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Solvable Models in the Conventional and Light-Front Field Theory: Recent Progress

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Abstract We discuss a few new results within the exactly solvable relativistic models studied in both the conventional and the light front field theory. The models include the Rothe–Stamatescu, the Thirring, the Federbush and the Thirring–Wess model. The unifying feature is that the corresponding field equations are solved in a simple and exact form. We work within the hamiltonian framework and pay a careful attention to the correct definition of interacting currents which are built from the known solutions in a point-split regularized manner. Quantum "anomalies" follow immediately. The Hamiltonians of the models are expressed in terms of the correct (dynamically independent) field variables, namely the free Heisenberg fields. Due to the simplicity of the models' dynamics, one can explicitly determine structure of the physical ground states.

1 Introduction

Solvable models are simple two-dimensional relativistic field theories in which solutions of the field equations can be written down on the quantum level in an exact (non-approximate) form. The models have played a major role in the development of quantum field theory (QFT) [1–6] and in particular in testing its methods and obtaining intuition about dynamical mechanisms potentially valid in far more complex realistic models in four dimensions. In spite of the simplicity of the solvable theories (such as the massless and massive models with derivative coupling [3,7,8], the Thirring [1], Thirring–Wess [9], Federbush [10] and the Schwinger model [2]), the generally accepted consensus on their physical content has not been achieved. Some features of their dynamics have even been overlooked or not completely understood. The cleanest physical picture of the models emerges in the hamiltonian treatment. The latter can further be refined by incorporating information about the solution of the field equations [11]. These solutions (which have to be properly regularized on the quantum level) identify in all cases free Heisenberg fields as the true "building blocks" of the dynamics without the need to introduce Ansaetze that bear a danger to obscure some properties of the theory.

Another feature which should be carefully incorporated is the correct mathematical treatment. This includes on the one hand the correctly defined (regularized) form of the product of the fermion field operators and of the exponential of the (elementary or composite) scalar field, on the other hand the fields themselves should be in principle considered as operator-valued distribution [12]. We will concentrate on the first aspect in the present treatment postponing the second issue to the future work. However, the finite-volume treatment with fields

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(anti)periodic in the space variable is a useful alternative which yields a consistent framework for detecting and analyzing some subtle points in the studied models [13].

While the exactly solvable models have a clear disadvantage of being not realistic, but merely "toy" models in an unphysical two-dimensional world (mostly even not belonging to the class of gauge theories), they also have some important advantages. These include the fact that non-perturbative (NP) studies are possible since explicit solutions of the Heisenberg equations are known. Hence a complete information on the NP dynamics and insights into the structure of QFT and its vacuum properties are available in principle. Last but not least, the models also make a comparison between the conventional space-like (SL) and light-front (LF) forms of the theory possible on a non-perturbative level.

Our formulation of the solvable models has one distinctive feature, namely reformulation of the dynamics in terms of the free fields. That the latter are the true degrees of freedom is easily seen from the corresponding form of the solutions of the field equations. As we will show, this step is crucial for removing discrepancies between physical predictions of the SL and LF forms of relativistic dynamics.

After an elementary example of the correct treatment of the free massive vector current, we will give an overview of our formulation of the Rothe–Stamatescu, Thirring and the Federbush model. A few remarks concerning the Thirring–Wess and the Schwinger model will also be added in the Summary.

2 Free Fermionic Vector Current

The vector current of the free fermion field in D = 1 + 1 is conventionally defined as a normal ordered product

$$j^\mu(x) =: \psi^\dagger(x)\gamma^0\gamma^\mu\psi(x) : . \tag{1}$$

Here we will show that this definition emerges naturally from the regularization of the product of two fields in (1) by the point splitting, $x \pm \frac{\epsilon}{2}$, if one splits in a manner not violating hermiticity of the current. The corresponding current is the $\epsilon \to 0$ limit of the non-local expression

$$j^\mu(x) = \frac{1}{2}\Big[\psi^\dagger\left(x+\frac{\epsilon}{2}\right)\gamma^0\gamma^\mu\psi\left(x-\frac{\epsilon}{2}\right) + \psi^\dagger\left(x-\frac{\epsilon}{2}\right)\gamma^0\gamma^\mu\psi\left(x+\frac{\epsilon}{2}\right)\Big]. \tag{2}$$

Evaluating the integral arising in momentum representation in course of normal-ordering, we get

$$\psi^\dagger\left(x+\frac{\epsilon}{2}\right)\gamma^0\gamma^\mu\psi\left(x-\frac{\epsilon}{2}\right) =: \psi^\dagger\left(x+\frac{\epsilon}{2}\right)\gamma^0\gamma^\mu\psi\left(x-\frac{\epsilon}{2}\right) : -\frac{i}{2\pi}\frac{\epsilon^\mu}{\epsilon^2}. \tag{3}$$

It follows that the singular parts in (2) cancel leaving (1) as the result. Cancelation of the singular parts appears also for the interacting currents as the example of the Rothe–Stamatescu model shows.

