Few-Body Syst (2014) 55:1223–1232
DOI 10.1007/s00601-014-0908-1

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Bound State Solutions of the Klein-Gordon Equation for the Mathews-Lakshmanan Oscillator

Received: 24 June 2014 / Accepted: 18 September 2014 / Published online: 30 September 2014

Abstract We study a boundary-value problem for the Klein-Gordon equation that is inspired by the well-known Mathews-Lakshmanan oscillator model. By establishing a link to the spheroidal equation, we show that our problem admits an infinite number of discrete energies, together with associated solutions that form an orthogonal set in a weighted L^2-Hilbert space.

1 Introduction

In this work we are concerned with the one-dimensional model of a particular nonlinear oscillator, often referred to as "Mathews-Lakshmanan oscillator". Originating from the classical theory, the latter model was first introduced and discussed in [12]. It presents two interesting features, namely nonlinearity of the potential and the presence of a term that can be interpreted as a position-dependent mass. This term produces a position-dependent spring constant, allowing for the existence of solutions that take a simple harmonic form. The classical model of the Mathews-Lakshmanan oscillator can be adapted to the nonrelativistic quantum context [13], governed by the Hamiltonian for a position-dependent mass system. The associated Schrödinger equation maintains solvability, admitting a discrete energy spectrum and a corresponding set of bound state solutions that can be constructed in closed form [4,6,13]. In particular, the solutions allow for a representation in terms of special functions [15,17]. The quantum version of the Mathews-Lakshmanan oscillator model was generalized several times [8,11], including more recent studies that involve the higher-dimensional case [5] and spaces of nonzero curvature [3]. Another generalization of the quantum Mathews-Lakshmanan oscillator model concerns its relativistic version, which to the best of our knowledge has not been considered yet. Therefore, the purpose of this note is to set up a simple relativistic model for the Mathews-Lakshmanan oscillator governed by the Klein–Gordon equation. Inspired by the classical model, we incorporate a position-dependent mass function and equip our equation with appropriate boundary conditions. Such kind of problems involving the Klein–Gordon equation and a posistion-dependent mass, have been studied for many particular cases, including Hulthen potentials [7], Coulombic systems [9] and kink-type interactions [10], just to mention a few. Surprisingly, the present problem will turn out to admit infinitely many discrete stationary energies, together with associated "bound state" solutions that form an orthogonal set in a weighted Hilbert space. Recall that such solutions cannot represent bound states in the strict sense, because the Klein–Gordon equation is not compatible with a probabilistic interpretation. However, for the sake of simplicity we will refer to such

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solutions as bound states. The remainder of this paper is organized as follows. In Sect. 2 we give a short review of the well-known Schrödinger model for the Mathews-Lakshmanan oscillator. Section 3 is devoted to the construction of our Klein–Gordon system and the solution method. Results and a discussion is presented in the final Sect. 4.

2 Preliminaries

We will now briefly review the quantum model of the Mathews-Lakshmanan oscillator within the one-dimensional Schrödinger theory. For details the reader may refer to [4,6] and references therein. The latter oscillator model originates from the classical Lagrangian

$$\mathcal{L} = \frac{\dot{x}^2}{2 - 2\,\mu\,x^2} - \frac{(1-\mu)\,x^2}{2 - 2\,\mu\,x^2}, \tag{1}$$

for a real-valued parameter $\mu > 0$. Let us point out that the initial problem leading to the Lagrangian (1) originally involved two independent parameters [12]. In order to preserve exact solvability in the nonrelativistic quantum case, the two parameters can be linked to each other [4], resulting in the Lagrangian (1) that we are considering here. The latter Lagrangian resembles a system under the presence of a position-dependent mass m and potential energy V, given by

$$m = \frac{1}{1-\mu\,x^2} \qquad\qquad V = \frac{(1-\mu)\,x^2}{2-2\,\mu\,x^2}. \tag{2}$$

