m_0$  is the rest mass of relativistic particles and  $m_1$  is the perturbed mass. The following can be deduced from the mass function as special cases

$$m(r) = \begin{cases} m_0, & m_1 \rightarrow 0 \\ m_0 + m_1, & \alpha \rightarrow \infty \\ m_0 + \frac{m_1}{1-q}, & \alpha \rightarrow 0 \end{cases} . \tag{4}$$

The  $q$ -parameter used in the deformed hyperbolic potential is defined as follows [56]

$$\begin{aligned} \sinh_q(x) &= \frac{e^x - qe^{-x}}{2}, & \cosh_q(x) &= \frac{e^x + qe^{-x}}{2}, \\ \tanh_q(x) &= \frac{\sinh_q(x)}{\cosh_q(x)}, & \operatorname{csc} h_q(x) &= \frac{1}{\sinh_q(x)}, \\ \cosh_q^2(x) - \sinh_q^2(x) &= q. \end{aligned} \tag{5}$$

## 2 Bound state Solution of Klein–Gordon Equation in D-Dimensions

The Klein–Gordon equation in higher dimension for spherically symmetric potential reads [57,58],

$$-\hbar^2 c^2 \Delta_D \psi_{n,l,m}(r, \Omega_D) = \{[E_{n,l} - V(r)]^2 - [mc^2 + S(r)]^2\} \psi_{n,l,m}(r, \Omega_D), \tag{6}$$

where  $E_{n,l}$ ,  $m$ ,  $V(r)$  and  $S(r)$  are the relativistic energy, rest mass, the repulsive vector potential and the attractive scalar potential respectively. Parameter  $\Delta_D$  is defined as

$$\Delta_D = \nabla_D^2 = \frac{1}{r^{D-1}} \frac{\partial}{\partial r} \left( r^{D-1} \frac{\partial}{\partial r} \right) - \frac{\Lambda_D^2(\Omega_D)}{r^2}. \quad (7)$$

The total wave function in D-dimensions is written as

$$\psi_{n,l,m}(r, \Omega_D) = R_{n,l_D}(r) Y_{l_D}^m(\Omega_D). \quad (8)$$

The term  $\frac{\Lambda_D^2(\Omega_D)}{r^2}$  is the generalization of the centrifugal term for the higher dimensional space. The eigenvalues of  $\Lambda_D^2(\Omega_D)$  are defined by the relation

$$\Lambda_D^2(\Omega_D) Y_{l_D}^m(\Omega_D) = l_D(l_D + D - 2) Y_{l_D}^m(\Omega_D), \quad (9)$$

where  $Y_{l_D}^m(\Omega_D)$ ,  $R_{n,l}$  and  $l_D$  represent the hyperspherical harmonics, the hyperradial wave function and the orbital angular momentum quantum number respectively. Now substituting ansatz form  $R_{n,l_D}(r) = r^{-\frac{(D-1)}{2}} F_{n,l_D}(r)$  for the wave function into Eq. (2) yields

$$\left\{ \hbar^2 c^2 \frac{d^2}{dr^2} + (E_{n,l} - V(r))^2 - (m(r)c^2 + S(r))^2 - \frac{(D + 2l_D - 1)(D + 2l_D - 3)\hbar^2 c^2}{4r^2} \right\} F_{n,l_D}(r) = 0. \quad (10)$$

Considering unequal scalar and vector potential for the Hulthen plus deformed hyperbolic potential, yields

$$\begin{aligned} V(r) &= -V_0 \frac{e^{-2\alpha r}}{1 - qe^{-2\alpha r}} + V_1 \left( \frac{1 + qe^{-2\alpha r}}{1 - qe^{-2\alpha r}} \right), \\ S(r) &= -S_0 \frac{e^{-2\alpha r}}{1 - qe^{-2\alpha r}} + S_1 \left( \frac{1 + qe^{-2\alpha r}}{1 - qe^{-2\alpha r}} \right). \end{aligned} \quad (11)$$

We obtain the following second-order Schrödinger-like equation

$$\begin{aligned} &\left\{ \hbar^2 c^2 \frac{d^2}{dr^2} + \left( E_{n,J_D} - \left( -V_0 \frac{e^{-2\alpha r}}{1 - qe^{-2\alpha r}} + V_1 \left( \frac{1 + qe^{-2\alpha r}}{1 - qe^{-2\alpha r}} \right) \right) \right)^2 \right\} U_{n,l_D}(r) \\ &- \left[ \left( \left( m_0 + \frac{m_1}{1 - qe^{-2\alpha r}} \right) c^2 + \left( -S_0 \frac{e^{-2\alpha r}}{1 - qe^{-2\alpha r}} + S_1 \left( \frac{1 + qe^{-2\alpha r}}{1 - qe^{-2\alpha r}} \right) \right) \right)^2 \right] U_{n,l_D}(r) \\ &- \left( \frac{(D + 2l_D - 1)(D + 2l_D - 3)\hbar^2 c^2}{4r^2} \right) U_{n,l_D}(r) = 0. \end{aligned} \quad (12)$$

It is well-known that Eq. (12) cannot be solved exactly except for  $J = 0$ . In order to find the solutions of Eq. (12), we apply a suitable approximation for the centrifugal term [56] since approximation in Eq. (1) is not suitable for both bound and scattering state problem. We invoke the suitable approximation as [56]

$$\frac{1}{r^2} \approx \frac{\alpha^2}{\sinh_q^2(\alpha r)} = \frac{4\alpha^2}{(1 - qe^{-2\alpha r})^2}. \quad (13)$$

Substituting Eq. (13) into Eq. (12) and after a little algebra, we get

$$-\frac{d^2 F_{n,J_D}}{dr^2} + \frac{1}{(1 - qe^{-2\alpha r})^2} [\omega_1 e^{-4\alpha r} + \omega_2 e^{-2\alpha r} + \omega_3] F_{n,J_D}(r) = \tilde{E}_{n,J_D} F_{n,J_D}, \quad (14)$$

where,

$$\tilde{E}_{n,J_D} = \frac{E_{n,J_D}^2 - m^2 c^4}{\hbar^2 c^2} \quad (15)$$

$$\begin{aligned} \omega_1 &= \frac{1}{\hbar^2 c^2} [2m_0 c^2 S_0 q - 2m_0 c^2 S_1 q^2 + 2E_{n,J_D} V_0 q - 2E_{n,J_D} V_1 q^2 + S_0^2 + S_1^2 q^2 - V_0^2 + V_1^2 q^2] \\ \omega_2 &= \frac{1}{\hbar^2 c^2} [-2m_0 c^2 S_0 - 2m_1 c^2 S_0 + 2m_1 c^2 S_1 q - 2E_{n,J_D} V_0 \\ &\quad - 2S_0 S_1 q + 2S_1^2 q + 2V_0 V_1 q + 2V_1^2 q - 2m_0 m_1 c^4 q + 4\gamma \hbar^2 c^2 \alpha^2] \\ \omega_3 &= \frac{1}{\hbar^2 c^2} [2m_1 c^2 S_1 + 2m_0 c^2 S_1 + 2E_{n,J_D} V_1 - 2S_0 S_1 + S_1^2 + 2V_0 V_1 + V_1^2 + 2m_0 m_2 c^4 + m_1^2 c^4] \\ \gamma &= \frac{(D + 2l_D - 1)(D + 2l_D - 3)}{4}. \end{aligned} \quad (16)$$