3 The Rothe–Stamatescu Model

The Lagrangian of the massive Rothe–Stamatescu model defines a gradient coupling between the pseudoscalar field of mass μ and the axial-vector current of the massive fermions

$$\mathcal{L} = \frac{i}{2}\overline{\Psi}\gamma^\mu \overleftrightarrow{\partial_\mu}\Psi - m\overline{\Psi}\Psi + \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}\mu^2\phi^2 - g\partial_\mu\phi J_5^\mu, \quad J_5^\mu = \overline{\Psi}\gamma^\mu\gamma^5\Psi. \tag{4}$$

Its massless version was studied in [8]. The corresponding field equations read

$$i\gamma^\mu\partial_\mu\Psi = m\Psi + g\partial_\mu\phi\gamma^\mu\gamma^5\Psi, \quad \partial_\mu\partial^\mu\phi + \mu^2\phi^2 = g\partial_\mu J_5^\mu = 2img\overline{\Psi}\gamma^5\Psi. \tag{5}$$

Scalar field is not free as it is the case for the vector-current interaction (the derivative-coupling model, DCM). The Dirac equation seems to have an operator solution [14] similar to the one from the DCM:

$$\Psi(x) = e^{-ig\gamma^5\phi(x)}\psi(x), \quad i\gamma^\mu\partial_\mu\psi = m\psi. \tag{6}$$

A simple check

$$i\gamma^\mu\partial_\mu\Psi(x) = i\gamma^\mu\big[e^{-ig\gamma^5\phi(x)}\partial_\mu\psi(x) - ig\gamma^5\partial_\mu\phi(x)\Psi(x)\big] = e^{+ig\gamma^5\phi(x)}i\gamma^\mu\partial_\mu\psi(x) + g\partial_\mu\phi(x)\gamma^\mu\gamma^5\Psi(x) \tag{7}$$

reveals the difficulty: the sign in the last exponential is reversed due to $\gamma^\mu\gamma^5 = -\gamma^5\gamma^\mu$ and hence (6) does not solve the field equation. Since it is the mass term that prevents solvability, the original massless RS model [8] is solvable and in fact almost trivial: the massless axial-vector current is conserved, hence the pseudoscalar field is free and the Dirac equation is exactly solved by (6). In the original treatment [8], starting from the definition of the vector current (VEV stands for the vacuum expectation value)

$$j^\mu_\epsilon(x) = \overline{\Psi}(x+\epsilon)\gamma^\mu\Psi(x)\exp\left(ig\int_x^{x+\epsilon} dy_\lambda \epsilon^{\lambda\nu}\partial_\nu\phi(y)\right) - VEV. \tag{8}$$

the vector and axial-vector quantum "anomaly" was found. The exponential of the line integral of the scalar field was introduced in (8) motivated by a similar construction in the Schwinger model [2]. However, in the RS model no symmetry is present and consequently this construction is not valid. Here we will briefly show that the results [8] are actually (and surprisingly) obtained also in the correct treatment.

In quantum theory, singular operator products (even in the Lagrangian) have to be regularized. Contrary to the standard practice, we do not define quantum solution $\Psi(x)$ (6) as a normal-ordered exponential, but simply regularize it by the point-splitting of the positive and negative-frequency part of the scalar field in the exponential and by applying the BCH operator identity $e^A e^B = e^{\frac{1}{2}[A,B]}e^{A+B}$:

$$\Psi(x) = Z^{1/2}(\epsilon)e^{-ig\gamma^5\phi^{(-)}(x)}e^{-ig\gamma^5\phi^{(+)}(x)}\psi(x), \tag{9}$$