In order to introduce a quantum model associated with the classical Lagrangian (1), we define the interval $D_\mu \subset \mathbb{R}$ by

$$D_\mu = \left(-\frac{1}{\sqrt{\mu}}\,,\,\frac{1}{\sqrt{\mu}}\right), \tag{3}$$

on which we consider the following boundary-value problem of Dirichlet type [4], obtained from a quantization process for position-dependent mass systems

$$(1-\mu\,x^2)\,\Psi'' - \mu\,x\,\Psi' + \left[2\,E - \frac{(1-\mu)\,x^2}{1-\mu\,x^2}\right]\Psi = 0, \quad x \in D_\mu \tag{4}$$

$$\Psi\left(-\frac{1}{\sqrt{\mu}}\right) = \Psi\left(\frac{1}{\sqrt{\mu}}\right) = 0, \tag{5}$$

where the stationary energy $E \in \mathbb{R}$ represents the spectral parameter and the boundary conditions are allowed to be understood in the sense of a limit. We refer to the potential of the above model as the function V in (2). Furthermore, we require $\Psi \in L^2_w\left(D_\mu\right)$ for the weight function w, given by

$$w = \frac{1}{\sqrt{1-\mu\,x^2}}. \tag{6}$$

The problem (4), (5) admits a discrete spectrum (E_n) and an infinite orthogonal set of corresponding solutions $(\Psi_n) \subset L^2_w\left(D_\mu\right)$, $n \in \mathbb{N} \cup \{0\}$, provided by the following expressions:

$$E_n = \frac{\mu\,n^2}{2} + n + \frac{1}{2} \tag{7}$$

$$\Psi_n = \left(1-\mu\,x^2\right)^{\frac{1}{4}} P^{-\frac{1}{\mu}+\frac{1}{2}}_{n+\frac{1}{\mu}+\frac{1}{2}}\left(-\sqrt{\mu}\,x\right), \tag{8}$$

where P stands for the associated Legendre function of the first kind [1].

3 The Klein–Gordon model

We will now set up a boundary-value problem that is governed by a one-dimensional Klein–Gordon equation for a quantum analogon of the nonlinear oscillator system described by (1). In its general form, the Klein–Gordon equation can be written as

$$\Psi'' + \left[(E - V)^2 - (m + S)^2\right]\Psi = 0. \tag{9}$$

Here, the real-valued constant E denotes the stationary energy, while the functions V and S represent vector and scalar potential, respectively. Let us reiterate that equations of the form (9) have been studied in several contexts, see for example [7,9,10]. Now, in order to adapt (9) to the classical context, the Lagrangian (1) dictates that we must incorporate a potential and a position-dependent mass in the form as given by (2). While substituting one or both of the potentials in (9) is a straightforward task, a conceptual problem arises when replacing the mass in our Klein–Gordon equation by a nonconstant function. In order to understand the issue, let us first recall how position-dependent masses are introduced in nonrelativistic quantum mechanics. There, the governing Schrödinger equation is constructed from an underlying Hamiltonian, which can be modified such as to include a nonconstant mass [19,20], while momentum and energy operators maintain their conventional definitions. In contrast to the latter scenario, the Klein–Gordon equation is generated from de Broglie's dispersion relation by substituting the momentum and energy operators. Since an ad-hoc introduction of a position-dependent mass function would render the dispersion relation invalid, a Klein–Gordon equation featuring a nonconstant mass cannot be justified from a physical viewpoint of a relativistic model. Despite this fact, there exists a variety of articles devoted to special cases of equation (9) for a position-dependent mass function, as mentioned above. Examples of such works can be found in [2,7,9,10] and references therein. Such studies are usually motivated by the question of integrability when passing from a well-known Schrödinger model for position-dependent mass to the corresponding Klein–Gordon system. A related question of particular interest is the persistence of a discrete spectrum. In the present case we will address both of the latter aspects. Besides the position-dependent mass, we must also replace the vector and scalar potential in (9) through expressions that are motivated by the classical potential in (2). While such an incorporation of the potentials can be done in several ways, in our approach we will consider a Klein–Gordon equation for position-dependent mass m and vector potential V as given in (2), and we set the scalar potential S to zero. If we instead chose to have equal (or equal up to a factor like in [2]) vector and scalar potentials of the form (2), then we would obtain a problem that is qualitatively equivalent to the situation we will be considering in this work. Below we shall comment further on implications resulting from incorporating (2) in a different way. Now, we define the following boundary-value problem on the same domain D_μ that was taken in the Schrödinger case (3).

$$\Psi'' + \left\{\left[E - \frac{(1-\mu)\,x^2}{2 - 2\,\mu\,x^2}\right]^2 - \left(\frac{1}{1-\mu\,x^2}\right)^2\right\}\Psi = 0, \quad x \in D_\mu \tag{10}$$

$$\Psi\left(-\frac{1}{\sqrt{\mu}}\right) = \Psi\left(\frac{1}{\sqrt{\mu}}\right) = 0, \tag{11}$$

where as before $\mu > 0$ and the real constant E stands for the stationary energy. We are interested in solutions that resemble bound states, so we impose the condition that Ψ be normalizable in an L^2-Hilbert space with an appropriate weight function to be determined.