In the SUSYQM formulation, the ground-state wave function  $F_{0,l}(r)$  is given by [14,18]

$$F_{0,l}(r) = \exp\left(-\int W(r)dr\right), \quad (17)$$

in which the integrand is called the superpotential and the Hamiltonian is composed of the raising and lowering operators

$$H_- = \hat{A}^+ \hat{A} = -\frac{d^2}{dr^2} + V_-(r), \quad (18)$$

$$H_+ = \hat{A} \hat{A}^+ = -\frac{d^2}{dr^2} + V_+(r), \quad (19)$$

with

$$\hat{A} = \frac{d}{dr} - W(r), \quad (20)$$

$$\hat{A}^+ = -\frac{d}{dr} - W(r), \quad (21)$$

$$V_{\pm}(r) = W^2(r) \mp W'(r). \quad (22)$$

Now substituting Eq. (17) into Eq. (14), we obtain the associated Riccati equation

$$W^2(r) - W'(r) = \frac{1}{(1 - qe^{-2\alpha r})^2} [\omega_1 e^{-4\alpha r} + \omega_2 e^{-2\alpha r} + \omega_3] - \tilde{E}_{0,l}. \quad (23)$$

Here, we choose the superpotential of the form

$$W(r) = P + \frac{Qe^{-2\alpha r}}{(1 - qe^{-2\alpha r})}. \quad (24)$$

Substituting Eq. (24) into Eq. (23) and solve explicitly, we obtain the following relations

$$P = -\left[\frac{-Q^2 + \omega_1 - \omega_3 q^2}{2qQ}\right] \quad (25)$$

$$Q = \alpha q \left[ -1 \pm \sqrt{1 + \frac{1}{\alpha^2 q^2} \{\omega_1 + \omega_2 q + \omega_3 q^2\}} \right] \quad (26)$$

$$\tilde{E}_{0,J_D} = -\left(\frac{-Q^2 + \omega_1 - \omega_3 q^2}{2qQ}\right)^2 + \omega_3. \quad (27)$$

We construct the pair of supersymmetric partner potentials  $V_+(r)$  and  $V_-(r)$  as follows,

$$\begin{aligned} V_+(r) &= W^2(r) + \frac{dW(r)}{dr}, \\ &= \frac{Q(Q-2\alpha)e^{-2\alpha r}}{(1-qe^{-2\alpha r})^2} + \frac{Q^2e^{-4\alpha r} - Q^2e^{-2\alpha r}}{(1-qe^{-2\alpha r})^2} + \frac{2PQe^{-2\alpha r}}{(1-qe^{-2\alpha r})} + \left(\frac{-Q^2 + \omega_1 - \omega_3q^2}{2qQ}\right)^2 \end{aligned} \quad (28)$$

$$\begin{aligned} V_-(r) &= W^2(r) - \frac{dW(r)}{dr}, \\ &= \frac{Q(Q+2\alpha)e^{-2\alpha r}}{(1-qe^{-2\alpha r})^2} + \frac{Q^2e^{-4\alpha r} - Q^2e^{-2\alpha r}}{(1-qe^{-2\alpha r})^2} + \frac{2PQe^{-2\alpha r}}{(1-qe^{-2\alpha r})} + \left(\frac{-Q^2 + \omega_1 - \omega_3q^2}{2qQ}\right). \end{aligned} \quad (29)$$

The partner potential are shape invariance via mapping of the form  $Q \rightarrow Q - 2\alpha$ . Also, it is easy to check the shape-invariance condition

$$V_+(r, \rho_0) = V_-(r, \rho_i) + R(\rho_i) \quad (30)$$

which holds via the mapping  $Q \rightarrow Q - 2\alpha$ . In this study  $\rho_0 = Q$  and  $\rho_i$  is a function of  $\rho_0$ , i.e.,  $\rho_i = f(\rho_0) = \rho_0 - 2\alpha$ . Thus we have  $\rho_n = \rho_0 - 2\alpha n$ . From Eq. (30), we write

$$R(\rho_n) = \left(\frac{-\rho_{n-1}^2 + \omega_1 - \omega_3q^2}{2q\rho_{n-1}}\right)^2 - \left(\frac{-\rho_n^2 + \omega_1 - \omega_3q^2}{2q\rho_n}\right)^2 \quad (31)$$

$$\tilde{E}_{n,l}^- = \sum_{k=1}^n R(\rho_k) = \left(\frac{-\rho_0^2 + \omega_1 - \omega_3q^2}{2q\rho_0}\right)^2 - \left(\frac{-\rho_n^2 + \omega_1 - \omega_3q^2}{2q\rho_n}\right)^2. \quad (32)$$

Using Eqs. (15), (26) and (31), we obtain the transcendental energy spectrum for the Hulthen plus deformed hyperbolic potential model for the Klein–Gordon equation in D-dimensions as follows

$$\tilde{E}_{n,l} = \tilde{E}_{n,l}^- + \tilde{E}_{0,l} = -\left(\frac{-\rho_n^2 + \omega_1 - \omega_3q^2}{2q\rho_n}\right)^2 + \omega_3. \quad (33)$$

More explicitly, we write the energy Eq. (33) as

$$E_{nJ_D}^2 - m_0^2 = -\frac{\hbar^2c^2}{4q^2} \left(\frac{\omega_3q^2 - \omega_1}{2\alpha(n+\sigma)} + 2\alpha(n+\sigma)\right)^2 + \hbar^2c^2\omega_3, \quad (34)$$

where

$$\sigma = \frac{q}{2} \left(1 + \sqrt{1 + \frac{1}{\alpha^2q^2}(\omega_1 + \omega_2q + \omega_3q^2)}\right). \quad (35)$$

In order to determine the corresponding wave function for the system, we use  $z = -qe^{-2\alpha r}$ . Then we can write

$$\frac{d^2U_{nJ_D}}{dz^2} + \frac{1}{z} \frac{dU_{nJ_D}}{dz} + \frac{1}{z^2(1-z)^2} (-\chi_1z^2 + \chi_2z + \chi_3) U_{nJ_D}(z) = 0, \quad (36)$$

where

$$\begin{aligned}\chi_1 &= \frac{1}{4\alpha^2 q^2 \hbar^2 c^2} [2m_0 c^2 S_0 q - 2m_0 c^2 S_1 q^2 + 2E_{n,J_D} V_0 q - 2E_{n,J_D} V_1 q^2 + S_0^2 + S_1^2 q^2 - V_0^2 + V_1^2 q^2] \\ &\quad - \frac{E_{n,J_D}^2 - m^2 c^4}{4\alpha^2 \hbar^2 c^2} \\ \chi_2 &= -\frac{1}{4\alpha^2 q \hbar^2 c^2} \left[ -2m_0 c^2 S_0 - 2m_1 c^2 S_0 + 2m_1 c^2 S_1 q - 2E_{n,J_D} V_0 - 2S_0 S_1 q + 2S_1^2 q + 2V_0 V_1 q \right] \\ &\quad - \frac{2(E_{n,J_D}^2 - m^2 c^4)}{4\alpha^2 \hbar^2 c^2} \\ \chi_3 &= \frac{1}{4\alpha^2 \hbar^2 c^2} [2m_1 c^2 S_1 + 2m_0 c^2 S_1 + 2E_{n,J_D} V_1 - 2S_0 S_1 + S_1^2 + 2V_0 V_1 + V_1^2 + 2m_0 m_2 c^4 + m_1^2 c^4] \\ &\quad - \frac{(E_{n,J_D}^2 - m^2 c^4)}{4\alpha^2 \hbar^2 c^2}.\end{aligned}\quad (37)$$

The solution of Eq. (36) becomes

$$U_{n,J_D}(r) = N_{n,J_D} (q e^{-2\alpha r})^{\sqrt{\chi_3}} (1 - q e^{-2\alpha r})^{\sqrt{\frac{1}{4} + \chi_1 - \chi_2 + \chi_3}} P_n^{(2\sqrt{\chi_3}, 2\sqrt{\frac{1}{4} + \chi_1 - \chi_2 + \chi_3})} (1 - 2q e^{-2\alpha r}), \quad (38)$$

where  $N_{n,l}$  is the normalization constant and  $P_n^{(\mu,\nu)}(x)$  is the Jacobi polynomial.