where $Z^{1/2}(\epsilon) = \exp\big\{g^2[\phi^{(+)}(x-\frac{\epsilon}{2}), \phi^{(-)}(x+\frac{\epsilon}{2})]\big\} = \exp\big\{-ig^2D^{(+)}(\epsilon)\big\}$ and $D^{(+)}(x-y)$ is the corresponding two-point function. The difference is that we keep the regularized (infinite) constant $Z(\epsilon)$. Then there is no need to define a renormalized solution, since the regularized factors automatically cancel in the point-split interacting currents:

$$J^\mu(x) = s\lim_{\epsilon\to 0}\frac{1}{2}\Big\{Z(\epsilon)\overline{\psi}(x+\frac{\epsilon}{2})e^{ig\gamma^5\phi^{(-)}(x+\frac{\epsilon}{2})}e^{ig\gamma^5\phi^{(+)}(x+\frac{\epsilon}{2})}$$
$$\times\gamma^\mu e^{-ig\gamma^5\phi^{(-)}(x-\frac{\epsilon}{2})}e^{-ig\gamma^5\phi^{(+)}(x-\frac{\epsilon}{2})}\psi(x-\frac{\epsilon}{2}) + H.c.\Big\} =: \overline{\psi}(x)\gamma^\mu\psi(x) : + \frac{g}{2\pi}\epsilon^{\mu\nu}\partial_\nu\phi(x). \tag{10}$$

$s\lim$ designates the symmetric limit. The free-field relation (3) has been used in (10).

Note that no vacuum subtractions have been used in the above derivation—all singular terms have automatically cancelled due to the manifestly hermitian definition of the current. The constant $Z(\epsilon)$ got cancelled by the factor $Z^{-1}(\epsilon)$ coming from normal-ordering of the two exponentials sandwiching γ^μ in (10). While $J^\mu(x)$ is conserved due to the presence of $\epsilon^{\mu\nu}$ in the quantum-correction term, the axial-vector current is not. It is obtained analogously as

$$J^\mu_5(x) =: \overline{\psi}(x)\gamma^5\gamma^\mu\psi(x) : + \frac{g}{2\pi}\partial^\mu\phi(x) \tag{11}$$

Its divergence does not vanish, $\partial_\mu J^\mu(x) = \frac{g}{2\pi}\partial_\mu\partial^\mu\phi(x)$. However, the only effect of the anomaly is to renormalize the scalar field mass,

$$\partial_\mu\partial^\mu\phi + \tilde{\mu}^2\phi = 0, \quad \tilde{\mu}^2 = \frac{\mu^2}{1-\frac{g^2}{2\pi}}. \tag{12}$$

The conjugate momenta $\Pi_\phi = \partial_0\phi(x) - gJ^0_5$, $\Pi_\Psi = \frac{i}{2}\Psi^\dagger$, $\Pi_{\Psi^\dagger} = -\frac{i}{2}\Psi$ lead from the Lagrangian (4) to the Hamiltonian $H = H_{0B} + H^{'}$. H_{0B} corresponds to the free massive scalar field and

$$H^{'} = \int_{-\infty}^{+\infty} dx^1\big[-i\Psi^\dagger\alpha^1\partial_1\Psi + g\partial_1\phi J^1_5\big]. \tag{13}$$

However, re-expressing the Lagrangian in terms of the true dynamical variables $\psi(x)$ and $\phi(x)$ leads to the total Hamiltonian given as a sum of free fermion and boson Hamiltonians. Consequently the spectrum of the model consists of free massless fermions and massive bosons (with mass renormalized by a finite amount).

Correlation functions are composed from the free fermion and boson two-point functions, but depend on the coupling constant. Note also that the momentum operator is ill-defined (it contains an interacting piece) if the knowledge of the operator solution is not taken into account.

4 The Massless Thirring Model

The model describes the current–current interaction of two-dimensional massless fermions. The operator solution of the model was given by Klaiber [15], who also constructed (sophistically regularized) n-point correlation functions. The solution gave basis to the Coleman's (perturbative) bosonization or rather to the discovery of the equivalence between the sine-Gordon model and massive Thirring model [16]. A legitimate question is: have indeed all aspects of the model been clarified? An extensive study of the model and of a few related topics has been made in a relatively recent series of papers by Faber and Ivanov (see [17] and references therein). In particular, they claimed to have discovered a broken phase of the model. Their analysis was based on a BCS-like Ansatz for the ground state. Here we will give a brief summary of our ab initio type of study emphasizing the distinctive new elements. The key observation is that the Hamiltonian following from the Klaiber's solution is non-diagonal in composite boson Fock operators $c(k^1), c^\dagger(k^1)$ (see below).

