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Relativistic Symmetries of (D + 1) Dimensional Dirac Equation with Multiparameter Exponential-Type Potentials Using Supersymmetric Quantum Mechanics

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Abstract In this article, we have presented the approximate solution of the Dirac equation with multiparameter exponential-type potential in (D + 1) dimensions within the framework of spin and pseudospin symmetries. We have used the supersymmetric quantum mechanics formalism to obtain the energy eigenvalues and the corresponding wave function in terms of the Jacobi polynomials. We have discussed in details the special cases of this potential which is consistent with those found in the literature.

1 Introduction

For many years the Dirac equation has been a subject of interest to study relativistic spin-1/2 fermions in nuclear and particle physics. The problem in this case, just as that of linear Schrödinger equation, appears as an ordinary second-order differential equation, which has been extensively discussed in the literature by various analytical and numerical techniques. The supersymmetry quantum mechanics (SUSYQM) and the concept of shape invariance in physics [1] are one of the most useful methods which help the authors to study the solvable potential models in both relativistic and non-relativistic quantum mechanics [2]. The concept of SUSYQM allows one to determine the eigenfunctions and eigenvalues analytically for solvable potentials model using algebraic operator formulation without solving the Schrödinger-like differential equation by standard series method [3]. The concept of SUSYQM was first introduced by Witten [1] for the first time as the simplest supersymmetric model of the quantum field theory. The supersymmetry predicts the degenerate super partner a state corresponding to every physical particle state of the theory [4]. The relativistic Dirac equation which describes the motion of spin-1/2 particle has been used successfully in solving many physical problems of nuclear and high-energy physics [5–10]. For about 40 years ago, to explain the phenomena of quasi-degeneracy between single-nucleon states in heavy nuclei, pseudospin symmetry was put forward in nuclear physics [11]. In three dimension, these degenerative single-nucleon states are $(n, l, j = l + \frac{1}{2})$ and $(n - 1, l + 2, j = l + \frac{3}{2})$, where n, l and j are the radial, orbital and total angular quantum numbers of the single nucleon respectively. The states $(\tilde{n} = n - 1, \tilde{l} = l + 1, \tilde{j} = \tilde{l} \pm \frac{1}{2})$ are considered as the doublet structures, where $\tilde{l}$ and $\tilde{s} = \frac{1}{2}$ are the pseudo-orbital angular momentum and the pseudospin quantum numbers respectively. In relativistic mean field theory, it is observed that one of the characteristics is that an attractive scalar potential $S(r)$ and

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a repulsive vector potential $V(r)$ are nearly equal in magnitude but different in sign [12]. Ginocchio shows that the near equality $V(r) + S(r) \sim 0$, leads to pseudospin symmetry in nuclei (see Ref. [13] and related references [14–16]). Within the framework of the Dirac theory, the spin symmetry occurs when the difference of the potential between the repulsive Lorentz vector potential $V(r)$ and attractive Lorentz scalar potential $S(r)$ is a constant, that is, $\Delta(r) = V(r) - S(r) = const$ [13]. Meng et al. [17] have proved that exact pseudospin symmetry occurs in the Dirac equation when $\frac{d\Sigma(r)}{dr} = 0$, $\Sigma(r) = V(r) + S(r) = const$. However, in order to investigate the nuclear shell model, the study of spin and pseudospin symmetries of the Dirac equation have become an important area of research in nuclear physics [18–20]. These symmetries have been used successfully to explain the feature of deformed nuclei [20], superdeformation and establish an effective shell-model coupling scheme [21]. Different techniques have been employed for dealing with the Dirac equation with the motivated potentials. Such methods include Nikiforov–Uvraov (NU) method [22], asymptotic iteration method (AIM) [23], shape invariance and SUSYQM [24,25], factorization method [26] and others. However, the recent advances in the search for the solutions of Dirac equation with physical potential models will lead to the discovery of a new phenomenon in addition to the spin and pseudopsin symmetry discover many years ago in the nuclei of atom in the Dirac theory. The investigated potentials include Coulomb potentials [27], Manning–Rosen potential [28], Deng–Fan potential [29], Mobius potential [30], shifted Hulthen potential [31] and others [32]. Recently, Garcia-Martinez [33] proposed solvable multiparameter exponential-type potential of the form (see Fig. 1)

$$V(r) = \frac{Ae^{-2\eta r}}{1 - e^{-2\eta r}} + \frac{Be^{-2\eta r}}{\left(1 - e^{-2\eta r}\right)^2} + \frac{Ce^{-4\eta r}}{\left(1 - e^{-2\eta r}\right)^2} \tag{1}$$

where A, B and C are potential parameters, and η is the screening parameter. The choice of the A, B and C parameters lead to specific exponential-type potentials [34].

In this work, we intend to use SUSYQM method to solve the Dirac equation for scalar and vector multiparameter exponential potential with pseudospin and spin symmetries. Because of the generality of the problem, we have considered Dirac equation in D dimension. In this case we can study the problem in special cases such as 1D, 2D and 3D space. Furthermore, recently in many research lines of theoretical physics the models in which the spacetime has more dimensions than the four dimensions observable in our daily experience have been studied extensively. The most analyzed models are those related to string theory [35]. Also the scrutiny of the properties and solutions of higher dimensional general relativity has attracted a lot of attention (see Ref. [36] and references therein). In several of these research lines we need to know the classical properties of the higher dimensional spacetimes to examine different phenomena. Therefore the investigation of these classical properties is an active research field.

