Few-Body Syst (2015) 56:369–374
DOI 10.1007/s00601-015-0966-z

Michele Viviani · Tobias Frederico · Giovanni Salmè

Numerical Studies of the Zero-Energy Bethe–Salpeter Equation in Minkowski Space

Received: 31 October 2014 / Accepted: 10 February 2015 / Published online: 1 March 2015

Abstract The inhomogeneous Bethe–Salpeter equation describing the zero-energy scattering of a system composed by two massive scalars exchanging a massive scalar is numerically investigated in ladder approximation, directly in Minkowski space. The solution is obtained by using the Nakanishi integral representation, as performed in Frederico et al. (Phys Rev D 89:016010, 2014) where the method was successfully applied to bound states. The scattering lengths are quantitatively investigated and the results compared with the corresponding ones present in literature.

1 Introduction

In the last years, the approach for solving the homogeneous Bethe–Salpeter equation (BSE) [1] directly in Minkowski space has made a substantial step forward due to the so-called Nakanishi perturbation-theory integral representation (PTIR) of the n-leg transition amplitudes [2,3]. In such an approach, a generic multi-leg transition amplitude, expressed in general through an infinite series of Feynman diagrams can be written as the folding of a non singular weight function, the so-called Nakanishi weight function, divided by a denominator containing the analytic structure of the amplitude.

As a matter of fact, the Nakanishi PTIR for the three-leg transition amplitude, although devised within the perturbative framework of the Feynman diagrams, has been proved to be a very effective tool for studying the bound state problem, within a non perturbative field-theory framework [4–10].

These successful achievements encourage to extend the Nakanishi representation to the study of the inhomogeneous BSE, i.e. the scattering states. Indeed, we have already presented the formal treatment of the BSE for scattering states, within the PTIR framework, in Ref. [11], obtaining an integral equation for the Nakanishi weight function after inserting the expression for the Bethe–Salpeter transition amplitude in the BSE. Moreover, by invoking the Nakanishi *uniqueness theorem* for the weight functions (see Ref. [3], Chapt. IV, pag. 141) we obtained a different equation for the weight function and one could wonder if and to what extent the two equations give the same solution. Note that the theorem was proven within the perturbative framework, where Nakanishi was able to formally resum the infinite Feynman diagrams contributing to a given multile-leg

M. Viviani (✉)
INFN, Sezione di Pisa, Largo Pontecorvo 3, 56100 Pisa, Italy
E-mail: michele.viviani@pi.infn.it

T. Frederico
Dep. de Física, Instituto Tecnológico de Aeronáutica, DCTA, São José dos Campos, São Paulo 12228-900, Brazil
E-mail: tobias@ita.br

G. Salmè
INFN, Sezione di Roma, P.le A. Moro 2, 00185 Rome, Italy
E-mail: salmeg@roma1.infn.it

transition amplitude. We will numerically explore to which extent the hypothesis of the uniqueness of the Nakanishi weight function is valid in a non perturbative framework. The attempt to start a quantitative investigation of the inhomogeneous BSE and to answer to the previous questions is the aim of the present work, where the problem of the solution of the zero-energy BSE will be discussed. Finally, we will compare the obtained scattering lengths with those recently appeared in literature obtained solving directly the BSE, and this would be a further stringent test to what extent the PTIR can be applied in a non-perturbative regime.

This contribution is organized as follows. In Sect. 2, we briefly recall the Nakanishi PTIR and write down the equations to solve. In Sect. 3 the expression for calculating the scattering length from the Nakanishi weight function is given and in Sect. 4 the obtained preliminary results are reported. Finally, in the last section, we discuss the perspectives of the present approach.