3.1 Connection with the Spheroidal Equation

In contrast to the Schrödinger equation (4), a direct approach to finding a closed-form solution of (10) in terms of special functions fails. Therefore, the most promising way of tracking down solutions is to check if (10) belongs to a known class of equations. This will in fact turn out to be true. Let us apply the following point transformation involving the coordinate change $x(y) = y/\sqrt{\mu}$:

$$\Psi\left[x(y)\right] = \sqrt{y^2 - 1}\;\; \Phi(y)\,. \tag{12}$$

After introduction of this transformation, our boundary-value problem (10), (11) can be written in the form

$$(1-y^2)\,\Phi'' - 2\,y\,\Phi' + \left(A - h^2\,y^2 + \frac{M^2}{y^2-1}\right)\Phi = 0, \quad y \in (-1,1) \tag{13}$$

$$\Psi(-1) \;=\; \Psi(1) = 0, \tag{14}$$

where the constants A, h and M stand for the following abbreviations

$$A = \frac{4\,E^2\,\mu^2 - \mu^2 + 2\,\mu - 1}{4\,\mu^3} \qquad h = \frac{2\,E\,\mu - \mu + 1}{2\,\mu^{\frac{3}{2}}} \qquad M = \sqrt{\frac{4\,\mu^3 + 3\,\mu^2 + 2\,\mu - 1}{4\,\mu^3}}. \tag{15}$$

The boundary-value problem (13), (14) is well-known. In particular, (13) can be identified as a spheroidal equation [18], which belongs to the confluent Heun class [16]. Before we construct solutions of the spheroidal equation, let us briefly return to the ambiguity in incorporating classical potential energy and mass (2) into the Klein–Gordon equation (9). Instead of having the scalar potential S vanish, we could also assign it the expression for V in (2) and/or at the same time set $V = 0$. However, each of these options results in a Klein–Gordon equation that can be converted into a spheroidal equation of the general form (13), where the expressions for the parameters A, h and M will vary depending on the actual approach. Returning to the present problem, if we can find solutions of the spheroidal equation, equipped with the boundary conditions (14), then these solutions can be converted by means of (12), such that they become applicable to our initial Klein–Gordon problem.

3.2 Solutions of the Spheroidal Equation

In order to make this work self-contained, let us summarize main facts about the solutions of the above problem (13), (14), for details the reader may refer to [14,18]. In general, the latter boundary-value problem is known to admit solutions only for a discrete set of parameter values for A. We will see below that these so-called characteristic values determine precisely the discrete bound state energies of our initial Klein–Gordon equation. Note that in order to facilitate calculations, we will not yet incorporate the settings (15) until we have fully developed our solution method. Returning to the spheroidal equation, it is known that there exists an infinite, discrete and ascending sequence of values $A = A_j$, $j \in \mathbb{N} \cup \{0\}$, for each of which the problem (13), (14) admits a solution that has exactly j zeros inside $(-1, 1)$. The solutions form an orthogonal set with respect to the usual inner product of the Hilbert space $L^2(-1, 1)$. All solutions have parity, but do not feature a closed-form representation. Instead, they can be written as a series expansion in terms of Gegenbauer polynomials T [1] via one of the formulas

$$\Phi_{2j} = \left(1-y^2\right)^{\frac{M}{2}} \sum_{n=0}^{\infty} (d_{2n})_{|A=A_{2j}}\; T_{2n}^{M}(y) \tag{16}$$

$$\Phi_{2j+1} = \left(1-y^2\right)^{\frac{M}{2}} \sum_{n=0}^{\infty} (d_{2n+1})_{|A=A_{2j+1}}\; T_{2n+1}^{M}(y), \tag{17}$$

for $j \in \mathbb{N} \cup \{0\}$, where (16) and (17) represent even and odd parity functions, respectively. The expansion coefficients in the series are determined through the following three-term recursion relation

$$a_n\,d_{n-2} + b_n\,d_n + c_n\,d_{n+2} \;=\; 0, \qquad n \in \mathbb{N} \cup \{0\},\; d_{-2} = d_{-1} = 0, \tag{18}$$

where the following abbreviations are involved:

$$a_n = \frac{n\,(n-1)\,h^2}{(2\,n+2\,M-1)(2\,n+2\,M-3)} \tag{19}$$

$$b_n = \frac{h^2\left\{2\left[(n+M)(n+M+1) - M^2\right] - 1\right\}}{(2\,n+2\,M+3)(2\,n+2\,M-1)) + (n+M)(n+M+1) - A} \tag{20}$$

$$c_n = \frac{h^2\,(n+2\,M+1)(n+2\,M+2)}{(2\,n+2\,M+3)(2\,n+2\,M+5)}. \tag{21}$$

It is important to point out that the series (16) and (17) are in general divergent. Convergence is guaranteed only if the parameter A in (20) takes a characteristic value $a = A_j$, $j \in \mathbb{N} \cup \{0\}$. Before we can use the point transformation (12) to convert the results of the present section back to our initial Klein–Gordon model, we must have a method at hand to find both the characteristic values for A, as well as the expansion coefficients in the solution series. Such a method will be presented in the next section.