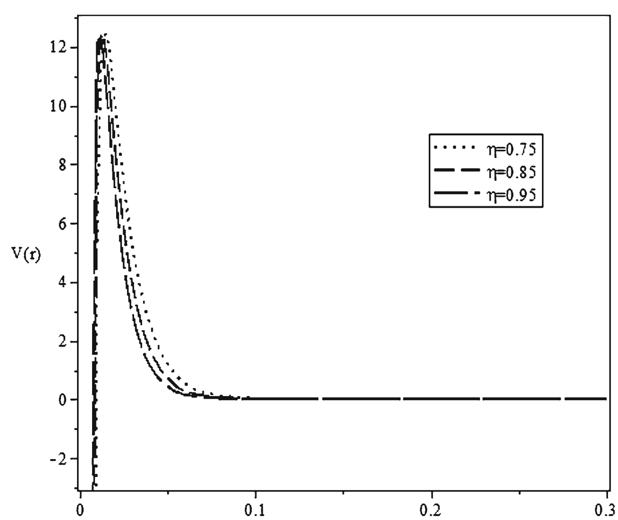

Fig. 1 The plot of the multiparameter exponential potential for A = 100, B = −50 and C = −50

2 The Dirac Equation in (D + 1)-Dimensions for Spin and Pseudospin Symmetries

The Dirac equation in $D + 1$ dimensions can be written as [36,37]

$$i \sum_{\mu=0}^{D} \gamma^{\mu} \left(\partial_{\mu} + ieA_{\mu}\right)\psi(x,t) = M\psi(x,t) \tag{2}$$

where M is the mass of the particle, and $D + 1$ matrices γ_{μ} satisfy the anticommutative relations:

$$\gamma^{\mu}\gamma^{\nu} + \gamma^{\nu}\gamma^{\mu} = 2\eta^{\mu\nu}1 \tag{3}$$

with

$$\eta^{\mu\nu} = \eta_{\mu\nu} = \begin{cases} \delta_{\mu\nu} & when\ \mu = 0 \\ -\delta_{\mu\nu} & when\ \mu \neq 0 \end{cases} \tag{4}$$

In the special case where only A_0 of A_{μ} is nonvanishing and spherically symmetric ($eA_0 = V(r)$, $A_a = 0\, when\, a \neq 0$), the Hamiltonian $H(\mathbf{x})$ of the system is expressed as

$$i\partial_0\psi(x,t) = H(x)\psi(x,t),\ H(x) = \sum_{c=1} \gamma^{0}\gamma^{c} p_c + V(r) + \gamma^{0}M \tag{5}$$

$$p_c = -i\partial_c = -i\frac{\partial}{\partial X^c}, c \in [1, D] \tag{6}$$

It is known that the spinor wave functions as well as those for the total angular momentum are different for $D = 2N + 1$ and $D = 2N$. In the case of **SO** (**2N+1**) (when D = 2N + 1) we can define

$$\gamma^{0} = \sigma_3 \times 1, \gamma^{a} = (i\sigma_2) \times \alpha_a \qquad a \in [1, 2N + 1] \tag{7}$$

with the Pauli matrix σ_a, the 2^N-dimensional unit matrix **1** and the $(2N{+}1)$ matrices α_a satisfying the following anticommutative relations:

$$\alpha_b\alpha_a + \alpha_a\alpha_b = 2\delta_{ab}1, \quad b, a = 1, 2, \ldots, (2N + 1) \tag{8}$$

The dimensions of $\alpha_a$matrices are 2^N. Thus the spinor operator S_{ab} becomes a block matrix

$$S_{ab} = 1 \times \bar{S}_{ab}, \quad \bar{S}_{ab} = -i\frac{\alpha_a\alpha_b}{2} \tag{9}$$

The relation between S_{ab} and $\bar{S}_{ab}$ is very similar to that between the spinor operators for the Dirac spinors and for the Pauli spinors. The operator κ becomes

$$\kappa = \sigma_3 \times \bar{\kappa}, \bar{\kappa} = -i\sum_{a<b} \alpha_a\alpha_b L_{ab} + \frac{D-1}{2} \tag{10}$$

In the presence of attractive scalar potential $S(r)$, repulsive vector potential $V(r)$ and the rest mass m in the relativistic unit $\hbar = c = 1$ the (D + 1)-dimensional Dirac equation within is written as,

$$H\psi(r) = E_{n\kappa}\psi(r), \tag{11}$$

where

$$H = \sum_{j=1}^{n} \hat{\alpha}_j p_j + \hat{\beta}\,[m + S(r)] + V(r), \tag{12}$$