2 The Nakanishi Integral Equation for Scattering States

As shown in great detail in Ref. [11], a Nakanishi integral representation can be introduced for the half-off-shell transition amplitude $\Phi^{(+)}(k, p)$ describing a scattering states of two identical scalar particles of mass m, with the same power of the denominator as in the case of the bound state analyzed in Refs. [7,8], viz

$$\begin{aligned}\Phi^{(+)}(k,p) &= (2\pi)^4\delta^{(4)}(k-k_i)\\ &\quad -i\int_{-1}^{1}dz'\int_{-1}^{1}dz''\int_{-\infty}^{\infty}d\gamma'\frac{g^{(+)}(\gamma',z',z'')}{\left[\gamma'+m^2-\frac{1}{4}M^2-k^2-p\cdot k\,z''-2k\cdot k_i\,z'-i\epsilon\right]^3},\\ &= (2\pi)^4\delta^{(4)}(k-k_i)-i\int_{-1}^{1}dz'\int_{-1}^{1}dz''\int_{-\infty}^{\infty}d\gamma'\\ &\quad\times\frac{g^{(+)}(\gamma',z',z'')}{\left[\gamma'+\gamma+\kappa^2-k^-(k^++\frac{M}{2}z''-\frac{M}{2}z_iz')-k^+\frac{M}{2}(z''+z_iz')+2z'cos\varphi\sqrt{\gamma\gamma_i}-i\epsilon\right]^3},\end{aligned} \tag{1}$$

where p is the incoming total four-momentum of the pair, k_i the (on-shell) incoming relative four-momentum, and k the final (eventually off-shell) relative four momentum. We will adopt the frame $p = \{M, \mathbf{0}\}$. The function $g^{(+)}(\gamma', z', z'')$ is the so-called Nakanishi weight function.

In the expression above, we introduced the following notation [7,11]: i) $z_i = -2k_i^+/M = 2k_i^-/M$, since $(p/2 \pm k_i)^2 = m^2$, and $1 \geq |z_i|$, since the incoming particles have positive longitudinal momenta, i.e. $p^+/2 \pm k_i^+ \geq 0$, ii) $cos\varphi = \widehat{\mathbf{k}}_\perp \cdot \widehat{\mathbf{k}}_{i\perp}$, iii) $\gamma = |\mathbf{k}_\perp|^2$, iv) $\gamma_i = |\mathbf{k}_{i\perp}|^2$, and v) $\kappa^2 = m^2 - M^2/4$. Let us recall that from $(p/2 \pm k_i)^2 = m^2 = M^2/4 - k_i^+k_i^- - \gamma_i$ one gets

$$M^2 = 4\,\frac{(m^2+\gamma_i)}{(1-z_i^2)}\,, \qquad \kappa^2 = -\gamma_i - z_i^2\frac{M^2}{4}. \tag{2}$$

Let us start recalling that the transition amplitude fulfills the following BSE [1]

$$\Phi^{(+)}(k,p) = (2\pi)^4\delta^{(4)}(k-k_i) + G_0^{(12)}(k,p)\int\frac{d^4k'}{(2\pi)^4}\,i\mathcal{K}(k,k',p)\Phi^{(+)}(k',p), \tag{3}$$

where $i\,\mathcal{K}$ is the interaction kernel that contains all the irreducible diagrams. In the present contribution the self-energy will be disregarded, and therefore $G_0^{(12)}$ is the free propagator of the two particles of mass m. Assuming a Yukawa interaction of the form $\mathcal{L}_I = -g\bar{\psi}\psi\phi$, where ψ is the field of particle of mass m and ϕ the field of another scalar particle of mass μ, the interaction kernel in the ladder approximation is given by

$$i\mathcal{K}(k,k',p) \approx i\mathcal{K}^{(L)}(k,k') = \frac{i(-ig)^2}{(k-k')^2-\mu^2+i\epsilon}. \tag{4}$$

Substituting the expression given in Eq. (1) in the BSE one obtains an integral equation for the weight function $g^{(+)}(\gamma', z', z'')$. The expression obtained in ladder approximation has been reported in Ref. [11].