3.3 The Solution Algorithm

We will now develop an algorithm, the purpose of which is to give a successive approximation to both the characteristic values and the expansion coefficients. This algorithm is not new, but was introduced in [14]. Since it is essential for understanding subsequent considerations, we shall briefly review the algorithm in two steps.

Step 1. We solve the recursion (18) with respect to d_2/d_0, first substituting $n = 0$ and the second time $n = 2$. This gives the following two relations

$$\frac{d_2}{d_0} = -\frac{b_0}{c_0} \qquad \frac{d_2}{d_0} = \frac{-a_2}{b_2 + c_2 \frac{d_4}{d_2}}. \tag{22}$$

Combining these two relations, we obtain

$$-\frac{b_0}{c_0} = \frac{-a_2}{b_2 + c_2 \frac{d_4}{d_2}}.$$

Now we will modify the right side of this equation. We shift all indices n in (18) up by two, set $n = 4$, solve for d_4/d_2, and substitute the result into the right-hand side of (22). An iteration of this process gives rise to the following continued fraction expansion:

$$-\frac{b_0}{c_0} = \cfrac{-a_2}{b_2 + c_2 \cfrac{-a_4}{b_4 + c_4 \cfrac{-a_6}{b_6 + c_6 \ldots}}}. \tag{23}$$

We stop this iteration after $N > 0$ steps by reasoning that d_{N+2}/d_N is very small, so we set $d_{N+2}/d_N = 0$. This turns (23) into an algebraic equation that can be solved for the parameter A, yielding approximations for the characteristic values belonging to solutions of even parity. Observe that (23) generally has several solutions, each of which is associated with a characteristic value. Also, the number of solutions increases with the number N of steps that are done. In a similar way we obtain a continued fraction expansion for odd indices:

$$-\frac{b_1}{c_1} = \cfrac{-a_3}{b_3 + c_3 \cfrac{-a_5}{b_5 + c_5 \cfrac{-a_7}{b_7 + c_7 \ldots}}}, \tag{24}$$

that leads to the characteristic values associated with the odd solutions.

Step 2. We set $d_0 = 1$. Relation (22) then gives d_2, which can be plugged into the recursion relation (18) for $n = 2$ in order to obtain d_4. The continuation of this process yields the following expansion coefficients $d_6, d_8, d_{10}, \ldots$. The coefficients for odd indices are constructed in a similar manner, starting out with $d_1 = 1$. The recursion (18) for $n = 1$ generates d_3, which can be substituted into (18) for $n = 3$ in order to give d_5. This iteration produces the subsequent expansion coefficients $d_7, d_9, d_{11}, \ldots$. The solutions of the boundary-value problem (13), (14) can now be approximated by plugging the expansion coefficients and the characteristic values into the series representations (16), (17).

3.4 Solutions of the Klein–Gordon Equation

While the algorithm described in the previous section is designed to work for the spheroidal equation, we can adapt it to our initial Klein–Gordon equation by applying a few minor modifications that will now be listed.

- The parameters A, h and M in (19)-(21) must be replaced through the corresponding expressions in (15). As a result, the recursion relation (18) and its coefficients depend entirely on the parameters μ, E and n.
- Equations (23) and (24) must now be solved for the energy E (instead of A, which has been replaced in the previous point). The resulting characteristic values $E = E_j$, where j is an integer, approximate the sought bound state energies of the Klein–Gordon model (10), (11). Recall that in general, the latter model features positive and negative energies, as is typical for the Klein–Gordon equation. Since we are interested in real energies only, we must make sure that the parameter M in (15) is real-valued, resulting in a restriction on μ that will be discussed in Sect. 4.
- The second step of the algorithm is performed exactly as described in Sect. 3.2, until the expansion coefficients have been found. As such, the algorithm is complete.