$E_{n\kappa}$ denotes the relativistic energy, $\{\hat{\alpha}_j\}$ and $\hat{\beta}$ are the Dirac matrices satisfying anti-commutation relations

$$\begin{aligned} &\hat{\alpha}_j\hat{\alpha}_k + \hat{\alpha}_k\hat{\alpha}_j = 2\delta_{jk}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \\ &\hat{\alpha}_j\hat{\beta} + \hat{\beta}\hat{\alpha}_j = 0, \\ &\hat{\alpha}_j^2 = \hat{\beta}^2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \end{aligned} \tag{13}$$

and

$$p_j = -i\frac{\partial}{\partial x_j}, \quad 1 \le j \le D. \tag{14}$$

We define the orbital angular momentum operator L_{jk}, the spinor operator $\hat{S}_{jk}$ and the total angular momentum operators $\hat{J}_{jk}$ as follows

$$\begin{aligned} L_{jk} &= ix_j\frac{\partial}{\partial x_k} - ix_k\frac{\partial}{\partial x_j}, \quad S_{jk} = i\hat{\alpha}_j\hat{\alpha}_k/2, \quad J_{jk} = L_{jk} + S_{jk}, \\ L^2 &= \sum_{j\prec k}^{D} L_{jk}^2, \quad S^2 = \sum_{j\prec k}^{D} S_{jk}^2, \quad J^2 = \sum_{j\prec k}^{D} J_{jk}^2. \quad 1 \le j \prec k \le D. \end{aligned} \tag{15}$$

For a spherically symmetric potential, the total angular momentum operator J_{jk} and spin-orbit operator $\hat{K} = -\hat{\beta}\left(J^2 - L^2 - S^2 + \frac{(D-1)}{2}\right)$ commute with the Dirac Hamiltonian. Thus, for a given total angular momentum j, the eigenvalues of $\hat{K}$ are $\kappa = -\left(j + \frac{(D-2)}{2}\right)$, for aligned spin $j = l + \frac{1}{2}$ and, $\kappa = \left(j + \frac{(D-2)}{2}\right)$ for unaligned spin $j = l - \frac{1}{2}$, respectively. In the hyper-spherical coordinates [26,37], we have

$$\begin{aligned} x_1 &= r\cos\theta_1, \\ x_\alpha &= r\sin\theta_1 \ldots \sin\theta_{\alpha-1}\cos\phi, \qquad 2 \le \alpha \le D-1, \\ x_D &= r\sin\theta_1 \ldots \sin\theta_{D-2}\sin\phi, \end{aligned} \tag{16}$$

and the volume element defined over the configuration space is

$$\prod_{j=1}^{D} dx_j = r^{D-1}drd\Omega, \tag{17}$$

where

$$d\Omega = \prod_{j=1}^{D-1}(\sin\theta_j)^{j-1}d\theta_j \tag{18}$$

and $0 \le r \le \infty$, $0 \le \theta_k \le \pi$ for $k = 1, 2, \ldots, D-2, 0 \le \phi \le 2\pi$. Thus, the wave functions in the hyper-spherical coordinates with the hyper-radial quantum number n and spin-orbit quantum number κ can be written as

$$\psi_{n\kappa}(r, \Omega_D) = r^{-\left(\frac{D-1}{2}\right)}\begin{pmatrix} F_{n\kappa}(r) & Y_{jm}^{l}(\Omega_D) \\ iG_{n\kappa}(r) & Y_{jm}^{\tilde{l}}(\Omega_D)\end{pmatrix}, \tag{19}$$

where $F_{n\kappa}(r)$ and $G_{n\kappa}(r)$ are the radial wave functions of the upper and lower spinors, respectively. In Eq. (19), $Y_{jm}^{l}(\Omega_D)$ and $Y_{jm}^{\tilde{l}}(\Omega_D)$ denote the hyper-spherical harmonics coupled to the total angular momentum j. The orbital angular momentum of the quasi-spin symmetry limit and the pseudo-orbital angular momentum of the quasi-pseudospin symmetry limit are respectively denoted by l and $\tilde{l}$. Substitution of Eq. (19) into Eq. (11) as well as using Eq. (12), give

$$\left(\frac{d}{dr} + \frac{\kappa}{r}\right)F_{n\kappa}(r) = (M + E_{n\kappa} - \Delta(r))\,G_{n\kappa}(r) \tag{20}$$

$$\left(\frac{d}{dr} - \frac{\kappa}{r}\right)G_{n\kappa}(r) = (M - E_{n\kappa} + \Sigma(r))\,F_{n\kappa}(r) \tag{21}$$

where $\Delta(r) = V(r) - S(r)$, $\Sigma(r) = V(r) + S(r)$ with $\kappa = \pm\frac{(2l+D-1)}{2}$.

Eliminating one component in favor of the other, we recover the decouples equations

$$\left\{\frac{d^2}{dr^2}-\frac{\kappa(\kappa+1)}{r^2}-[M+E_{n\kappa}-\Delta(r)][M-E_{n\kappa}+\Sigma(r)]+\frac{\frac{d\Delta(r)}{dr}\left(\frac{d}{dr}+\frac{\kappa}{r}\right)}{[M+E_{n\kappa}-\Delta(r)]}\right\}F_{n\kappa}(r)=0, \quad (22)$$

$$\left\{\frac{d^2}{dr^2}-\frac{\kappa(\kappa-1)}{r^2}-[M+E_{n\kappa}-\Delta(r)][M-E_{n\kappa}+\Sigma(r)]+\frac{\frac{d}{dr}\Sigma(r)\left(\frac{d}{dr}-\frac{\kappa}{r}\right)}{[M-E_{n\kappa}+\Sigma(r)]}\right\}G_{n\kappa}(r)=0. \quad (23)$$

The radial wave functions have to vanish at the origin and tend to zero for extremely large r values. At this stage, we take $\Delta(r)$ or $\Sigma(r)$ as the multiparameter potential. Before proceeding further, it should be clearly mentioned that Eqs. (22) and (23) can be only exactly solved for $\kappa = 0, -1$ and $\kappa = 0, 1$, respectively.