### 3 Scattering State Solutions

By using of  $x = 1 - z$  in Eq. (36), we get

$$x(1-x)U''(x) - xU'(x) + \left( \frac{\Xi_1}{x} + \frac{\Xi_2}{1-x} + \Xi_3 \right) U(x) = 0, \quad (39)$$

where,

$$\begin{aligned}\Xi_1 &= \chi_2 + \chi_3 - \chi_1 \\ \Xi_2 &= \chi_3 \\ \Xi_3 &= \chi_1.\end{aligned}\quad (40)$$

By taking the transformation  $F_{n,l} = y^{\lambda_1} (1-y)^{\lambda_2} \varphi(z)$  in Eq. (39), we obtain the hypergeometric function in the form [36]

$$z(1-z)\varphi''(z) + (1+2\eta_1 - (1+2\eta_1+2\eta_2)z)\varphi'(z) - \eta_1\eta_2\varphi(z) = 0, \quad (41)$$

where,

$$\begin{aligned}\lambda_1 &= \frac{1}{2} \left( 1 + \sqrt{1 - 4\Xi_1} \right), \\ \lambda_2 &= -\frac{ik}{2\alpha}, \quad k = \sqrt{4\alpha^2 \Xi_2}.\end{aligned}\quad (42)$$

The solutions of Eq. (41) is the hypergeometric function given by

$$\varphi(y) = {}_2F_1(\eta_1, \eta_2, \eta_3, y), \quad (43)$$

where,

$$\begin{aligned}\eta_1 &= \frac{1}{2} \left( 1 + \sqrt{1 - 4\Xi_1} \right) - \frac{ik}{\alpha} + \sqrt{\Xi_3}, \\ \eta_2 &= \frac{1}{2} \left( 1 + \sqrt{1 - 4\Xi_1} \right) - \frac{ik}{\alpha} - \sqrt{\Xi_3}, \\ \eta_3 &= 1 + \sqrt{1 - 4\Xi_1}.\end{aligned}\quad (44)$$

We can write the complete wave function as

$$F_{n,l}(r) = N_{n,l} (-qe^{-2\alpha r})^{\lambda_1} (1 - qe^{-2\alpha r})^{\lambda_2} {}_2F_1(\eta_1, \eta_2, \eta_3; 1 - qe^{-2\alpha r}). \quad (45)$$

The scattering state for  $E > 0$ , are defined as follows [59,60]

$$\begin{aligned} U(r) &\rightarrow 0, \\ r &\rightarrow 0 \\ U(r) &\rightarrow 2 \sin \left[ kr + \delta - \frac{l_D \pi}{2} \right], \\ r &\rightarrow \infty \end{aligned} \quad (46)$$

in 3D case of the boundary conditions. However, the extension to the D-dimensional case takes the form [59,60]

$$\begin{aligned} U(r) &\rightarrow 0, \\ r &\rightarrow 0 \\ U(r) &\rightarrow 2 \sin \left[ kr + \delta - \frac{\pi}{2} \left( l_D + \frac{(D-3)}{2} \right) \right], \\ r &\rightarrow \infty \end{aligned} \quad (47)$$

Eq.(47) ensures us that the radial wave functions of the scattering state for exponential type potential are also normalized, where  $\delta_{l_D}$  is the phase shift in D-dimensions.

Now we study the asymptotic form of Eq. (45) for large values of  $r$  and calculate the normalization constant  $N_{nl_D}$  and the phase shift. From Eq. (44), we get

$$\eta_3 - \eta_1 - \eta_2 = \frac{2ik}{\alpha} = (\eta_1 + \eta_2 - \eta_3)^* \quad (48)$$

$$\eta_3 - \eta_1 = \frac{1}{2} \left( 1 + \sqrt{1 - 4\Xi_1} \right) + \frac{ik}{\alpha} - \sqrt{\Xi_3} = \eta_2^* \quad (49)$$

$$\eta_3 - \eta_2 = \frac{1}{2} \left( 1 + \sqrt{1 - 4\Xi_1} \right) - \frac{ik}{\alpha} + \sqrt{\Xi_3} = \eta_1^*. \quad (50)$$

Now applying the transformation properties for the hypergeometric function [61]

$${}_2F_1(\eta_1, \eta_2, \eta_3; 0) = 1, \quad (51)$$

$$\begin{aligned} {}_2F_1(\eta_1, \eta_2, \eta_3; z) &= \frac{\Gamma(\eta_3)\Gamma(\eta_3 - \eta_1 - \eta_2)}{\Gamma(\eta_3 - \eta_1)\Gamma(\eta_3 - \eta_2)} {}_2F_1(\eta_3; \eta_2; \eta_1 + \eta_2 - \eta_3 + 1; 1 - z) \\ &+ (1 - z)^{\eta_3 - \eta_1 - \eta_2} \frac{\Gamma(\eta_3)\Gamma(\eta_1 + \eta_2 - \eta_3)}{\Gamma(\eta_1)\Gamma(\eta_2)} {}_2F_1(\eta_3 - \eta_1; \eta_3 - \eta_2; \eta_3 - \eta_1 - \eta_2 + 1; 1 - z). \end{aligned} \quad (52)$$

Thus, the term  ${}_2F_1(\eta_1, \eta_2, \eta_3; 1 - qe^{-2\alpha r})$  in Eq. (45) becomes

$$\begin{aligned} {}_2F_1(\eta_1, \eta_2, \eta_3; 1 - qe^{-2\alpha r}) &= \Gamma(\eta_3) \left[ \frac{\Gamma(\eta_3 - \eta_1 - \eta_2)}{\Gamma(\eta_3 - \eta_1)\Gamma(\eta_3 - \eta_2)} \right. \\ &\quad \left. + \left[ \frac{\Gamma(\eta_3 - \eta_1 - \eta_2)}{\Gamma((\eta_3 - \eta_1))\Gamma(\eta_3 - \eta_2)} \right]^* q^{(\eta_3 - \eta_1 - \eta_2)} e^{-\alpha(\eta_3 - \eta_1 - \eta_2)r} \right], \\ &= \Gamma(\eta_3) q^{i(\eta_3 - \eta_1 - \eta_2)} \left[ \frac{\Gamma(\eta_3 - \eta_1 - \eta_2) q^{-i(\eta_3 - \eta_1 - \eta_2)}}{\Gamma(\eta_3 - \eta_1)\Gamma(\eta_3 - \eta_2)} \right. \\ &\quad \left. + \left[ \frac{\Gamma(\eta_3 - \eta_1 - \eta_2) q^{-i(\eta_3 - \eta_1 - \eta_2)}}{\Gamma(\eta_3 - \eta_1)\Gamma(\eta_3 - \eta_2)} \right]^* e^{-i\alpha(\eta_3 - \eta_1 - \eta_2)r} \right] \end{aligned} \quad (53)$$



for  $r \rightarrow \infty$ . We can simplify Eq. (53) as follows

$$\begin{aligned} {}_2F_1(\eta_1, \eta_2, \eta_3; 1 - qe^{-2\alpha r}) &= \Gamma(\eta_3)q^{\frac{ik}{\alpha}} \left[ \frac{\Gamma(\eta_3 - \eta_1 - \eta_2)q^{-\frac{ik}{\alpha}}}{\Gamma(\eta_3 - \eta_1)\Gamma(\eta_3 - \eta_2)} + \left[ \frac{\Gamma(\eta_3 - \eta_1 - \eta_2)q^{-\frac{ik}{\alpha}}}{\Gamma(\eta_3 - \eta_1)\Gamma(\eta_3 - \eta_2)} \right]^* \right] e^{-ikr} \\ &= \Gamma(\eta_3)q^{\frac{ik}{\alpha}} \left[ \frac{\Gamma(2ik/\alpha)q^{-\frac{ik}{\alpha}}}{\Gamma(\eta_2^*)\Gamma(\eta_1^*)} + \left[ \frac{\Gamma(2ik/\alpha)q^{-\frac{ik}{\alpha}}}{\Gamma(\eta_2^*)\Gamma(\eta_1^*)} \right]^* \right] e^{-ikr}. \end{aligned} \quad (54)$$