The remaining task is to construct the solutions of our initial boundary-value problem (10), (11) that are associated with the characteristic values for the energy parameter E. We will enumerate these characteristic values in the following way. First, we apply the convention that E_0 corresponds to the lowest positive energy ("ground state"), while E_{-1} is the highest negative energy. As a consequence, the solutions associated with the energies E_0 and E_{-1} do not have any zeros inside D_μ. Furthermore, a characteristic value $E = E_j$ for a $j > 0$ is associated with a solution that has exactly j zeros inside D_μ. Finally, a characteristic value $E = E_j$ for a $j < 0$ is associated with a solution that has exactly $j + 1$ zeros inside D_μ. This situation is illustrated in Sect. 4 within Table 1 below. Now, the sought solutions of the problem (10), (11) can be found by reversing the point transformation (12), applying it to the solution series (16), (17) of the spheroidal equation. In addition, all terms must be expressed through the initial variable x, recall that $y(x) = \sqrt{\mu}x$, and the parameters A, h, M are to be replaced by (15), where E attains a characteristic value. This gives

$$\Psi_{2j} = \left(1-\mu\, x^2\right)^{\frac{1}{2}+\sqrt{\frac{4\mu^3+3\mu^2+2\mu-1}{16\mu^3}}} \sum_{n=0}^{\infty} (d_{2n})_{|E=E_{2j}}\; T_{2n}^{\sqrt{\frac{4\mu^3+3\mu^2+2\mu-1}{4\mu^3}}}\left(\sqrt{\mu}\, x\right) \tag{25}$$

$$\Psi_{2j+1} = \left(1-\mu\, x^2\right)^{\frac{1}{2}+\sqrt{\frac{4\mu^3+3\mu^2+2\mu-1}{16\mu^3}}} \sum_{n=0}^{\infty} (d_{2n+1})_{|E=E_{2j+1}}\; T_{2n+1}^{\sqrt{\frac{4\mu^3+3\mu^2+2\mu-1}{4\mu^3}}}\left(\sqrt{\mu}\, x\right), \tag{26}$$

for $j \in \mathbb{N}\cup\{0\}$. Observe that the functions (25) are even, while their counterparts (26) are odd. Note further that the square roots stem from the incorporation of the definition for M, see (15). Next, we investigate on which space the functions (25), (26) are orthogonal. Such a space must exist, because the initial series solutions (16) and (17) form an orthogonal set in $L^2(-1, 1)$. The solutions of the spheroidal equation and its Klein–Gordon counterpart are connected by the point transformation (12). We now set up the orthogonality relation for (16), (17) and apply the latter point transformation for nonnegative integers k, k':

$$(\Phi_k, \Phi_{k'}) = \delta_{kk'} = \int_{-1}^{1} \Phi_k^*\, \Phi_{k'}\, dy = \int_{-\sqrt{\frac{1}{\mu}}}^{\sqrt{\frac{1}{\mu}}} \Psi_k^*\, \Psi_{k'}\, \frac{1}{1-\mu\, x^2}\, dx = \int_{-\sqrt{\frac{1}{\mu}}}^{\sqrt{\frac{1}{\mu}}} \Psi_k^*\, \Psi_{k'}\, w^2\, dx,$$

where we used the abbreviation w that was introduced in (6). We conclude that the functions (25), (26) form an orthogonal set on the weighted Hilbert space $L^2_{w^2}\left(D_\mu\right)$.

4 Results

We will now apply the theoretical findings developed in the previous sections to concrete examples. Before we do so, some technical remarks are in order. At first let us mention that all symbolic calculations leading to the results reported in this section were performed using the MuPAD language in MATLAB R2013a, where the number of significant decimal digits was set to 40. Next, recall that the bound state energies (characteristic

values) of our boundary value problem (10), (11) must be determined through the approximative scheme described in Sects. 3.3 and 3.4. Thus, we cannot state a closed-form relation for the bound state energies in terms of the constant μ, as it is possible for the Schrödinger scenario (7). Instead, the present problem requires the parameter μ to be specified beforehand, such that the bound state energies can be determined numerically through our algorithm. Since the latter energies are required to be real, we must guarantee that the parameters (15) entered in the algorithm are real-valued. While this is always true for A and h, the constant M can become imaginary if the argument of its square root is negative. In order to avoid this, we must request that

$$\mu \geq -\frac{1}{4} - \frac{5}{3^{\frac{1}{3}}\, 4\, \left(99 + 8\sqrt{159}\right)^{\frac{1}{3}}} + \frac{\left(99 + 8\sqrt{159}\right)^{\frac{1}{3}}}{3^{\frac{2}{3}}\, 4} \approx 0.304481. \tag{27}$$

Hence, in contrast to the Schrödinger case outlined in section 2, the parameter $\mu > 0$ cannot be arbitrary, but must obey the restriction (27) in order to ensure reality of the bound state energies. Next, let us compute these energies for the case $\mu = 3/5$, observe that this choice is compatible with (27).