2.1 Pseudospin Symmetry for the Multiparameter Potential

We take the difference in the potentials as

$$\Delta(r)=\frac{Ae^{-2\eta r}}{\left(1-e^{-2\eta r}\right)}+\frac{Be^{-2\eta r}}{\left(1-e^{-2\eta r}\right)^2}+\frac{Ce^{-4\eta r}}{\left(1-e^{-2\eta r}\right)^2} \quad (24)$$

and the spin orbit coupling term as [37] (see Fig. 2)

$$\frac{1}{r^2}\approx\frac{4\eta^2e^{-2\eta r}}{\left(1-e^{-2\eta r}\right)^2} \quad (25)$$

$$\frac{1}{r^2}\approx\frac{4\eta^2e^{-\eta r}}{\left(1-e^{-2\eta r}\right)^2} \quad (26)$$

Substituting Eqs. (24–26) into Eq. (23) and after a little algebraic with absolute care, we obtain the second order Schrödinger-like equation

$$-\frac{dG^{ps}_{n,\kappa}(r)}{dr^2}+V_{eff}(r)G^{ps}_{n,\kappa}=\tilde{E}^{ps}_{n,\kappa}G^{ps}_{n,\kappa} \quad (27)$$

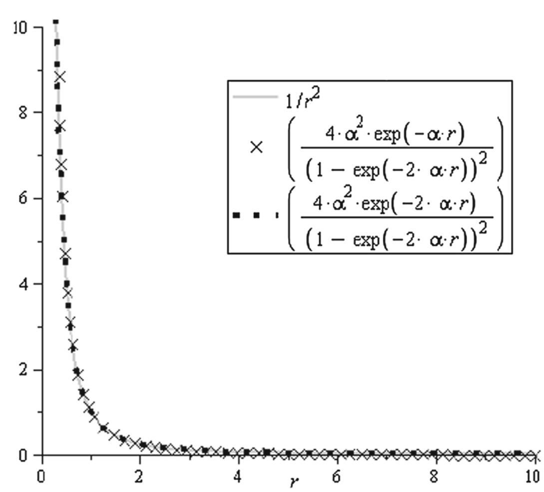

Fig. 2 $\frac{1}{r^2}$ and its approximations for $\alpha = 0.01$

where,

$$V_{eff}(r) = \frac{\rho_1^{ps} e^{-4\eta r} + \rho_2^{ps} e^{-2\eta r}}{\left(1 - e^{-2\eta r}\right)^2} \tag{28}$$

$$\rho_1^{ps} = \left(M - E_{n,\kappa}^{ps} + C_{ps}\right)(A - C) \tag{29}$$

$$\rho_2^{ps} = 4\eta^2 \kappa(\kappa - 1) - \left(M - E_{n,\kappa}^{ps} + C_{ps}\right)(A + B) \tag{30}$$

$$\tilde{E}_{n,\kappa}^{ps} = M^2 - ME_{n,\kappa}^{ps} + MC_{ps} + ME_{n,\kappa}^{ps} - (E_{n,\kappa}^{ps})^2 + E_{n,\kappa}^{ps} C_{ps} \tag{31}$$

Using the concept of the SUSYQM [24,25], we proposed the superpotential of the form,

$$W(r) = \frac{Q_1^{ps} e^{-2\eta r}}{\left(1 - e^{-2\eta r}\right)} + Q_2^{ps} \tag{32}$$

The superpotential $W(r)$ satisfies the Riccati equation,

$$W^2(r) \mp W'(r) = V_{eff}(r) - \tilde{E}_{0,\kappa}^{ps} \tag{33}$$

Substituting Eq. (32) into Eq. (33), we obtained the following three set of relationship

$$Q_1^{ps} = -\eta \pm \eta \sqrt{1 + \frac{\left(\rho_1^{ps} + \rho_2^{ps}\right)}{\eta^2}} \tag{34}$$

$$Q_2^{ps} = \frac{\left(Q_1^{ps}\right)^2 - \rho_1^{ps}}{2Q_1^{ps}} \tag{35}$$

$$\tilde{E}_{0,\kappa} = -\left(Q_2^{ps}\right)^2 \tag{36}$$

We can obtain the supersymmetric partner potentials as

$$V_+(r) = \frac{Q_1^{ps}\left(Q_1^{ps} - 2\eta\right) e^{-4\eta r}}{\left(1 - e^{-2\eta r}\right)^2} + \frac{\frac{2Q_1^{ps}(Q_1^{ps} - \rho_1^{ps})e^{-2\eta r}}{2Q_1^{ps}}}{\left(1 - e^{-2\eta r}\right)} + \left(\frac{Q_1^{ps} - \rho_1^{ps}}{2Q_1^{ps}}\right)^2 \tag{37}$$

$$V_-(r) = \frac{Q_1^{ps}\left(Q_1^{ps} + 2\eta\right) e^{-4\eta r}}{\left(1 - e^{-2\eta r}\right)^2} + \frac{\frac{2Q_1^{ps}(Q_1^{ps} - \rho_1^{ps})e^{-2\eta r}}{2Q_1^{ps}}}{\left(1 - e^{-2\eta r}\right)} + \left(\frac{Q_1^{ps} - \rho_1^{ps}}{2Q_1^{ps}}\right)^2 \tag{38}$$