Let us now specialize to the BSE in the zero-energy limit. In this case one has $M^2 = 4m^2$, $\kappa^2 = 0$ and $\gamma_i = z_i = 0$. The equation in ladder approximation for the weight function in this case becomes [11]

$$\int_0^\infty d\gamma' \frac{g_{0L}^{(+)}(\gamma', z)}{[\gamma + \gamma' + z^2 m^2 - i\epsilon]^2} = \frac{g^2}{\gamma + z^2 m^2 - i\epsilon} \left[\frac{\theta(z)(1-z)}{\gamma + m^2 z^2 + (1-z)\mu^2} + \frac{\theta(-z)(1+z)}{\gamma + m^2 z^2 + (1+z)\mu^2} \right]$$
$$+ \frac{g^2}{2(4\pi)^2} \frac{1}{\gamma + z^2 m^2 - i\epsilon} \int_0^\infty d\gamma' \int_{-1}^1 dz'\, g_{0L}^{(+)}(\gamma', z') \mathcal{V}(\gamma, z, \gamma', z') , \tag{5}$$

where

$$g_{0L}^{(+)}(\gamma, z) = \int dz' g^{(+)}(\gamma, z', z) , \tag{6}$$

and

$$\mathcal{V}(\gamma, z, \gamma', z') = \int_0^1 dv\, v^2 \left[\frac{(1-z)^2 \theta(z - z')}{D(v, \gamma, z, \gamma', z')^2} + \frac{(1+z)^2 \theta(z' - z)}{D(v, \gamma, -z, \gamma', -z')^2} \right] , \tag{7}$$

$$D(v, \gamma, z, \gamma', z') = \gamma v(1-v)(1-z') + \gamma' v(1-z) + m^2 v(1-v) z^2 (1-z')$$
$$+ m^2 v^2 (z')^2 (1-z) + \mu^2 (1-v)(1-z). \tag{8}$$

Eq. (5) can be rearranged as follows

$$\int_0^\infty d\gamma'' \frac{g_{0L}^{(+)}(\gamma'', z)}{[\gamma + \gamma'' + z^2 m^2 - i\epsilon]^2} = \frac{g^2}{\mu^2} \int_0^\infty d\gamma'' \frac{\theta(z)\, \theta\left(1 - z - \gamma''/\mu^2\right) + \theta(-z)\, \theta\left(1 + z - \gamma''/\mu^2\right)}{\left[\gamma + \gamma'' + z^2 m^2 - i\epsilon\right]^2}$$
$$- \frac{g^2}{2(4\pi)^2} \int_0^\infty d\gamma'' \frac{1}{\left[\gamma + \gamma'' + z^2 m^2 - i\epsilon\right]^2} \int_0^\infty d\gamma' \int_{-1}^1 dz'\, g_{0L}^{(+)}(\gamma', z')$$
$$\times \left[\frac{(1+z)}{(1+z')} \theta(z' - z)\, h_0'(\gamma, z; \gamma', z') + \frac{(1-z)}{(1-z')} \theta(z - z')\, h_0'(\gamma, -z; \gamma', -z') \right]. \tag{9}$$

Notably, $h_0'(\gamma, z; \gamma', z')$ is the proper kernel for determining a bound state with vanishing energy (cf Ref. [10]) from the homogeneous BSE, and is given by

$$h_0'(\gamma, z; \gamma', z') = \theta\left(\gamma \frac{1+z'}{1+z} - \gamma' - \mu^2 - 2\mu\sqrt{z'^2 m^2 + \gamma'} \right) \times$$
$$\left[- \frac{\mathcal{B}_0(\gamma, z; \gamma', z')}{\mathcal{A}_0(\gamma', z')\, \Delta_0(\gamma, z; \gamma', z')} \frac{1}{\gamma} + \frac{1+z'}{1+z} \int_{y_-}^{y_+} dy \frac{y^2}{\left[y^2 \mathcal{A}_0(\gamma', z') + y(\mu^2 + \gamma') + \mu^2\right]^2} \right]$$
$$- \frac{1+z'}{1+z} \int_0^\infty dy \frac{y^2}{\left[y^2 \mathcal{A}_0(\gamma', z') + y(\mu^2 + \gamma') + \mu^2\right]^2} \tag{10}$$

with

$$\mathcal{A}_0(\gamma', z') = z'^2 m^2 + \gamma' \geq 0 , \tag{11}$$

$$\mathcal{B}_0(\gamma, z; \gamma', z') = \mu^2 + \gamma' - \gamma \frac{(1+z')}{(1+z)} \leq 0 , \tag{12}$$

$$\Delta_0^2(\gamma, z; \gamma', z') = \mathcal{B}_0^2(\gamma, z; \gamma', z') - 4\mu^2 \mathcal{A}_0(\gamma', z') \geq 0 , \tag{13}$$

$$y_\pm = \frac{1}{2\mathcal{A}_0(\gamma', z')} \left[-\mathcal{B}_0(\gamma, z; \gamma', z') \pm \Delta_0(\gamma, z; \gamma', z') \right] \geq 0. \tag{14}$$