The Lagrangian density of the massless Thirring model and the corresponding field equations read

$$\mathcal{L} = \frac{i}{2}\overline{\Psi}\gamma^\mu \overset{\leftrightarrow}{\partial_\mu} \Psi - \frac{1}{2}g J_\mu J^\mu, \quad i\gamma^\mu\partial_\mu\Psi(x) = gJ^\mu(x)\gamma_\mu\Psi(x) \tag{14}$$

The Klaiber's solution of the Dirac equation (14) has the form

$$\Psi(x) = e^{i(g/\sqrt{\pi})\left(\alpha\tilde{j}(x)-\beta\gamma^5 j(x)\right)}\psi(x), \quad \gamma^\mu\partial_\mu\psi(x) = 0. \tag{15}$$

The coefficients α and β satisfy $\alpha + \beta = 1$. The "potentials" $j(x)$ and $\tilde{j}(x)$ are connected to the (normal-ordered) free current $j^\mu(x)$ according to $\partial_\mu\tilde{j}(x) = -\sqrt{\pi}j_\mu(x), \partial_\mu j(x) = \sqrt{\pi}\epsilon_{\mu\nu}j^\nu(x)$. This corresponds to replacing $J^\mu(x)$ by $j^\mu(x)$ in the field equation (14) which is a rather restrictive assumption.

A more general treatment is possible. We set $\beta = 0$ for simplicity and consider the solution

$$\Psi(x) = e^{i(g/\sqrt{\pi})\tilde{J}(x)}\psi(x), \tag{16}$$

with the unknown potential $\tilde{J}(x)$ of the interacting current $J^\mu(x)$, i.e. defining $\partial_\mu\tilde{J}(x) = -\sqrt{\pi}J_\mu(x)$. Compute the interacting current from the solution (16) using the point-splitting regularization as in Eq.(10):

$$J^\mu(x) =: \overline{\psi}(x)\gamma^\mu\psi(x) : + \frac{g}{2\pi}J^\mu(x) \Rightarrow J^\mu(x) = G(g)j^\mu(x), \quad G(g) = \left(1 - \frac{g}{2\pi}\right)^{-1}. \tag{17}$$

Thus the interacting current is simply the renormalized free current. In the Klaiber's solution the factor $G(g)$ was missed. This may have consequences for bosonization of the massive Thirring model [16].

The rest of the study consists in a bosonization of the free vector current and a subsequent Bogoliubov transformation to diagonalize the Hamiltonian and to find the lowest-energy eigenstate [11].

The fermion field is expanded using the "spinors" $u^\dagger = (\theta(-p^1), \theta(p^1)), \ v^\dagger = (-\theta(-p^1), \theta(p^1))$ as

$$\psi(x) = \int \frac{dp^1}{\sqrt{2\pi}}\left\{b(p^1)u(p^1)e^{-i\hat{p}\cdot x} + d^\dagger(p^1)v(p^1)e^{i\hat{p}\cdot x}\right\}, \quad \hat{p}\cdot x = |p^1|t - p^1x^1,$$
$$\left\{b(p^1), b^\dagger(q^1)\right\} = \left\{d(p^1), d^\dagger(q^1)\right\} = \delta(p^1 - q^1), \quad b(k^1)|0\rangle = d(k^1)|0\rangle = 0. \tag{18}$$

After the Fourier transformation, the current $j^\mu(x)$ is expressed in terms of boson operators $c(k^1)$:

$$j^\mu(x) = \frac{-i}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}\frac{dk^1k^\mu}{\sqrt{2|k^1|}}\left\{c(k^1)e^{-i\hat{k}\cdot x} - c^\dagger(k^1)e^{i\hat{k}\cdot x}\right\}, \quad \left[c(p^1), c^\dagger(q^1)\right] = \delta(p^1 - q^1), \quad c(k^1)|0\rangle = 0,$$
$$c(k^1) = \frac{i}{\sqrt{|k^1|}}\int_{-\infty}^{+\infty}dp^1\left\{\theta\left(p^1k^1\right)\left[b^\dagger(p^1)b(p^1+k^1) - (b \to d)\right] + \epsilon(p^1)\theta\left(p^1(k^1-p^1)\right)d(k^1-p^1)b(p^1)\right\}. \tag{19}$$