The shape invariancy in this case holds via the mapping

$$a_n = f(a_0) = a_0 - 2n\eta = Q_1 - 2n\eta. \tag{39}$$

and therefore the residuals are

$$R(a_1) = \left(\frac{(a_0)^2 - \rho_1^{ps}}{2a_0}\right)^2 - \left(\frac{(a_1)^2 - \rho_1^{ps}}{2a_1}\right)^2, \tag{40}$$

$$R(a_2) = \left(\frac{(a_1)^2 - \rho_1^{ps}}{2a_1}\right)^2 - \left(\frac{(a_2)^2 - \rho_1^{ps}}{2a_2}\right)^2, \tag{41}$$

$$\vdots$$

$$R(a_n) = \left(\frac{(a_{n-1})^2 - \rho_1^{ps}}{2a_{n-1}}\right)^2 - \left(\frac{(a_n)^2 - \rho_1^{ps}}{2a_n}\right)^2 \tag{42}$$

The energy eigenvalues can be obtained as follows

$$\tilde{E}_{n,\kappa}^{ps} = \tilde{E}_{n,\kappa}^{-} + \tilde{E}_{0,\kappa} \quad , \tag{43}$$

where $\tilde{E}_{n,\kappa}^{-}$ are defined as follows:

$$\tilde{E}_{n,\kappa}^{-} = \sum_{k=1}^{n} R(a_k) = \left(\frac{(a_0)^2 - \rho_1^{ps}}{2a_0}\right)^2 - \left(\frac{(a_n)^2 - \rho_1^{ps}}{2a_n}\right)^2, \tag{44}$$

Using Eqs. (34–36) and (44), we obtain the complicated transcendental energy equation for the multiparameter exponential-type potential for the pseudopsin symmetry in the Dirac theory as,

$$M^2 - ME_{n,\kappa}^{ps} + MC_{ps} + ME_{n,\kappa}^{ps} - (E_{n,\kappa}^{ps})^2 + C_{ps}E_{n,\kappa}^{ps} = -\frac{1}{4}\left[\frac{\left(M - E_{n,\kappa}^{ps} + C_{ps}\right)(A - C)}{2\eta\,(n+\sigma)} - 2\eta\,(n+\sigma)\right]^2 \tag{45}$$

where,

$$\sigma = \frac{1}{2}\left(1 + \sqrt{1 + \frac{\left(4\eta^2\kappa(\kappa-1) - \left(M - E_{n,\kappa}^{ps} + C_{ps}\right)(B + C)\right)}{\eta^2}}\right) \tag{46}$$

The corresponding lower component of the wave function is obtain as follow

$$G_{n\kappa}^{ps}(r) = N_{n\kappa}^{ps} e^{-2\eta\sqrt{\lambda_3^{ps}}\,r}(1 - e^{-2\eta r})^{1/2+\sqrt{\lambda_1^{ps}-\lambda_2^{ps}+\lambda_3^{ps}+1/4}} P_n^{\left(2\sqrt{\lambda_3^{ps}},\,2\sqrt{\lambda_1^{ps}-\lambda_2^{ps}+\kappa_3^{ps}+1/4}\right)}(1 - 2e^{-2\eta r}), \tag{47}$$

with

$$\lambda_1^{ps} = \frac{1}{4\eta^2}\left(\rho_1^{ps} + (M + E_{n\kappa}^{ps})(M - E_{n\kappa}^{ps} + C_{ps})\right), \tag{48}$$

$$\lambda_2^{ps} = \frac{1}{4\eta^2}\left(-\rho_2^{ps} + 2(M + E_{n\kappa}^{ps})(M - E_{n\kappa}^{ps} + C_{ps})\right), \tag{49}$$

$$\lambda_3^{ps} = \frac{1}{4\eta^2}\left((M + E_{n\kappa}^{ps})(M - E_{n\kappa}^{ps} + C_{ps})\right). \tag{50}$$

where $N_{n\kappa}$ is the normalization constant. For the upper component, we can simply use

$$F_{n\kappa}^{ps}(r) = \frac{1}{M - E_{n\kappa}^{ps} + C_{ps}}\left(\frac{d}{dr} - \frac{\kappa}{r} + U(r)\right)G_{n\kappa}^{ps}(r). \tag{51}$$

2.2 Spin Symmetry for the Multiparameter Potential

In the case of the spin symmetry, we take the difference in potential as the multiparameter exponential-type potential. However, in order to avoid repetition the positive spin symmetry case can be obtained directly from the pseudospin symmetric solution via the following transformations:

$$\begin{gathered} G_{n,\kappa}^{ps} \leftrightarrow F_{n,\kappa}^{s}, \quad V(r) \to -V(r), \\ \kappa \to \kappa + 1, \quad E_{n,\kappa}^{ps} \to -E_{n,\kappa}^{s}, \quad C_{ps} \to -C_s \end{gathered} \tag{52}$$