The conditions above are enforced by the θ function in Eq. (10). From the uniqueness of the Nakanishi weight function [3], one then can obtain the following equation (see Refs. [10–12])

$$g_{0L}^{(+)}(\gamma, z) = \frac{g^2}{\mu^2} \theta(\gamma) \left[\theta(z)\, \theta(1 - z - \gamma/\mu^2) + \theta(-z)\, \theta(1 + z - \gamma/\mu^2) \right] - \frac{g^2}{2(4\pi)^2} \int_{-1}^1 dz' \int_0^\infty d\gamma'$$
$$\times \left[\frac{1+z}{1+z'} \theta(z' - z)\, h_0'(\gamma, z; \gamma', z') + \frac{1-z}{1-z'} \theta(z - z')\, h_0'(\gamma, -z; \gamma', -z') \right] g_{0L}^{(+)}(\gamma', z'). \tag{15}$$

3 The Scattering Length

The scattering amplitude in ladder approximation can be expressed in terms of the Nakanishi function as follows [10–12]:

$$f^{(Ld)}(s,\theta) = \frac{2m^2}{M}\,\alpha\,\Bigg\{\frac{1}{-2\kappa^2(1-cos\theta)+\mu^2-i\epsilon} + \frac{1}{2(4\pi)^2}\int_0^\infty d\gamma\int_{-1}^{1}dz\int_{-1}^{1}dz'\; g^{(+)}_{(L)}(\gamma,z,z') \times\int_0^\infty dy\;\frac{y^2}{\left[y^2\,\mathcal{A}(\gamma,z,z')+y\;\left(\mu^2+\gamma-2z\kappa^2\,cos\theta\right)+\mu^2-i\epsilon\right]^2}\Bigg\}_{\gamma_i=0}, \tag{16}$$

where $s\equiv M^2$, θ is the scattering angle and the dimensionless constant α is given by

$$\alpha = \frac{g^2}{16\pi m^2}\;.$$

Above $\mathcal{A}(\gamma,z,z')$ is the generalization for positive energies of the function given in Eq. (11) [11,12]. In the zero-energy scattering limit,

$$\lim_{s\to 4m^2} f_0^{(Ld)}(s,\theta) = -a = m\,\alpha\,\Bigg\{\frac{1}{\mu^2} + \frac{1}{2(4\pi)^2}\int_0^\infty d\gamma\int_{-1}^{1}dz\; g^{(+)}_{0L}(\gamma,z) \times\int_0^\infty dy\;\frac{y^2}{\left[y^2\,\mathcal{A}_0(\gamma,z)+y\;\left(\mu^2+\gamma\right)+\mu^2-i\epsilon\right]^2}\Bigg\}. \tag{17}$$

The scattering length in Born approximation (by adopting the convention of Ref. [13]) is

$$a^{BA} = -\,m\,\frac{\alpha}{\mu^2}. \tag{18}$$

4 Results

Following Ref. [10], we search the solutions of the zero-energy BSE expanding $g^{(+)}_{(L0)}(\gamma,z)$ as follows

$$g^{(+)}_{(L0)}(\gamma,z) = \sum_{\ell=1}^{N_z}\sum_{j=1}^{N_g} A_{\ell j}\;G_\ell(z)\;\mathcal{L}_j(\gamma). \tag{19}$$