The Hamiltonian is derived from the Lagrangian (14) after inserting the solution (15) into it. The contribution of the term $(i/2)\overline{\Psi}\gamma^{\mu} \overset{\leftrightarrow}{\partial}_{\mu} \Psi$ reverses the sign of the interacting term,

$$H = \int_{-\infty}^{+\infty} \mathrm{d}x^1 \Big[-i\psi^{\dagger}\alpha^1 \partial_1 \psi - \frac{1}{2} g \big(J^0 J^0 - J^1 J^1 \big) \Big]. \tag{20}$$

In Fock representation, $H = H_0 + H_g$ has the form

$$H = \int_{-\infty}^{+\infty} \mathrm{d}p^1 |p^1| \Big[b^{\dagger}(p^1) b(p^1) + d^{\dagger}(p^1) d(p^1) \Big] + G^2(g) \frac{g}{\pi} \int_{-\infty}^{+\infty} dk^1 |k^1| \Big[c^{\dagger}(k^1) c^{\dagger}(-k^1) + c(k^1) c(-k^1) \Big]. \tag{21}$$

H_g is not diagonal and thus $|0\rangle$ is not an eigenstate of the full H. Its diagonalization is performed by a Bogoliubov transformation e^{iS} [11,18], yielding simultaneously the physical ground state $|\Omega\rangle = e^{-iS}|0\rangle$:

$$\hat{H}_g^d = \frac{1}{\cosh 2\gamma_d} \int_{-\infty}^{+\infty} \mathrm{d}k^1 |k^1| c^{\dagger}(k^1) c(k^1), \quad |\Omega\rangle = N \exp\Bigg[-\kappa \int_{-\infty}^{+\infty} \mathrm{d}p^1 c^{\dagger}(p^1) c^{\dagger}(-p^1) \Bigg] |0\rangle, \tag{22}$$

where $\gamma_d = \frac{1}{2}\mathrm{artanh}\, G(g)\frac{2g}{\pi}$ and $\kappa = \frac{1}{2}\tanh \gamma_d$. The new vacuum has the form of a coherent state of pairs of composite bosons with zero total momentum $P^1|\Omega\rangle = 0$ and is invariant under axial $U(1)$ transformations

$$V(\beta)|\Omega\rangle = |\Omega\rangle, \quad V(\beta) = e^{i\beta Q_5}, \quad Q_5 = \int_{-\infty}^{+\infty} \mathrm{d}k^1 \epsilon(k^1) \big[b^{\dagger}(k^1) b(k^1) - d^{\dagger}(k^1) d(k^1) \big]. \tag{23}$$

Thus, no chiral symmetry breaking occurs (contrary to some claims in literature [17,19]). Correlation functions have to be calculated using the vacuum $|\Omega\rangle$ and the "integrated current" (with the infra-red cutoff η)

$$J(x) = \frac{G(g)}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \mathrm{d}q^1 \frac{\theta\big(|q^1| - \eta\big)}{\sqrt{2|q^1|}} \eta(q^1) \big[c(q^1) e^{-i\hat{q}\cdot x} + c^{\dagger}(q^1) e^{i\hat{q}\cdot x} \big]. \tag{24}$$

However, a simpler formulation is obtained if we keep the Fock vacuum as the physical ground state and transform all operators including the solution (15) into the new (inequivalent) representation defined by $O \rightarrow e^{-iS} O e^{iS}$. This transformation is of course the Bogoliubov transformation used to diagonalize the Hamiltonian in Eq. (22). One has to keep in mind that the operator e^{-iS} in not well defined in the continuum theory (it has a zero norm) without a suitable regularization. Another option is to reformulate the present solution of the Thirring model in a finite volume [20].

5 The Federbush Model

We will give here a very brief description of the main steps of the solution of the model in the hamiltonian form including its massless version.