Substituting Eq. (52) into Eq. (45) yields the energy equation for the spin symmetry with the multiparameter exponential potential as,

$$M^2 + ME_{n,\kappa}^{s} - MC_s - ME_{n,\kappa}^{s} - (E_{n,\kappa}^{s})^2 + C_sE_{n,\kappa}^{s} = -\frac{1}{4}\left[\frac{\left(M + E_{n,\kappa}^{s} - C_s\right)(A - C)}{2\eta\,(n+\sigma)} - 2\eta\,(n+\sigma)\right]^2 \tag{53}$$

where,

$$\sigma = \frac{1}{2}\left(1 + \sqrt{1 + \frac{\left(4\eta^2\kappa(\kappa+1) - \left(M + E_{n,\kappa}^{s} - C_s\right)(B + C)\right)}{\eta^2}}\right) \tag{54}$$

The corresponding upper component of the wave function for the multiparameter exponential-type potential becomes,

$$F_{n\kappa}^{s}(r)=D_{n\kappa}^{s}e^{-2\eta\sqrt{\alpha_3^s}r}(1-e^{-2\eta r})^{1/2+\sqrt{\alpha_1^s-\alpha_2^s+\alpha_3^s+1/4}}P_n^{\left(2\sqrt{\alpha_3^s},2\sqrt{\alpha_1^s-\alpha_2^s+\alpha_3^s+1/4}\right)}(1-2e^{-2\eta r}), \tag{55}$$

with

$$\alpha_1^s=\frac{1}{4\eta^2}\left(\chi_1^s+(M-E_{n\kappa}^s)(M+E_{n\kappa}^s-C_s)\right), \tag{56}$$

$$\alpha_2^s=\frac{1}{4\eta^2}\left(-\chi_2^s+2(M-E_{n\kappa}^s)(M+E_{n\kappa}^s-C_s)\right), \tag{57}$$

$$\alpha_3^s=\frac{1}{4\eta^2}\left((M-E_{n\kappa}^s)(M+E_{n\kappa}^s-C_s)\right). \tag{58}$$

The lower component of the wave function can be obtained as follows:

$$G_{n\kappa}^{s}(r)=\frac{1}{M+E_{n\kappa}^s-C_s}\left(\frac{d}{dr}+\frac{\kappa}{r}-U(r)\right)F_{n\kappa}^{s}(r). \tag{59}$$

3 Discussions and Special Cases

In this section we will investigate the energy eigenvalues and corresponding eigenfunctions of the special cases of the multiparameter exponential-type potential.

3.1 Hulthen Potential

The Hulthén potential is very important in atom and molecular fields [39]. This potential has been used to explain the electronic properties of F-colour centre in alkali halides [38]. In this special case, we choose $B=C=0$, $A=-Ze^2\delta$, $\eta=\frac{\delta}{2}$, where δ is the screening parameter and the multiparameter exponential-type potential turns into the Hulthen potential as,

$$V(r)=\frac{-Ze^2\delta e^{-\delta r}}{1-e^{-\delta r}} \tag{60}$$

From Eq. (45), we obtained the energy spectrum for Hulthen potential including Coulomb and Yukawa tensor interactions within the framework of pseudopsin symmetry as,

$$M^2-ME_{n,\kappa}^{ps}+MC_{ps}+ME_{n,\kappa}^{ps}-(E_{n,\kappa}^{ps})^2+C_{ps}E_{n,\kappa}^{ps}=-\frac{1}{4}\left[-\frac{\left(M-E_{n,\kappa}^{ps}+C_{ps}\right)Ze^2\delta}{2\eta\,(n+\sigma)}-2\eta\,(n+\sigma)\right]^2 \tag{61}$$

where,

$$\sigma=\frac{1}{2}\left(1+\sqrt{1+4\kappa(\kappa-1)}\right) \tag{62}$$

and the corresponding wave function becomes,

$$G_{n\kappa}^{ps}(r)=N_{n\kappa}^{ps}e^{-\delta\sqrt{\lambda_3^{ps}}r}(1-e^{-\delta r})^{1/2+\sqrt{\lambda_1^{ps}-\lambda_2^{ps}+\lambda_3^{ps}+1/4}}P_n^{\left(2\sqrt{\lambda_3^{ps}},2\sqrt{\lambda_1^{ps}-\lambda_2^{ps}+\kappa_3^{ps}+1/4}\right)}(1-2e^{-\delta r}), \tag{63}$$

with

$$\lambda_1^{ps}=\frac{1}{\delta^2}\left(-\left(M-E_{n,\kappa}^{ps}+C_{ps}\right)Ze^2\delta+(M+E_{n\kappa}^{ps})(M-E_{n\kappa}^{ps}+C_{ps})\right), \tag{64}$$

$$\lambda_2^{ps}=\frac{1}{\delta^2}\left(-\left(\delta^2\kappa(\kappa-1)+\left(M-E_{n,\kappa}^{ps}+C_{ps}\right)Ze^2\delta\right)+2(M+E_{n\kappa}^{ps})(M-E_{n\kappa}^{ps}+C_{ps})\right), \tag{65}$$

$$\lambda_3^{ps}=\frac{1}{\delta^2}\left((M+E_{n\kappa}^{ps})(M-E_{n\kappa}^{ps}+C_{ps})\right). \tag{66}$$