with

$$G_\ell(z) = 4\,(1-z^2)\,\Gamma(5/2)\sqrt{\frac{\left(2(\ell-1)+5/2\right)\,\left(2(\ell-1)\right)!}{\pi\,\Gamma\left((2(\ell-1)+5)\right)}}\;C^{(5/2)}_{2(\ell-1)}(z)\;,\qquad \mathcal{L}_j(\gamma) = \sqrt{b}\,L_{j-1}(b\gamma)\,e^{-b\gamma/2}\;, \tag{20}$$

where the functions $G_\ell(z)$ are given in terms of even Gegenbauer polynomials, $C^{(5/2)}_{2(\ell-1)}(z)$, and the functions $L_j(b\gamma)$ are Laguerre polynomials. With this choice, the basis is orthonormal, i.e.,

$$\int_{-1}^{1}dz\;G_\ell(z)\;G_n(z) = \delta_{\ell n}\;,\quad \int_0^\infty d\gamma\;\mathcal{L}_j(\gamma)\;\mathcal{L}_k(\gamma) = b\int_0^\infty d\gamma\;e^{-b\gamma}\;L_{j-1}(b\gamma)\;L_{k-1}(b\gamma) = \;\delta_{jk}. \tag{21}$$

The parameter b, controlling the range of the Laguerre polynomials, is chosen in order to speed up the convergence. This kind of expression ignores possible discontinuities present in $g^{(+)}_{(L0)}(\gamma,z)$ (an improved approach will be presented elsewhere [16]).

Table 1 Comparison between the scattering lengths, in ladder approximation, evaluated in Ref. [14], a_{CK}, with those obtained in the present work, a_{FSV} and a_{UNI}. The values a_{FSV} are calculated by inserting the Nakanishi weight function solution of Eq. (5) in Eq. (17). The values a_{UNI} are calculated by inserting the solution of Eq. (15) in Eq. (17). The value of the exchanged scalar mass is $\mu/m = 0.5$ and 1.0, and the coupling constant $\alpha = g^2/(16\pi m^2)$, has been varied as shown in the first column. The scattering lengths are given in units of m^{-1}

α	a_{CK} [14]	a_{FSV}	a_{UNI}	a_{CK} [14]	a_{FSV}	a_{UNI}
	$\mu/m = 0.5$			$\mu/m = 1.0$		
0.01	$-0.403\ 10^{-1}$	$-0.403\ 10^{-1}$	$-0.404\ 10^{-1}$	$-0.100\ 10^{-1}$	$-0.100\ 10^{-1}$	$-0.100\ 10^{-1}$
0.05	-0.209	-0.209	-0.209	$-0.510\ 10^{-1}$	$-0.510\ 10^{-1}$	$-0.510\ 10^{-1}$
0.10	-0.438	-0.438	-0.437	-0.104	-0.104	-0.104
0.20	-0.971	-0.971	-0.968	-0.217	-0.217	-0.217
0.30	$-0.164\ 10^{1}$	$-0.164\ 10^{1}$	$-0.163\ 10^{1}$	-0.339	-0.339	-0.339
0.40	$-0.250\ 10^{1}$	$-0.250\ 10^{1}$	$-0.248\ 10^{1}$	-0.474	-0.474	-0.473
0.50	$-0.366\ 10^{1}$	$-0.366\ 10^{1}$	$-0.363\ 10^{1}$	-0.621	-0.621	-0.620
0.60	$-0.534\ 10^{1}$	$-0.533\ 10^{1}$	$-0.529\ 10^{1}$	-0.784	-0.784	-0.782
0.70	$-0.798\ 10^{1}$	$-0.796\ 10^{1}$	$-0.790\ 10^{1}$	-0.965	-0.965	-0.962
0.80	$-0.128\ 10^{2}$	$-0.128\ 10^{1}$	$-0.126\ 10^{2}$	$-0.117\ 10^{1}$	$-0.117\ 10^{1}$	$-0.116\ 10^{1}$
0.90	$-0.247\ 10^{2}$	$-0.245\ 10^{2}$	$-0.241\ 10^{2}$	$-0.140\ 10^{1}$	$-0.140\ 10^{1}$	$-0.139\ 10^{1}$
1.00	$-0.103\ 10^{3}$	$-0.994\ 10^{2}$	$-0.993\ 10^{2}$	$-0.166\ 10^{1}$	$-0.166\ 10^{1}$	$-0.165\ 10^{1}$
1.10	$0.620\ 10^{2}$	$0.634\ 10^{2}$	$0.618\ 10^{2}$	$-0.195\ 10^{1}$	$-0.196\ 10^{1}$	$-0.195\ 10^{1}$
1.20	$0.261\ 10^{2}$	$0.263\ 10^{2}$	$0.257\ 10^{2}$	$-0.230\ 10^{1}$	$-0.230\ 10^{1}$	$-0.229\ 10^{1}$
1.50	$0.110\ 10^{2}$	$0.110\ 10^{2}$	$0.107\ 10^{2}$	$-0.379\ 10^{1}$	$-0.380\ 10^{1}$	$-0.377\ 10^{1}$
2.00	$0.634\ 10^{1}$	$0.635\ 10^{1}$	$0.617\ 10^{1}$	$-0.111\ 10^{2}$	$-0.112\ 10^{2}$	$-0.111\ 10^{2}$
2.50	$0.454\ 10^{1}$	$0.454\ 10^{1}$	$0.438\ 10^{1}$	$0.568\ 10^{2}$	$0.526\ 10^{2}$	$0.522\ 10^{2}$
3.00	$0.332\ 10^{1}$	$0.330\ 10^{1}$	$0.317\ 10^{1}$	$0.108\ 10^{2}$	$0.106\ 10^{2}$	$0.104\ 10^{2}$