The Lagrangian of the Federbush model

$$\mathcal{L} = \frac{i}{2} \overline{\Psi} \gamma^{\mu} \overset{\leftrightarrow}{\partial}_{\mu} \Psi - m \overline{\Psi} \Psi + \frac{i}{2} \overline{\Phi} \gamma^{\mu} \overset{\leftrightarrow}{\partial}_{\mu} \Phi - \mu \overline{\Phi} \Phi - g \epsilon_{\mu\nu} J^{\mu} H^{\nu}, \tag{25}$$

encodes the dynamics of two species of coupled fermion fields with masses m and μ. Both currents $J^{\mu} = \overline{\Psi}\gamma^{\mu}\Psi$, $H^{\mu} = \overline{\Phi}\gamma^{\mu}\Phi$ are conserved. The coupled field equations

$$i\gamma^{\mu}\partial_{\mu}\Psi(x) = m\Psi(x) + g\epsilon_{\mu\nu}\gamma^{\mu}H^{\nu}(x)\Psi(x), \quad i\gamma^{\mu}\partial_{\mu}\Phi(x) = \mu\Phi(x) - g\epsilon_{\mu\nu}\gamma^{\mu}J^{\nu}(x)\Phi(x) \tag{26}$$

are exactly solvable:

$$\Psi(x) = e^{-i(g/\sqrt{\pi})h(x)}\psi(x),\ \ i\gamma^\mu\partial_\mu\psi(x) = m\psi(x),$$
$$\Phi(x) = e^{i(g/\sqrt{\pi})j(x)}\varphi(x),\ \ i\gamma^\mu\partial_\mu\varphi(x) = \mu\varphi(x). \tag{27}$$

In quantum theory, the above exponentials are regularized by the "triple-dot ordering" [4]. The potentials $j(x)$ and $h(x)$ are defined as $\partial_\mu j(x) = \sqrt{\pi}\epsilon_{\mu\nu}j^\nu(x)$, $\partial_\mu h(x) = \sqrt{\pi}\epsilon_{\mu\nu}h^\nu(x)$. They enter into the solutions (27) in an "off-diagonal" way. After inserting the solutions into the Lagrangian (25), the interaction term changes its sign yielding the Hamiltonian (H_0 is the sum of two free Hamiltonians)

$$H = H_0 + g\int_{-\infty}^{+\infty} \mathrm{d}x^1\big(j^0h^1 - j^1h^0\big). \tag{28}$$

The LF field equations are also solved by (27) with the free LF fields $\psi(x)$, $\varphi(x)$ and $j(x)$, $h(x)$ given by $2\partial_- j(x) = \sqrt{\pi}j^+(x)$, $2\partial_- h(x) = \sqrt{\pi}h^+(x)$. The conventional LF treatment is to insert the solution of the fermionic constraint into $\mathcal{L}$. This however generates the free LF Hamiltonian! Only if one inserts the full solution like in the SL case, the four-fermion interaction term persists also in the LF case:

$$P^-_g = \frac{1}{2}g\int_{-\infty}^{+\infty}\frac{\mathrm{d}x^-}{2}\big(j^+h^- - j^-h^+\big). \tag{29}$$

The interacting SL Hamitonian (28) contains terms composed solely from creation or annihilation operators, so the Fock vacuum is not its eigenstate. The diagonalization can be performed by a Bogoliubov transformation using a *massive* current bosonization. This is considerably more complicated than the massless case. The massive analog (up to the kinematical factors) of the boson operator $c(k^1)$ (19) is

$$\begin{aligned}
A(k^1,t) = i\int_{-\infty}^{+\infty}\frac{dp^1}{\sqrt{E(k^1)}}\Big\{&\big[b^\dagger(p^1)b(k^1+p^1) - (b\to d)\big]\tilde{f}_1\left(p^1,p^1+k^1\right)e^{i\left(E(p^1)-E(k^1+p^1)\right)t}\theta(k^1p^1)\\
&+\frac{1}{2}\big[b^\dagger(-p^1)b(k^1-p^1) - (b\to d)\big]\theta(p^1(k^1-p^1))\tilde{f}_1(-p^1,k^1-p^1)e^{i\left(E(p^1)-E(k^1-p^1)\right)t}\\
&+d(p^1)b(k^1-p^1)\times\epsilon(p^1)\theta\left(p^1(k^1-p^1)\right)\tilde{f}_2(p^1,k^1-p^1)e^{-i\left(E(p^1)+E(k^1-p^1)\right)t}\\
&+d(p^1+k^1)b(-p^1)\theta(p^1k^1)\tilde{f}_2\left(-p^1,p^1+k^1\right)\times e^{-i\left(E(p^1)+E(k^1+p^1)\right)t}\\
&-b(p^1)d\left(-(p^1-k^1)\right)\theta\left(k^1(p^1-k^1)\right)\tilde{f}_2\left(p^1,-(p^1-k^1)\right)e^{-i\left(E(p^1)+E(k^1-p^1)\right)t}\Big\}.
\end{aligned} \tag{30}$$