Results of this calculation are shown in Table 1. The first and second column display a selection of bound state energies (recall that there are infinitely many of these energies in total). The remaining columns show parity of the associated solutions, as well as their number of zeros. The explicit form of the solutions can be obtained from (25), (26), therefore we omit to state them here again. Instead we present a plot of three solutions Ψ_j, for $j = 0, 1, 2$, in the left part of Fig. 1, note that these functions are normalized in the Hilbert space $L^2_{w^2}\left(D_\mu\right)$. It is interesting to compare the Klein–Gordon solutions with their Schrödinger counterparts that are given in (8). The corresponding plots are shown in the right part of Fig. 1, where we normalized the Schrödinger solutions with respect to the space $L^2_w\left(D_\mu\right)$. The plots in Fig. 1 demonstrate the similarity between the two systems, both of which admit an infinite set of bound state energies with associated solutions that have parity. The qualitative shape of the solutions shown in Fig. 1 does not change significantly if the value of our parameter μ varies. Next, let us have a closer look at the bound state energies shown in Table 1. These energies are displayed in the left part of Fig. 2, where the dashed line indicates the zero energy level. The tick marks on the x-axis stand for the index of the respective bound state energy. The right part of Fig. 2 shows the bound state energies (7) of the Schrödinger model. We observe from Fig. 2 that the sequence of bound state energies for the Klein–Gordon model behaves approximately like a linear function if we ignore the gap between positive and negative energies. Despite this general tendency, it is clearly visible that the bound state energies are not equidistant. In contrast to this fact, the sequence of Schrödinger energies increases quadratically, as we know from (7). As far as the general behaviour of the Klein–Gordon bound state energies is concerned, we observe from Fig. 3 that for large indices the energy sequence becomes linear and the energy levels become equidistant. Before we conclude this section, let us study the dependence of the bound state energies on the parameter μ. To this end, we calculate the ground state energy E_0 for several values of μ. The result is displayed in Fig. 4, where the x-axis represents the values of μ. Inspection of the figure shows that the energy E_0 increases strictly with μ, where the rate of increase corresponds to a square root. Numerical calculations for different bound state energies confirm this finding, such that we conjecture $E_n \sim \sqrt{\mu}$ for all integers n.

Table 1 Bound state energies and solution properties of the boundary-value problem (10), (11) for the parameter setting $\mu = 3/5$

Stationary energy	Numerical value	Associated solution	Parity	Number of zeros
E_{-6}	−5.518046	Ψ_{-6}	Odd	5
E_{-5}	−4.751330	Ψ_{-5}	Even	4
E_{-4}	−3.797747	Ψ_{-4}	Odd	3
E_{-3}	−3.110552	Ψ_{-3}	Even	2
E_{-2}	−2.448525	Ψ_{-2}	Odd	1
E_{-1}	−1.664147	Ψ_{-1}	Even	0
E_0	1.682410	Ψ_0	Even	0
E_1	2.492612	Ψ_1	Odd	1
E_2	3.008767	Ψ_2	Even	2
E_3	3.649014	Ψ_3	Odd	3
E_4	4.749066	Ψ_4	Even	4
E_5	5.503584	Ψ_5	Odd	5

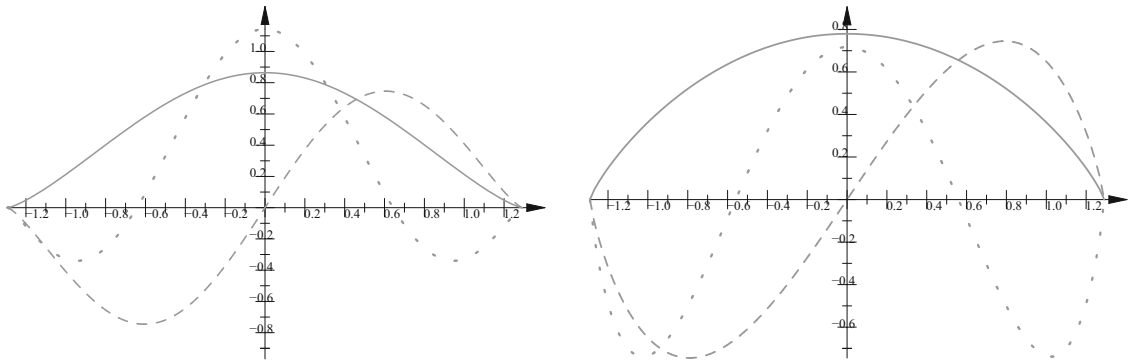

Fig. 1 The solutions Ψ_0 (solid curve), Ψ_1 (*dashed curve*) and Ψ_2 (*dotted curve*) of the Klein–Gordon model (*left plot*) and of its Schrödinger counterpart (*right plot*) for the parameter setting $\mu = 3/5$