Also by using the relations

$$\begin{aligned} \frac{\Gamma(\eta_3 - \eta_1 - \eta_2)}{\Gamma(\eta_3 - \eta_1)\Gamma(\eta_3 - \eta_2)} &= \left| \frac{\Gamma(\eta_3 - \eta_1 - \eta_2)}{\Gamma(\eta_3 - \eta_1)\Gamma(\eta_3 - \eta_2)} \right| e^{i\delta}, \\ &= \left| \frac{\Gamma(2ik/\alpha)}{\Gamma(\eta_2^*)\Gamma(\eta_1^*)} \right| e^{i\delta} \end{aligned} \quad (55)$$

$$q^{-\frac{ik}{\alpha}} = \left| q^{-\frac{ik}{\alpha}} \right| e^{i\delta'}, \quad (56)$$

and Eq. (53), we have

$$\begin{aligned} {}_2F_1(\eta_1, \eta_2, \eta_3; 1 - qe^{-2\alpha r}) &= \Gamma(\eta_3)q^{\frac{ik}{\alpha}} \left| \frac{\Gamma(2ik/\alpha)q^{-\frac{ik}{\alpha}}}{\Gamma(\eta_2^*)\Gamma(\eta_1^*)} \right| \left[ e^{i(\delta+\delta')} + e^{-2ikr} e^{-i(\delta+\delta')} \right] \\ &= \Gamma(\eta_3)q^{\frac{ik}{\alpha}} \left| \frac{\Gamma(2ik/\alpha)q^{-\frac{ik}{\alpha}}}{\Gamma(\eta_2^*)\Gamma(\eta_1^*)} \right| e^{-ikr} \left[ e^{i(\delta+\delta' - \frac{k\ell n 2}{\alpha} + ikr)} + e^{-i(\delta+\delta' - \frac{k\ell n 2}{\alpha} + ikr)} \right] \\ &= \Gamma(\eta_3)q^{\frac{ik}{\alpha}} \left| \frac{\Gamma(2ik/\alpha)q^{\frac{ik}{\alpha}}}{\Gamma(\eta_2^*)\Gamma(\eta_1^*)} \right| e^{-ikr} \sin \left( kr + \delta + \delta' - \frac{k\ell n 2}{\alpha} \right. \\ &\quad \left. + \frac{\pi}{2} \left[ l_D + \frac{(D-3)}{2} \right] + \frac{\pi}{2} \left[ l_D + \frac{(D-1)}{2} \right] \right) \end{aligned} \quad (57)$$

Therefore, the asymptotic form of Eq. (45) becomes

$$\begin{aligned} F_{n,l_D} &= 2N_{n,l_D}\Gamma(\eta_3) \left| \frac{\Gamma(2ik/\alpha)q^{\frac{ik}{\alpha}}}{\Gamma(\eta_2^*)\Gamma(\eta_1^*)} \right| e^{-ikr} \sin \left( kr + \delta + \delta' - \frac{k\ell n 2}{\alpha} + \frac{\pi}{2} \left[ l_D + \frac{(D-3)}{2} \right] \right. \\ &\quad \left. + \frac{\pi}{2} \left[ l_D + \frac{(D-1)}{2} \right] \right), \end{aligned} \quad (58)$$

for  $r \rightarrow \infty$ . Now comparing Eq. (58) with the boundary conditions [62–64],  $r \rightarrow \infty \Rightarrow F(\infty) \rightarrow 2 \sin \left( kr + \delta_{l_D} - \frac{\pi}{2} \left( l_D + \frac{(D-3)}{2} \right) \right)$ , we get the phase shift and the normalization constant as follows

$$\begin{aligned} \delta_{l_D} &= \frac{\pi}{2} \left( l_D + \frac{(D-3)}{2} \right) + \delta + \delta' \\ &\quad \frac{\pi}{2} \left[ l_D + \frac{(D-1)}{2} \right] - \frac{k\ell n 2}{\alpha} + \arg \Gamma(\eta_3 - \eta_2 - \eta_1) - \arg \Gamma(\eta_3 - \eta_1) \\ &\quad - \arg \Gamma(\eta_3 - \eta_2) + \arg \left( q^{-\frac{ik}{\alpha}} \right) \end{aligned} \quad (59)$$

$$N_{n,l_D} = \frac{1}{\Gamma(\eta_3)} \left| \frac{\Gamma(\eta_2^*)\Gamma(\eta_1^*)}{\Gamma(2ik/\alpha)q^{\frac{ik}{\alpha}}} \right| \quad (60)$$

It had been shown that the poles of the S-matrix in the complex energy plane determined the bound states for the real poles and the scattering states for the complex poles in the lower half of the energy plane [65]. In order to obtain the bound state solutions, we used the following definition of the gamma functions,

$$\Gamma(z) = \frac{\Gamma(z+1)}{z} = \frac{\Gamma(z+2)}{z(z+1)} = \frac{\Gamma(z+3)}{z(z+1)(z+2)} = \dots, \quad (61)$$

**Table 1** Energy of the system for different states and dimensions  $m_0 = -5, m_1 = -0.2, V_0 = 2, \alpha = 0.01, S_1 = 3, V_1 = 0.5, q = 1$

	$l$	$E_{n,l}^D(D = 1)$	$E_{n,l}^D(D = 2)$	$E_{n,l}^D(D = 3)$	$E_{n,l}^D(D = 4)$
$n = 0$	0	2.569172676	2.569156234	2.569172676	2.569221997
		-2.203041293	-2.203011271	-2.203041203	-2.203130998
	1	2.569172676	2.569221997	2.569304187	2.569419235
		-2.203041203	-2.203130998	-2.203280640	-2.203490115
	2	2.569304187	2.569419235	2.569567123	2.569747824
		-2.203280640	-2.203490115	-2.203759397	-2.204088452
	3	2.569567123	2.569747824	2.569961309	2.570207541
		-2.203759397	-2.204088452	-2.204477240	-2.204925711
	4	2.569961309	2.570207541	2.570486483	2.570798087
		-2.204477240	-2.204925711	-2.205433819	-2.206001495
$n = 1$	0	2.5953363031	2.595347227	2.595363031	2.595410441
		-2.251003944	-2.250974832	-2.251003944	-2.251091282
	1	2.595363031	2.595410441	2.595489446	2.595600036
		-2.251003944	-2.251091282	-2.251236828	-2.251440570
	2	2.595489446	2.595600036	2.595742192	2.595915893
		-2.251236828	-2.251440570	-2.251702481	-2.252022535
	3	2.595742192	2.595915893	2.596121109	2.596357804
		-2.251702481	-2.252022535	-2.252400688	-2.252836894
	4	2.596121109	2.596357804	2.596625945	2.596925485
		-2.252400688	-2.252836894	-2.253331107	-2.253883264
$n = 2$	0	2.620547699	2.620532497	2.620547699	2.620593305
		-2.297667736	-2.297639403	-2.297667736	-2.297752733
	1	2.620547699	2.620593305	2.620669304	2.620775688
		-2.297667736	-2.297752733	-2.297894378	-2.2980922663
	2	2.620669304	2.620775688	2.620912436	2.621079530
		-2.297894378	-2.298092663	-2.298347556	-2.298659035
	3	2.620912436	2.621079530	2.621276940	2.621504637
		-2.298347556	-2.298659035	-2.299027058	-2.299451584
	4	2.621276940	2.621504637	2.621762582	2.622050736
		-2.299027058	-2.299451584	-2.299932564	-2.300469940

where  $z = 0, -1, -2 \dots$  are the first order poles of the gamma function  $\Gamma(z)$ . Thus, the first order poles in Eq. (61) is  $\Gamma(\eta_3 - \eta_1)$  and satisfy the condition

$$\frac{1}{2} \left( 1 + \sqrt{1 - 4\Xi_1} \right) - \frac{ik}{\alpha} - \sqrt{\Xi_3} = -n_r, \tag{62}$$

where,  $n_r = 0, 1, 2 \dots$ . At the poles of the scattering amplitude the bound state energy level in D-dimensions is given by Eq. (62) which is the same as Eq. (33).