The quantities

$$\tilde{f}_1(p^1,q^1) = \frac{\sqrt{p^+q^+}+\sqrt{p^-q^-}}{\sqrt{2E(p^1)}\sqrt{2E(q^1)}},\quad \tilde{f}_2(p^1,q^1) = \frac{\sqrt{p^+q^+}-\sqrt{p^-q^-}}{\sqrt{2E(p^1)}\sqrt{2E(q^1)}} \tag{31}$$

are two coefficient functions appearing in four spinor products of the form $u^\dagger(p^1)\gamma^0\gamma^\mu u(q^1)$ etc., which arise when one calculates the free vector current from the Fock expansion of the free massive fermion field:

$$\begin{aligned}
j^0(x) = \iint dp^1dq^1\big[&\big(b^\dagger(p^1)b(q^1) - (b\to d)\big)e^{i(\hat{p}-\hat{q})\cdot x}\tilde{f}_1(p^1,q^1)\\
&+\big(b^\dagger(p^1)d^\dagger(q^1)e^{i(\hat{p}+\hat{q})\cdot x} + H.c.\big)\tilde{f}_2(p^1,q^1)\big].
\end{aligned} \tag{32}$$

For the component $j^1(x)$, the functions $\tilde{f}_1$ and $\tilde{f}_2$ are interchanged. Although the operators $A(k^1,t)$, $A^\dagger(k^1,t)$ (which for $m = 0$ reduce to Klaiber's $c(k^1)$, $c^\dagger(k^1)$) have a complicated structure (for example, there is no common $\hat{k}^\mu$ factor as in the massless current and each term has a separate time dependence), still they represent a useful concept since their algebraic properties are simple at equal times and the Hamiltonian of the model becomes quadratic when expressed in terms of them.

The corresponding massive charge density in the bosonized form is then written as

$$j^0(x) = \frac{-i}{\sqrt{2}\pi} \int_{-\infty}^{+\infty} \frac{dk^1 E(k^1)}{\sqrt{2E(k^1)}} \big[A(k^1,t)e^{ik^1x^1} + A^\dagger(k^1,t)e^{-ik^1x^1}\big]. \tag{33}$$

The analogous LF operators are much simpler and have a structure similar to the massless SL case (19):

$$\hat{A}(k^+,x^+) = i\int_0^{+\infty} \frac{dp^+}{\sqrt{k^+}} \Big\{\big[\hat{b}^\dagger(p^+)\hat{b}(k^+ + p^+) - (\hat{b}\to\hat{d})\big]e^{\frac{i}{2}\frac{m^2k^+x^+}{p^+(k^+ + p^+)}} + \hat{d}(p^+)\hat{b}(k^+ - p^+)e^{-\frac{i}{2}\frac{m^2k^+x^+}{p^+(k^+ - p^+)}}\Big\}. \tag{34}$$

k^+ and $x^-(x^+)$ is the LF momentum and space (time) variable, respectively. We use the hat notation to distinguish the LF operators from the conventional ones. The bosonized form of the LF current is given in terms of $\hat{A}$ as

$$j^+(x) = \frac{-i}{2\pi}\int_0^{\infty} \frac{dk^+}{\sqrt{k^+}} k^+\big[\hat{A}(k^+,x^+)e^{-\frac{i}{2}k^+x^-} + \hat{A}^\dagger(k^+,x^+)e^{\frac{i}{2}k^+x^-}\big],\ \Big[\hat{A}(k^+,x^+),\hat{A}^\dagger(l^+,x^+)\Big] = \delta(k^+ - l^+). \tag{35}$$

In deriving $\hat{A}(k^+,x^+)$, we have used the Fock expansion of the dynamical fermion field component

$$\psi_2(x) = \int_0^{\infty} \frac{dp^+}{4\pi}\big[\hat{b}(p^+)e^{-i\hat{p}\cdot x} + \hat{d}^\dagger(p^+)e^{i\hat{p}\cdot x}\big],\quad \Big\{\hat{b}(p^+),\hat{b}^\dagger(q^+)\Big\} = \Big\{\hat{d}(p^+),\hat{d}^\dagger(q^+)\Big\} = \delta(p^+ - q^+). \tag{36}$$