3.2 Manning–Rosen Potential

Manning–Rosen potential is one of the short range potential and it has been used to describe the diatomic molecular vibration [28]. The special case of Manning–Rosen potential is obtained from the multiparameter potential by considering, $B = 0$, $A = -\frac{V_0}{b^2}$, $C = \frac{\alpha(\alpha-1)}{b^2}$ and $\eta = \frac{1}{2b}$. Thus, the Manning–Rosen potential becomes,

$$V(r) = \frac{1}{b^2}\left(\frac{\alpha(\alpha-1)e^{-\frac{2r}{b}}}{\left(1-e^{-\frac{2r}{b}}\right)^2} - \frac{V_0 e^{-\frac{r}{b}}}{1-e^{-\frac{r}{b}}}\right) \tag{67}$$

Substituting these parameters into Eq. (45), we obtain the energy eigenvalues for the Manning–Rosen follow:

$$\begin{aligned} &M^2 - ME_{n,\kappa}^{ps} + MC_{ps} + ME_{n,\kappa}^{ps} - (E_{n,\kappa}^{ps})^2 + C_{ps}E_{n,\kappa}^{ps} \\ &= -\frac{1}{4b^2}\left[\frac{-\left(M - E_{n,\kappa}^{ps} + C_{ps}\right)(V_0 + \alpha(\alpha-1))}{(n+\sigma)} - (n+\sigma)\right]^2 \end{aligned} \tag{68}$$

where,

$$\sigma = \frac{1}{2}\left(1 + \sqrt{1 + 4\kappa(\kappa-1) - 4\alpha(\alpha-1)\left(M - E_{n,\kappa}^{ps} + C_{ps}\right)}\right) \tag{69}$$

and the corresponding wave function becomes,

$$G_{n\kappa}^{ps}(r) = N_{n\kappa}^{ps} e^{-\frac{1}{b}\sqrt{\chi_3^{ps}}r}(1-e^{-\frac{r}{b}})^{1/2+\sqrt{\chi_1^{ps}-\chi_2^{ps}+\chi_3^{ps}+1/4}} P_n^{\left(2\sqrt{\chi_3^{ps}},2\sqrt{\chi_1^{ps}-\chi_2^{ps}+\chi_3^{ps}+1/4}\right)}(1-2e^{-\frac{r}{b}}), \tag{70}$$

with

$$\chi_1^{ps} = b^2\left(-\frac{\left(M - E_{n,\kappa}^{ps} + C_{ps}\right)(V_0 + \alpha(\alpha-1))}{b^2} + (M + E_{n\kappa}^{ps})(M - E_{n\kappa}^{ps} + C_{ps})\right), \tag{71}$$

$$\chi_2^{ps} = b^2\left(-\left(\frac{1}{b^2}\kappa(\kappa-1) + \frac{\left(M - E_{n,\kappa}^{ps} + C_{ps}\right)V_0}{b^2}\right) + 2(M + E_{n\kappa}^{ps})(M - E_{n\kappa}^{ps} + C_{ps})\right), \tag{72}$$

$$\chi_3^{ps} = b^2\left((M + E_{n\kappa}^{ps})(M - E_{n\kappa}^{ps} + C_{ps})\right). \tag{73}$$

3.3 Eckart Potential

The Eckart potential is one of the solvable exponential-type potential in quantum mechanics since it has been introduced by Eckart [40] in 1930. Eckart potential is one of most important potential model in physics and chemical physics [40] and the bound state solution of the Dirac and Schrödinger equation and the scattering states of this potential has been investigated in Refs. [41] and [42], respectively. The Eckart potential is obtained from the multiparameter potential by setting $A = -\alpha$, $B = \beta$, $C = 0$ and $\eta = \frac{2}{a}$ as,

$$V(r) = -\frac{\alpha e^{-\frac{r}{a}}}{1-e^{-\frac{r}{a}}} + \frac{\beta e^{-\frac{r}{a}}}{\left(1-e^{-\frac{r}{a}}\right)^2} \tag{74}$$

The energy equation and the corresponding wave function for the Eckart potential with the generalized tensor equation becomes,

$$M^2 - ME_{n,\kappa}^{ps} + MC_{ps} + ME_{n,\kappa}^{ps} - (E_{n,\kappa}^{ps})^2 + C_{ps}E_{n,\kappa}^{ps} = -\frac{1}{4}\left[\frac{-\left(M - E_{n,\kappa}^{ps} + C_{ps}\right)\alpha}{(4/a)(n+\sigma)} - (4/a)(n+\sigma)\right]^2 \tag{75}$$

$$G_{n\kappa}^{ps}(r) = N_{n\kappa}^{ps} e^{-2\eta\sqrt{\lambda_3^{ps}}r}(1-e^{-2\eta r})^{1/2+\sqrt{\lambda_1^{ps}-\lambda_2^{ps}+\lambda_3^{ps}+1/4}} P_n^{\left(2\sqrt{\lambda_3^{ps}},2\sqrt{\lambda_1^{ps}-\lambda_2^{ps}+\kappa_3^{ps}+1/4}\right)}(1-2e^{-2\eta r}), \tag{76}$$