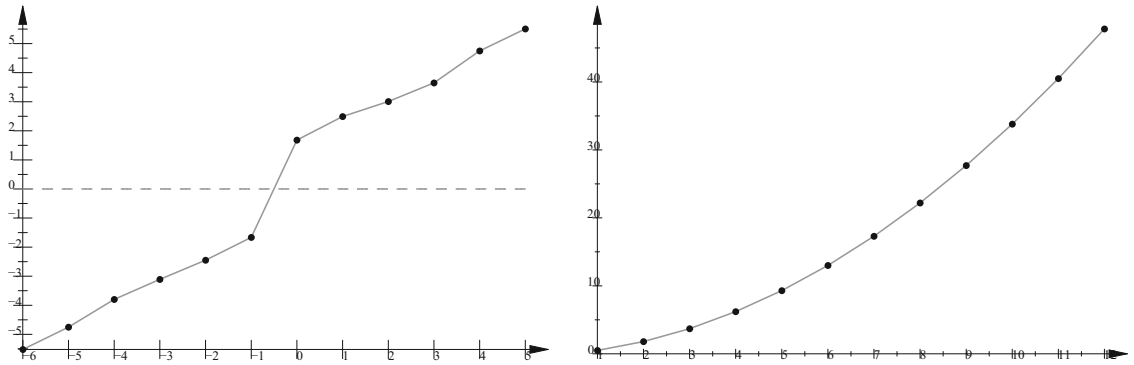

Fig. 2 Bound state energies for the Klein–Gordon system (*left plot*) and for the Schrödinger model (*right plot*), taken from (7). In both cases we have $\mu = 3/5$

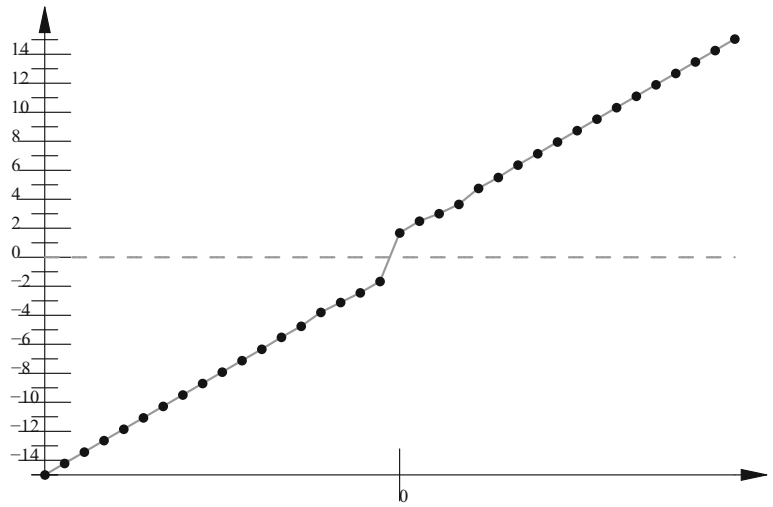

Fig. 3 The Klein–Gordon bound state energies for large indices and $\mu = 3/5$. We see that the energy sequence becomes linear and equidistant