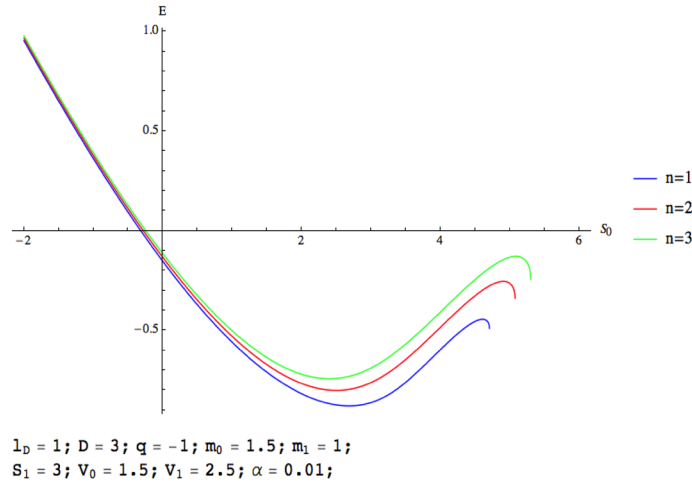
### 4 Results and Discussion

In order to test the accuracy of our work, we computed numerically the bound state energy for the Hulthen plus deformed hyperbolic potential for arbitrary  $l$ -state in D-dimensions in Table 1 and we also show the plot of the behaviour of the energy level as a function of the potential parameters in Figs. 3, 4, 5, 6, 7, 8, 9 and 10. We also examine the special cases of the phase shift factor and normalization constant and compare our results with the available literature.

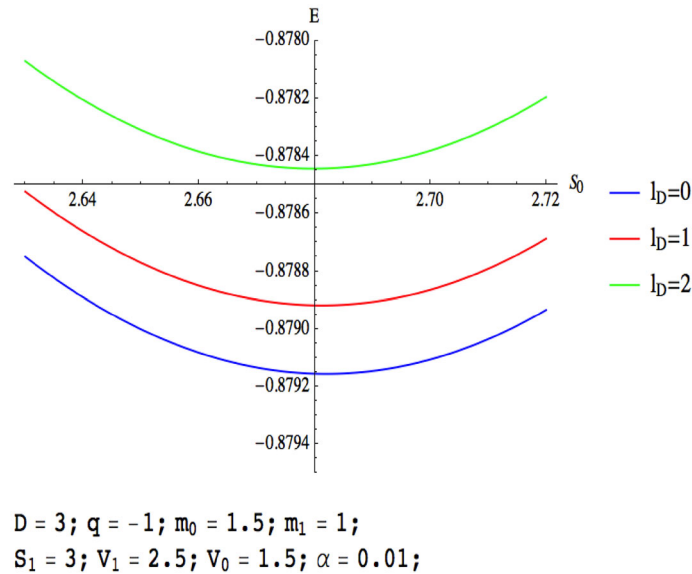
#### 4.1 Scattering State Solutions of Generalized Hulthen Potential in D-dimensions

If we choose,  $V_1 = 0, S_0 = S_1 = m_1 = 0$  and  $\alpha \rightarrow \frac{\alpha}{2}$  in Eq. (2), we obtain the generalized Hulthen potential as

$$V_H(r) = -V_0 \frac{e^{-\alpha r}}{1 - qe^{-\alpha r}}. \tag{63}$$



**Fig. 3** Behaviour of the energy level of the system versus  $S_0$  for various values of  $n = 1, 2$  and  $3$  respectively



**Fig. 4** Variation of the energy spectrum with  $S_0$  for different values of the angular momentum quantum number  $l_D = 0, 1$  and  $2$  respectively

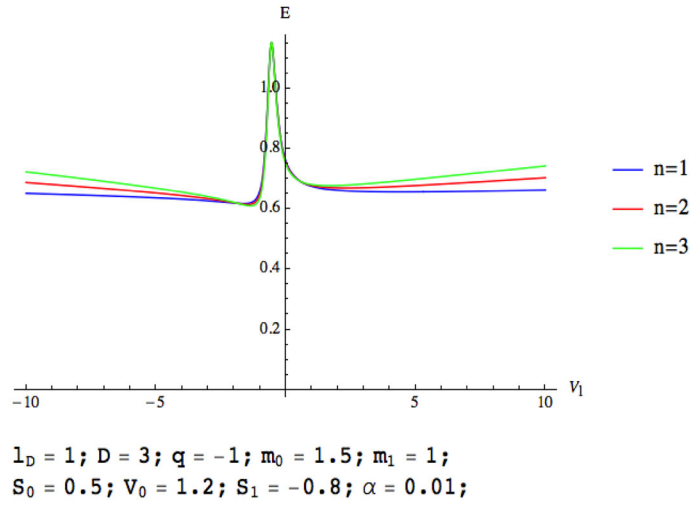
If we substitute these parameters into Eqs. (59) and (60) we obtain the phase factor and normalization constant for the Hulthen potential in  $D$ -dimensions as

$$\delta_J^H = \frac{\pi}{2} \left[ J_D + \frac{(D-1)}{2} \right] - \frac{k \ell n 2}{\alpha} + \arg \Gamma \left( \frac{2ik}{\alpha} \right) - \arg \Gamma \left( \lambda_1 - \frac{ik}{\alpha} - \sqrt{\Xi_3} \right) - \arg \Gamma \left( \lambda_1 - \frac{ik}{\alpha} - \sqrt{\Xi_3} \right) + \arg \left( q \frac{-ik}{\alpha} \right) \tag{64}$$

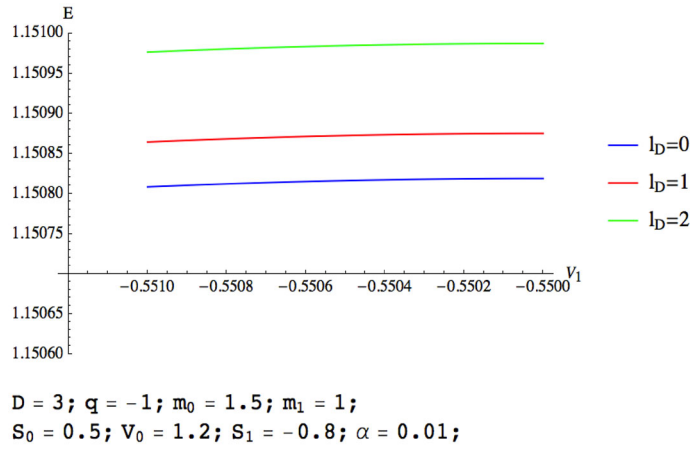
$$N_{nJ_D}^H = \frac{\Gamma(\lambda_1^H - \frac{ik}{\alpha} - \sqrt{\Xi_3})}{\Gamma(1 + 2\lambda_1^H)} \times \frac{\Gamma(\lambda_1^H - \frac{ik}{\alpha} - \sqrt{\Xi_3})}{\Gamma(2ik/\alpha) q^{\frac{ik}{\alpha}}} \tag{65}$$