where

$$\sigma = \frac{1}{2}\left(1+\sqrt{1+\frac{\left((4/a)^2\kappa(\kappa-1)-\left(M-E_{n,\kappa}^{ps}+C_{ps}\right)\beta\right)}{(2/a)^2}}\right) \tag{77}$$

$$\lambda_1^{ps} = \left(\frac{a}{4}\right)^2\left(-\left(M-E_{n,\kappa}^{ps}+C_{ps}\right)\alpha+(M+E_{n\kappa}^{ps})(M-E_{n\kappa}^{ps}+C_{ps})\right), \tag{78}$$

$$\lambda_2^{ps} = \left(\frac{a}{4}\right)^2\left(-\left((4/a)^2\kappa(\kappa-1)-\left(M-E_{n,\kappa}^{ps}+C_{ps}\right)(\beta-\alpha)\right)+2(M+E_{n\kappa}^{ps})(M-E_{n\kappa}^{ps}+C_{ps})\right), \tag{79}$$

$$\lambda_3^{ps} = \left(\frac{a}{4}\right)^2\left((M+E_{n\kappa}^{ps})(M-E_{n\kappa}^{ps}+C_{ps})\right). \tag{80}$$

where $N_{n\kappa}$ is the normalization constant.

3.4 Deng–Fan potential

The Deng–Fan potential [43,44] discovery more than 50 years ago is the simplest modified form of Morse potential and is related to the Manning–Rosen and Eckart potentials. This potential is used to describe diatomic molecular energy spectra and electromagnetic transition. It is usually regarded as the true inter nuclear potential in diatomic molecules. In this case, the choice of $A = -2bD_e$, $B = 0$, $C = D_e b^2$ and $\eta = \frac{\alpha}{2}$, where D_e is the dissociation energy, turns the multiparameter exponential-type potential into the Deng–Fan potential from Eq. (6) as [45]

$$V(r) = \frac{-2bD_e e^{-\alpha r}}{1-e^{-\alpha r}} + \frac{D_e b^2 e^{-2\alpha r}}{\left(1-e^{-2\alpha r}\right)^2} \tag{81}$$

Substituting these parameters into Eqs. (45) and (47), we obtain the energy spectrum and the corresponding eigenfunction for the Deng–Fan potential in the Dirac theory as follow:

$$\begin{aligned} &M^2 - ME_{n,\kappa}^{ps} + MC_{ps} + ME_{n,\kappa}^{ps} - (E_{n,\kappa}^{ps})^2 + C_{ps}E_{n,\kappa}^{ps} \\ &= -\frac{1}{4}\left[\frac{-\left(M-E_{n,\kappa}^{ps}+C_{ps}\right)D_e b(b+2)}{\alpha\,(n+\sigma)} - \alpha\,(n+\sigma)\right]^2 \end{aligned} \tag{82}$$

$$\begin{aligned} G_{n\kappa}^{ps}(r) &= N_{n\kappa}^{ps} e^{-2\eta\sqrt{\lambda_3^{ps}}r}(1-e^{-2\eta r})^{1/2+\sqrt{\lambda_1^{ps}-\lambda_2^{ps}+\lambda_3^{ps}+1/4}} \\ &P_n^{\left(2\sqrt{\lambda_3^{ps}},2\sqrt{\lambda_1^{ps}-\lambda_2^{ps}+\kappa_3^{ps}+1/4}\right)}(1-2e^{-2\eta r}), \end{aligned} \tag{83}$$

where,

$$\sigma = \frac{1}{2}\left(1+\sqrt{1+\frac{\left(\alpha^2\kappa(\kappa-1)-\left(M-E_{n,\kappa}^{ps}+C_{ps}\right)D_e b^2\right)}{(\alpha/2)^2}}\right) \tag{84}$$

and

$$\lambda_1^{ps} = \frac{1}{\alpha^2}\left(-\left(M-E_{n,\kappa}^{ps}+C_{ps}\right)D_e b(b+2)+(M+E_{n\kappa}^{ps})(M-E_{n\kappa}^{ps}+C_{ps})\right), \tag{85}$$

$$\lambda_2^{ps} = \frac{1}{\alpha^2}\left(-\left(\alpha^2\kappa(\kappa-1)+2\left(M-E_{n,\kappa}^{ps}+C_{ps}\right)D_e b\right)+2(M+E_{n\kappa}^{ps})(M-E_{n\kappa}^{ps}+C_{ps})\right), \tag{86}$$

$$\lambda_3^{ps} = \frac{1}{\alpha^2}\left((M+E_{n\kappa}^{ps})(M-E_{n\kappa}^{ps}+C_{ps})\right). \tag{87}$$

4 Conclusions

The approximate solution of the Dirac equation for mutiparameter exponential-type potential within the framework of spin and pseudospin symmetry limits is obtained using the supersymmetric quantum mechanics formalism. We have obtained explicitly the energy levels in a closed form and the corresponding wave functions expressed in terms of the Jacobi polynomials for this potential within the spin and pseudospin symmetry limits. We deduced well known potentials by adjusting the potential parameter of the multiparameter exponential-type potential. The results of our work will find many applications in both nuclear, Hadron and high energy physics.

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