Inserting Eq. (19) in Eq. (5) or (15), the problem of determining the unknown coefficients $A_{\ell j}$ is reduced to the solution of a linear system. Once $g^{(+)}_{(L0)}(\gamma, z)$ is found, Eq. (17) can be used to compute the scattering length. The results obtained in this way for the cases $\mu/m = 0.5$ and 1.0 have been reported in Table 1. The accuracy of the calculation can be checked increasing the values of N_g and N_z, namely the number of terms included in the expansion given in Eq. (19). In the case of solution of Eq. (5) a good convergence has been found for values $b \approx 10\,m^{-2}$, $N_g \approx 24$ and $N_z \approx 10$. The solution of Eq. (15) is found to converge more slowly. In this case, values around $N_g \approx 48$ and $N_z \approx 10$ have been employed. Such a behavior is related to the presence of discontinuities in the Nakanishi function, made manifest by the θ-functions appearing in the inhomogeneous term of Eq. (15).

In Table 1, the preliminary results for the scattering lengths calculated for various values of α and for $\mu/m = 0.5$ and 1.0 are reported. The values a_{FSV} and a_{UNI} are the scattering lengths obtained inserting the solution of either Eq. (5) or Eq. (15) in Eq. (17), respectively. It is important to notice that for a given mass of the exchanged scalar and for particular values of α, states with zero binding energy appear, as can be verified considering the homogeneous BSE. While approaching such values of α, the scattering length becomes (in modulus) larger and larger, as expected.

Our results have been compared with the corresponding values, a_{CK}, obtained in Ref. [14], where the direct solution of the BSE was implemented [15]. As it can be seen, the results obtained solving the two equations are very close and in rather good agreement with the values reported in Ref. [14]. The small differences found are due to the above mentioned difficulties in the solution of Eq. (15). More sizable differences are observed when the scattering lengths become very large, namely when α approaches the value necessary to obtain the zero-energy state.

For smaller values of μ/m, the convergence of the expansion becomes much slower and with the method of solution considered in the present work we have not been able to find stable results. Such a problem is again connected to the presence of discontinuities in the Nakanishi function which become more severe as $\mu \to 0$. Work is in progress to take into account exactly of such discontinuities and the corresponding results will be published in a forthcoming paper [16]. Such discontinuities of the Nakanishi weight function should produce the expected singularities of the scattering BS amplitude in Minkowski space.