These results are consistent with that reported by Feng et al. [59] in the case of  $D = 3$ . The normalized wave function for the Hulthen potential becomes

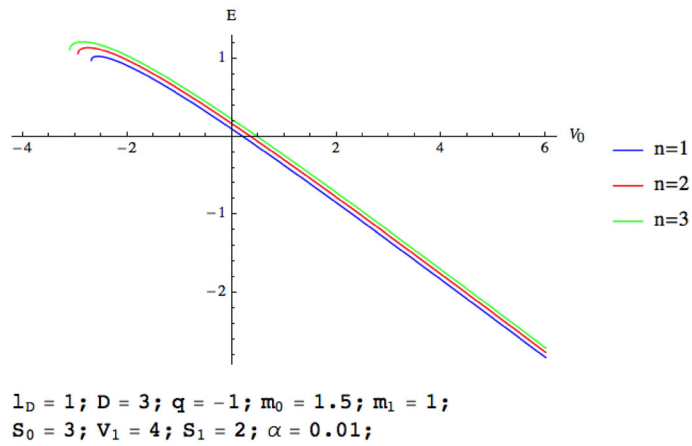
$$U_{nJ_D}^H(r) = N_{nJ_D}^H (1 - qe^{-ar})^{\lambda_1^H} (q)^{\frac{ik}{\alpha}} e^{-ikr} {}_2F_1 \left( \eta_1^H, \eta_2^H, \eta_3^H, 1 - qe^{-ar} \right). \tag{66}$$



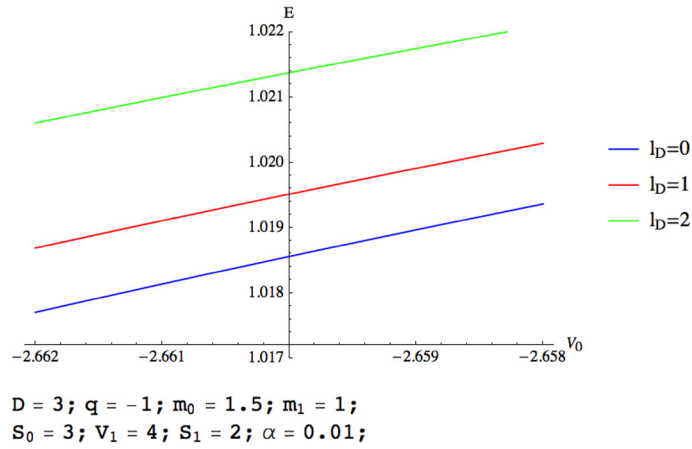
**Fig. 5** Plot of the energy level versus  $V_1$  for various values of the principal quantum number  $n = 1, 2$  and  $3$  respectively



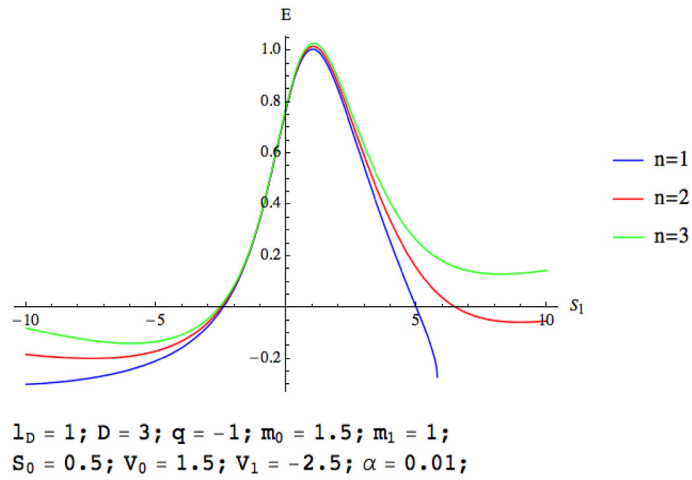
**Fig. 6** Variation of the energy eigenvalues with  $V_1$  for different values of  $l_D = 0, 1$  and  $2$  respectively



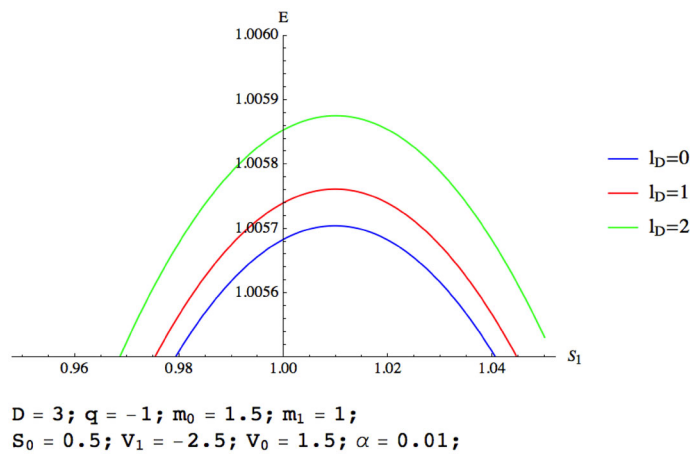
**Fig. 7** Energy variation with  $V_0$  for  $n = 1, 2$  and  $3$  respectively



**Fig. 8** Energy changes with  $V_0$  for different values of  $l_D = 0, 1$  and  $2$  respectively



**Fig. 9** Plot of the energy spectrum with  $S_1$  for  $n = 1, 2$  and  $3$  respectively



**Fig. 10** Plot of the energy with  $S_1$  for different values of  $l_D = 0, 1$  and  $2$  respectively

If  $q = 1$  then the generalized Hulthen potential reduced to the standard potential reported by Chen et al. [65].

#### 4.2 Scattering State Solutions of Woods–Saxon Potential in D-dimensions

Again if we set  $\alpha = \frac{1}{R}$ ,  $q = e^{\frac{\theta}{R}}$  and  $V_0 \rightarrow V_0 q = V_0 e^{\frac{\theta}{R}}$  as given in Ref. [59], the generalized Hulthen potential reduces to the standard Woods-Saxon potential [66],

$$V(r) = \frac{V_0}{1 - \exp\left[\left(\frac{r-\theta}{R}\right)\right]} \quad (67)$$

where  $V_0$  is the potential depth,  $\theta$  is the width of the potential and  $R$  is the surface thickness whose values correspond to the ionization energies. We obtain the corresponding phase factor and normalization for the Woods–Saxon potential in D-dimensions as follows,

$$\delta_J^{WS} = \frac{\pi}{2} \left[ J_D + \frac{(D-1)}{2} \right] - kR \ln 2 + \arg \Gamma(2ikR) - \arg \Gamma\left(\lambda_1^{WS} - iRk + \sqrt{\Xi_3}\right) - \arg \Gamma\left(\lambda_1^{WS} + iRk - \sqrt{\Xi_3}\right) + \arg \left( \left( e^{-\frac{\theta}{R}} \right)^{-iRk} \right) \quad (68)$$

$$N_{nJ_D}^{WS} = \frac{\Gamma(\lambda_1^{WS} - iRk - \sqrt{\Xi_3})}{\Gamma(1 + 2\lambda_1^{WS})} \times \frac{\Gamma(\lambda_1^{WS} + iRk - \sqrt{\Xi_3})}{\Gamma(2ikR) \left( e^{\frac{\theta}{R}} \right)^{ikR}} \quad (69)$$

$$U_{nJ_D}^{WS}(r) = N_{nJ_D}^{WS} \left( 1 - \exp\left(\frac{\theta-r}{R}\right) \right)^{\mu^{WS}} \left( \exp\left(\frac{\theta}{r}\right) \right)^{ikR} e^{-ikr} {}_2F_1\left(a, b, c, 1 - \exp\left(\frac{\theta-r}{R}\right)\right).$$

The above result is the same as the one reported by Feng et al. [59] when  $D = 3$ .