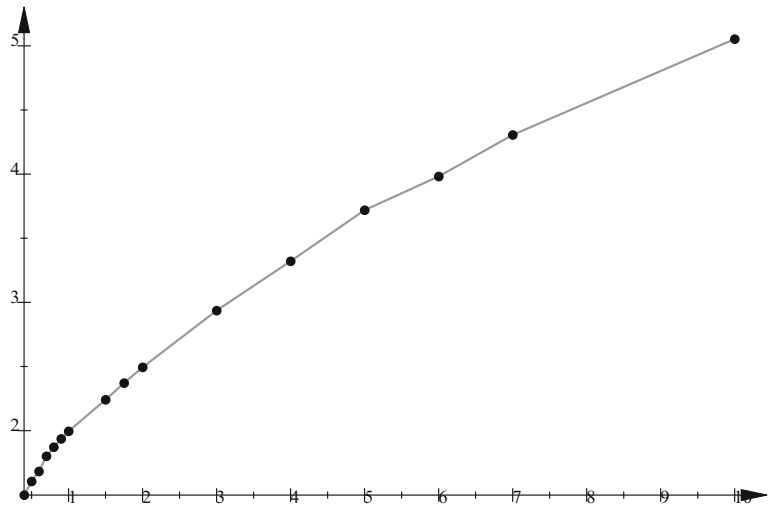

Fig. 4 Dependence of the Klein–Gordon ground state energy E_0 on the parameter μ

5 Concluding Remarks

We have studied a boundary-value problem for the Klein–Gordon equation in a setting that resembles the classical interaction of the Mathews-Lakshmanan oscillator. Similar to the corresponding Schrödinger system that was studied previously, our Klein–Gordon model turned out to admit discrete bound state energies and an associated orthogonal set of solutions. A possible next step within the present line of work is to set up a Dirac equation for the Mathews-Lakshmanan oscillator in order to study existence and form of bound states.

References

1. Abramowitz, M., Stegun, I.: Handbook of mathematical functions with formulas, graphs, and mathematical tables. Dover Publications, New York (1964)
2. Arda, A., Sever, R., Tezcan, C.: Approximate analytical solutions of the Klein Gordon equation for the Hulthen potential with the position dependent mass. Phys. Scr. **79**, 015006 (2009)
3. Carinena, J.F., Ranada, M.F., Santander, M.: The quantum free particle on spherical and hyperbolic spaces: A curvature dependent approach. II. J. Math. Phys. **53**, 102109 (2012)
4. Carinena, J.F., Ranada, M.F., Santander, M.: A quantum exactly-solvable nonlinear oscillator with quasi-harmonic behaviour. Ann. Phys. **322**, 434–459 (2007)
5. Carinena, J.F., Ranada, M.F., Santander, M., Senthilvelan, M.: A non-linear oscillator with quasi-harmonic behaviour: two- and n-dimensional oscillators. Nonlinearity **17**, 1941–1963 (2004)
6. Carinena, J.F., Ranada, M.F., Santander, M.: One-dimensional model of a quantum non-linear harmonic oscillator. Rep. Math. Phys. **54**, 285–293 (2004)
7. Farrokh, M., Shojaei, M.R., Rajabi, A.A.: Klein–Gordon equation with Hulthen potential and position-dependent mass. Eur. Phys. J. Plus **128**, 14 (2013)
8. Higgs, P.W.: Dynamical symmetries in a spherical geometry. J. Phys. A **12**, 309–323 (1979)
9. Ikhdair, S.M.: Exact Klein–Gordon equation with spatially dependent masses for unequal scalar-vector Coulomb-like potentials. Eur. Phys. J. A **40**, 143–149 (2009)
10. Jia, C.-S., Li, X.-P., Zhang, L.-H.: Exact solutions of the Klein–Gordon equation with position-dependent mass for mixed vector and scalar kink-like potentials. Few-Body Syst. **52**, 11–18 (2012)
11. Lakshmanan, M., Eswaran, K.: Quantum dynamics of a solvable nonlinear chiral model. J. Phys. A **8**, 1658–1669 (1975)
12. Mathews, P.M., Lakshmanan, M.: On a unique nonlinear oscillator. Quart. Appl. Math. **32**, 215–218 (1974)
13. Mathews, P.M., Lakshmanan, M.: A quantum-mechanically solvable nonpolynomial Lagrangian with velocity-dependent interaction. Nuovo Cimento A **26**, 299–316 (1975)
14. Morse, P.M., Feshbach, H.: Methods of theoretical physics, vol. 2. Mc Graw-Hill, New York (1953)
15. Midya, B., Roy, B.: A generalized quantum nonlinear oscillator. J. Phys. A **42**, 285301 (2009)
16. Ronveaux, A.: Heun's differential equations. Oxford University Press, New York (1995)
17. Schulze-Halberg, A., Morris, J.R.: Special function solutions of a spectral problem for a nonlinear quantum oscillator. J. Phys. A **45**, 305301 (2012)

18. Slavianov, S.Y., Lay, W.: Special functions: a unified theory based on singularities. Oxford University Press, New York (2000)
19. Roos, O.von : Position-dependent effective masses in semiconductor theory. Phys. Rev. B **27**, 7547–7552 (1983)
20. Roos, O.von , Mavromatis, H.: Position-dependent effective masses in semiconductor theory. II. Phys. Rev. B **31**, 2294–2298 (1985)