5 Conclusions

We have quantitatively investigated the zero-energy inhomogeneous BSE in ladder approximation, directly in Minkowski space within the perturbation-theory integral representation of the multi-leg transition amplitudes, proposed by Nakanishi in the 60s [2,3]. Using the light-front framework, as shown in Ref. [7] for bound states and in Ref. [11] for scattering states, it is possible to derive an equation for the Nakanishi weight function, see Eq. (5). Moreover, the Nakanishi theorem [3] on the uniqueness of the non singular weight function, related to the vertex function in PTIR, leads to an equation for $g_{0L}^{(+)}$, Eq. (15), much simpler than Eq. (5), from the formal point of view, but more demanding from the numerical side. These two equations allow for the numerical evaluation of the weight function corresponding to a given value of the coupling constant of the interacting system and the exchanged-boson mass. We have shown that the scattering lengths obtained by solving Eqs. (5) and (15) can be substantially taken as the same. This gives us great confidence in the validity of the uniqueness theorem also in a non perturbative regime, as already checked for the bound case [10]. Work is in progress to incorporate the presence of discontinuities in the Nakanishi weight function [16].

In perspective, the numerical analysis we have performed appears very encouraging, and it makes compelling the next steps, represented by the study of positive energy scattering states and the inclusion of the crossed-box diagrams as already done in Ref. [8].

Acknowledgments We gratefully thank Jaume Carbonell and Vladimir Karmanov for very stimulating discussions. MV acknowledges the warm hospitality of INFN, Sezione di Roma, and of the Instituto Tecnológico de Aeronáutica, São José dos Campos (Brazil). TF thanks the partial financial support from CNPq, FAPESP and CAPES.

References

1. Salpeter, E.E., Bethe, H.A.: A relativistic equation for bound-state problems. Phys. Rev. **84**, 1232 (1951)
2. Nakanishi, N.: Partial-wave Bethe–Salpeter equation. Phys. Rev. **130**, 1230 (1963)
3. Nakanishi, N.: Graph theory and Feynman integrals. Gordon and Breach, New York (1971)
4. Task, K., Williams, A.G.: Solving the Bethe–Salpeter equation for scalar theories in Minkowski space. Phys. Rev. **D 51**, 7026 (1995)
5. Kusaka, K., Simpson, K., Williams, A.G.: Solving the Bethe–Salpeter equation for bound states of scalar theories in Minkowski space. Phys. Rev. **D 56**, 5071 (1997)
6. Sauli, V., Adam, J.: Study of relativistic bound states for scalar theories in the Bethe–Salpeter and Dyson-Schwinger formalism. Phys. Rev. **D 67**, 085007 (2003)
7. Karmanov, V.A., Carbonell, J.: Solving Bethe–Salpeter equation in Minkowski space. Eur. Phys. J. **A27**, 1 (2006)
8. Carbonell, J., Karmanov, V.A.: Cross-ladder effects in Bethe–Salpeter and light-front equations. Eur. Phys. J. **A27**, 11 (2006)
9. Carbonell, J., Karmanov, V.A.: Solving the Bethe–Salpeter equation for two fermions in Minkowski space. Eur. Phys. J **A46**, 387 (2010)
10. Frederico, T., Salmè, G., Viviani, M.: Quantitative studies of the homogeneous Bethe–Salpeter equation in Minkowski space. Phys. Rev. D **89**, 016010 (2014)
11. Frederico, T., Salmè, G., Viviani, M.: Two-body scattering states in Minkowski space and the Nakanishi integral representation onto the null plane. Phys. Rev. **D 85**, 036009 (2012)
12. Frederico, T., Salmè, G., Viviani, M.: Solutions of the Bethe–Salpeter Equation in Minkowski space: a comparative study. Few-Body Sys. **55**, 693 (2014)
13. Newton, R.: Scattering Theory of Waves and Particles. Monographs in Physics. Springer, New York, vol. **2**, (2002)
14. Carbonell, J., Karmanov, V.A.: Bethe–Salpeter scattering amplitude in Minkowski space. Phys. Lett. **B 727**, 319 (2013)
15. Carbonell, J., Karmanov, V.A.: Bethe–Salpeter scattering state equation in Minkowski space. Phys. Rev. D **90**, 056002 (2014)
16. Frederico, T., Salmè, G., Viviani, M.: in preparation