## 5 Conclusions

In this paper, we have studied the Klein–Gordon equation with Hulthen plus deformed hyperbolic potential and obtain the bound state energy eigenvalues and the scattering state phase factor. Special cases of the potential are discussed in details. The present findings may have many applications in different branches of physics and quantum chemistry [67].

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## References

1. Dong, S.H.: Wave equations in higher dimensions. Springer, Dordrecht (2011)
2. Ikot, A.N., Obong, H.P., Hassanabadi, H., Sahehi, N., Thomas, O.S.: Solutions of D-dimensional Klein–Gordon equation for multiparameter exponential-type potential using supersymmetric quantum mechanics. *Indian J. Phys.* **69**, 649 (2015)
3. Lu, L.L., Yazarloo, B.H., Zarrinkamar, S., Liu, G., Hassanabadi, H.: Calculation of the oscillator strength for the Klein–Gordon equation with tietz potential. *Few Body Syst.* **53**, 573–581 (2012)
4. Oyewumi, K.J., Akinpelu, F.O., Agboola, A.D.: Exactly complete solutions of the pseudoharmonic potential in N-dimensions. *Int. J. Theor. Phys.* **47**, 1039 (2008)
5. Dong, S.H., Chen, C.Y., Cassou, M.L.: Generalized hypervirial and Blanchard’s recurrence relations for radial matrix elements. *J. Phys. B* **38**, 2211 (2007)
6. Garcia, M.G., de Catro, A.S.: Scattering and bound states of spinless particles in a mixed vector–scalar smooth step potential. *Ann. Phys.* **324**, 2372 (2009)
7. Alhaidari, A.D., Bahlouli, H., Al Hassan, A.: Dirac and Klein–Gordon equations with equal scalar and vector potentials. *Phys. Lett. A* **349**, 87 (2006)
8. Qiang, W.C., Dong, S.H.: Analytical approximations to the l-wave solutions of the Klein–Gordon equation for a second Pöschl–Teller like potential. *Phys. Lett. A* **322**, 285 (2006)

9. Sun, G.H., Dong, S.H.: Quantum information entropies of the eigenstates for a symmetrically trigonometric Rosen–Morse potential. *Phys. Scr.* **87**, 045003 (2013)
10. Arda, A., Sever, R.: Step-up and Step-down operators of a two-term molecular potential via Nikiforov–Uvarov method. *Few Body Syst.* **55**, 265 (2014)
11. Akcay, H., Sever, R.: Analytical solutions of Schrodinger equation for the diatomic molecular potentials with any angular momentum. *J. Math. Chem.* **50**, 1973 (2012)
12. Ikot, A.N., Obong, H.P., Owate, I.O., Onjeaju, M.C., Hassanabadi, H.: Scattering state of Klein-Gordon particles by q-parameter hyperbolic Pöschl-Teller potential. *Adv. High Energy. Phys.* **2015**, 1–7 (2015)
13. Nikiforov, A.F., Uvarov, V.B.: *Special functions of mathematical physics*. Birkhauser, Basel (1988)
14. Cooper, F., Khare, A., Sukhatme, U.: *Supersymmetry and quantum mechanics*. *Phys. Rep.* **251**, 267 (1995)
15. Ciftci, H., Hall, R.L., Saad, N.: Construction of exact solutions to eigenvalue problems by the asymptotic iteration method. *J. Phys. A Math. Gen.* **38**, 1147 (2005)
16. Ma, Z.Q., Gonzalez-Cisneros, A., Xu, B.W., Dong, S.H.: Energy spectrum of the trigonometric Rosen–Morse potential using an improved quantization rule. *Phys. Lett. A* **371**, 180 (2007)
17. Ikot, A.N., Maghsoodi, E., Zarrinkamar, S., Hassanabadi, H.: Supersymmetry quantum mechanics to Dirac equation with a modified Yukawa potential and a Yukawa tensor term. *Indian J. Phys.* **88**, 283 (2014)
18. Liu, J.Y., Zhang, L.H., Jia, C.S.: Calculation of the interaction potential energy curve and vibrational levels for the  $a^3\Sigma_u +$  state of  ${}^7\text{Li}_2$  molecule. *Phys. Lett. A* **377**, 1444 (2013)
19. Xie, X.J., Jia, C.S.: Solutions of the Klein–Gordon equation with the Morse potential energy model in higher spatial dimensions. *Phys. Scr.* **90**, 035207 (2015)
20. Nieto, M.M.: Hydrogen atom and relativistic pi-mesic atom in N-space dimensions. *Am. J. Phys.* **47**, 1067 (1979)
21. Dong, S.H., Gu, X.Y., Ma, Z.Q., Yu, J.: The Klein–Gordon equation with the coulomb potential in D dimensions. *Int. J. Mod. Phys. E* **12**, 555–565 (2003)
22. Ma, Z.Q., Dong, S.H., Gu, X.Y., Yu, J., Lozada-Cassou, M.: The Klein–Gordon equation with a coulomb plus scalar potential in D-dimensions. *Int. J. Mod. Phys. E* **13**, 597–610 (2004)
23. Qiang, W.C., Dong, S.H.: Analytical approximations to the l-wave solutions of the Klein–Gordon equation for a second Pöschl–Teller like potential. *Phys. Lett. A* **372**, 4789 (2008)
24. Dong, S.H., Dong, S.H., Bahl Ouli, H., Bezerra, V.B.: Algebraic approach to the Klein–Gordon equation with hyperbolic scarf potential. *Int. J. Mod. Phys. E* **20**, 55 (2011)
25. Isonguyo, C.N., Ituen, I.B., Ikot, A.N., Hassanabadi, H.: Solution of Klein Gordon equation for some diatomic molecules with new generalized morse-like potential using SUSYQM. *Bull. Korean Chem. Soc.* **35**, 3443 (2014)
26. Ikot, A.N., Ita, B.I., Awoga, O.A.: Exact solutions of the Klein–Gordon equation with Hylleraas potential. *Few Body Syst.* **53**, 539 (2012)
27. Antia, A.D., Ikot, A.N., Hassanabadi, H., Maghsoodi, E.: Bound state solutions of Klein-Gordon equation with Mobius square plus Yukawa potentials. *Indian J. Phys.* **87**, 1133 (2013)
28. Kratzer, A.: Die ultraroten Rotationsspektren der Halogenwasserstoffe. *Z. Phys.* **3**, 289 (1920)
29. Tezcan, C., Sever, R.: Exact solutions of the Schrödinger equation with position-dependent effective mass via general point canonical transformation. *J. Math. Chem.* **42**, 387 (2006)
30. Zhang, L.H., Li, X.P., Jia, C.S.: Approximate solutions of the Schrödinger equation with the generalized Morse potential model including the centrifugal term. *Int. J. Quantum Chem.* **111**, 1870 (2011)
31. Sever, R., Tezcan, C., Aktas, M., Yesiltas, O.: Exact solution of Schrodinger equation for pseudoharmonic potential. *J. Math. Chem.* **43**, 845 (2007)
32. Ikot, A.N., Hassanabadi, H., Obong, H.P., Chad-Umoren, Y.E., Isonguyo, C.N., Yazarloo, B.H.: Approximate solutions of Klein–Gordon equation with improved Manning–Rosen potential in D-dimensions using SUSYQM. *Chin. Phys. B* **23**, 120303 (2014)
33. Hassanabadi, H., Yazarloo, B.H.: Bound and scattering states of spinless particles under the generalized Pöschl-Teller potential. *Indian J. Phys.* **87**, 1017 (2013)
34. You, Y., Lu, F.L., Sun, D.S., Chen, C.Y.: Improved analytical approximations to the scattering solutions of the Schrödinger equation with a hyperbolic potential. *Commun. Theor. Phys.* **62**, 315 (2014)
35. Ikot, A.N., Obong, H.P., Olisa, J.D., Hassanabadi, H.: Scattering state of coupled Hulthen–Woods–Saxon potentials for the Duffin–Kemmer–Petiau equation with Pekeris approximation for the centrifugal term. *Z. Naturforsch* **70**, 185 (2015)
36. Arda, A., Sever, R.: Effective-mass Klein–Gordon–Yukawa problem for bound and scattering states. *J. Math. Phys.* **52**, 092101 (2011)
37. Chen, C.Y., Lu, F.L., Sun, D.S.: Relativistic scattering states of coulomb potential plus a new ring-shaped potential. *Commun. Theor. Phys.* **45**, 889 (2006)
38. You, Y., Lu, F.L., Sun, D.S., Chen, C.Y., Dong, S.H.: Solutions of the second Pöschl-Teller potential solved by an improved scheme to the centrifugal term. *Few Body Syst.* **54**, 2125 (2013)
39. Xu, Y., He, S., Jia, C.S.: Approximate analytical solutions of the Dirac equation with the Pöschl–Teller potential including the spin–orbit coupling term. *J. Phys. A Math. Gen.* **41**, 255302 (2008)
40. Wei, G.F., Dong, S.H.: A novel algebraic approach to spin symmetry for Dirac equation with scalar and vector second Pöschl–Teller potentials. *Eur. Phys. J. A* **43**, 185 (2010)
41. Ikot, A.N., Awoga, O.A., Hassanabadi, H., Maghsoodi, E.: Analytical approximate solution of Schrödinger equation in D dimensions with quadratic exponential-type potential for arbitrary l-State. *Commun. Theor. Phys.* **61**, 457 (2014)
42. Dong, S.H.: Relativistic treatment of spinless particles subject to a rotating Deng–Fan oscillator. *Commun. Theor. Phys.* **55**, 969 (2011)
43. Jia, C.S., Chen, T., Cui, L.G.: Approximate analytical solutions of the Dirac equation with the generalized Pöschl–Teller potential including the pseudo-centrifugal term. *Phys. Lett. A* **373**, 1621–1626 (2009)
44. Wang, Z., Long, Z.W., Long, C.Y., Wang, L.Z.: Analytical solutions of position-dependent mass Klein–Gordon equation for unequal scalar and vector Yukawa potentials. *Indian J. Phys.* **89**, 1059 (2012)

45. Ikot, A.N., Awoga, O.A., Antia, A.D., Hassanabadi, H., Maghsoodi, E.: Approximate solutions of D-dimensional Klein–Gordon equation with modified Hylleraas potential. *Few Body syst.* **54**, 2041 (2013)
46. Alhaidari, A.: Solutions of the nonrelativistic wave equation with position-dependent effective mass. *Phys. Rev. A* **66**, 042116 (2002)
47. Xiang, J.G.: Non-hypergeometric type of polynomials and solutions of Schrödinger equation with position-dependent mass. *Commun. Theor. Phys.* **56**, 235 (2011)
48. de Souza, A., Almeida, C.A.S.: Exact solvability of potentials with spatially dependent effective masses. *Phys. Lett. A* **275**, 25 (2000)
49. Jia, C.S., de Souza Dutra, A.: Extension of PT-symmetric quantum mechanics to the Dirac theory with position-dependent mass. *Ann. Phys.* **323**, 566–579 (2008)
50. Yu, J., Dong, S.H., Sun, G.H.: Series solutions of the Schrödinger equation with position-dependent mass for the morse potential. *Phys. Lett. A* **322**, 290 (2004)
51. Yu, J., Dong, S.H.: Exactly solvable potentials for the Schrödinger equation with spatially dependent mass. *Phys. Lett. A* **325**, 194 (2004)
52. Chen, X.Y., Chen, T., Jia, C.S.: Solutions of the Klein–Gordon equation with the improved Manning–Rosen potential energy model in D dimensions. *Eur. Phys. J. Plus* **129**, 75 (2014)
53. Tan, M.S., He, S., Jia, C.S.: Molecular spinless energies of the improved Rosen–Morse potential energy model in D dimensions. *Eur. Phys. J. Plus* **129**, 264 (2014)
54. Jiang, L., Yi, L.Z., Jia, C.S.: Exact solutions of the Schrödinger equation with position-dependent mass for some Hermitian and non-Hermitian potentials. *Phys. Lett. A* **345**, 279 (2005)
55. Panahi, H., Bakshii, Z.: Solvable potentials with position-dependent effective mass and constant mass Schrödinger equation. *Acta Phys. Polo. B* **4111**, (2010)
56. Kurniawan, A., Suparmi, A., Cari, C.: Approximate analytical solution of the Dirac equation with q-deformed hyperbolic Pöschl–Teller potential and trigonometric Scarf II non-central potential. *Chin. Phys. B* **24**, 030302 (2015)
57. Hassanabadi, H., Zarrinkamar, S., Rahimov, H.: Approximate solution of D-dimensional Klein–Gordon equation with Hulthén-type potential via SUSYQM. *Commun. Theor. Phys.* **56**, 423 (2011)
58. Chen, C.Y., Liu, L.F., Yuan, Y.: Scattering states of modified Pöschlben–Teller potential in D-dimension. *Chin. Phys. B* **21**, 030302 (2012)
59. Feng, W.G., Li, C.W., Ying, W.H., Yuan, L.Y.: The scattering states of the generalized Hulthén potential with an improved new approximate scheme for the centrifugal term. *Chin. Phys. B* **18**, 3663 (2009)
60. Ikot, A.N., Hassanabadi, H., Maghsoodi, E., Yazarloo, B.H.: Yukawa potential under relativistic spin and pseudospin symmetries with three tensor interactions. *Eur. J. Phys. Plus* **129**, 218 (2014)
61. Abramowitz, M., Stegun, I.A.: *Handbook of mathematical functions*. Dover, New York (1965)
62. Chen, C.Y., Sun, D.S., Liu, C.L., Lu, F.L.: Scattering states of n-dimensional hydrogen atom. *Acta Phys. Sin.* **52**, 781 (2003)
63. Chen, C.Y., Sun, D.S., Liu, C.L., Lu, F.L.: Approximate analytical solutions for scattering states of D-dimensional Hulthén potentials. *Commun. Theor. Phys.* **55**, 399 (2011)
64. Chen, C.Y., Sun, D.S., Lu, F.L.: Scattering states of the Klein-Gordon equation with coulomb-like scalar plus vector potentials in arbitrary dimension. *Phys. Lett. A* **330**, 424 (2004)
65. Chen, C.Y., Lu, F.L., Sun, D.S.: Exact solutions of scattering states for the s-wave Schrödinger equation with the manning-rosen potential. *Acta Phys. Sin.* **56**, 0204 (2007)
66. Woods, R.D., Saxon, D.S.: Diffuse surface optical model for nucleon–nuclei scattering. *Phys. Rev.* **95**, 577 (1964)
67. Ikot, A.N., Ibanga, E.J., Hassanabadi, H.: Scattering state of the multiparameter potential with an improved approximation for the centrifugal term in D-dimensions. *Int. J. Quantum Chem.* **116**, 81 (2016)