The field $\varphi_2(x)$ is expanded analogously. Similar formulae hold for the solution $\Psi(x)$ built from the operators $\hat{B}(k^+,x^+)$, $\hat{B}^\dagger(k^+,x^+)$ which are constructed from $h^+(x)$. The j^- and h^- currents contain the boson operators $\hat{C}(k^+,x^+)$, $\hat{D}(k^+,x^+)$ and their conjugates, related to $\hat{A}$, $\hat{A}^\dagger$, $\hat{B}$, $\hat{B}^\dagger$ via the current conservation. Unlike the SL Hamiltonian, its LF analog is diagonal and therefore $|0\rangle$ is its lowest-energy eigenstate:

$$P_g^- = \frac{g}{8\pi}\int_0^{+\infty} \mathrm{d}k^+k^+\Big[\hat{A}^\dagger(k^+)\hat{D}(k^+) + \hat{D}^\dagger(k^+)\hat{A}(k^+) - \hat{B}^\dagger(k^+)\hat{C}(k^+) - \hat{C}^\dagger(k^+)\hat{B}(k^+)\Big]. \tag{37}$$

The next step is to compute the correlation functions in both schemes. This task is far from being simple since one needs to know the commutators of the composite boson operators at unequal times. This is the place where complexities of the usual triple-dot ordering technique enter into our bosonization approach. Irrespectively of this, the LF calculation will be much simpler: it works with a Fock vacuum and simple operator structures while the SL formalism requires nontrivial coherent-state vacuum and complicated operator terms.

Let us look at the massless version of the model in the conventional form of the theory. One can derive the truly interacting currents in a full analogy with the calculation in the Thirring model. The currents are built from the operator solutions

$$\Psi(x) = e^{-i\frac{g}{\sqrt{\pi}}\left(\alpha H(x) - \beta\gamma^5\tilde{H}(x)\right)}\psi(x),\quad \Phi(x) = e^{i\frac{g}{\sqrt{\pi}}\left(\alpha J(x) - \beta\gamma^5\tilde{J}(x)\right)}\phi(x), \tag{38}$$

where

$$\partial_\mu\tilde{J} = -\sqrt{\pi}J_\mu(x),\quad \partial_\mu J = \sqrt{\pi}\epsilon_{\mu\nu}J^\nu,\quad \partial_\mu\tilde{H} = -\sqrt{\pi}H_\mu(x),\quad \partial_\mu H = \sqrt{\pi}\epsilon_{\mu\nu}H^\nu. \tag{39}$$

The interacting currents obtained by the point-splitting regularization are

$$\begin{aligned} J^\mu(x) &= \big(1 + \frac{g^2}{4\pi^2}\big)^{-1}\big[j^\mu(x) + \frac{g}{2\pi}(\alpha - \beta)\epsilon^{\mu\nu}h_\nu(x)\big],\\ H^\mu(x) &= \big(1 + \frac{g^2}{4\pi^2}\big)^{-1}\big[h^\mu(x) - \frac{g}{2\pi}(\alpha - \beta)\epsilon^{\mu\nu}j_\nu(x)\big]. \end{aligned} \tag{40}$$

The rest of the analysis will proceed as in the Thirring model case.

6 Summary and Outlook

- Solvable models represent a perfect laboratory for studying subtleties of QFT including the vacuum problem and for comparison between the SL and LF forms of the relativistic dynamics.
- It is crucial to work with the right field variables (the free fields) and correct Hamiltonians.
- The fermion currents have to be computed in a consistent way—a regularization by the point-splitting of the hermitian sum of fermion operator product is required.
- The Rothe–Stamatescu model was reformulated in a "minimal way", quantum corrections to the currents were found directly from the operator solution of the field equation.
- A generalization of the Klaiber's Thirring-model solution was found, truly interacting currents were employed, a diagonalization of the Hamiltonian by a Bogoliubov transformation was performed and the true physical vacuum state was derived in the form of a coherent state. This gives a simple example of the complicated ground state in the SL field theory.
- A Hamiltonian approach to the Federbush model was sketched, including bosonization of the massive current. It was demonstrated that the LF treatment is much simpler. A few elements of the massless SL version of the Federbush model were derived.
- A similar treatment of the Thirring–Wess and Schwinger model can be given [13]. Quantum currents and their "anomalies" are again computed from the regularized operator solutions of the corresponding field equations. The subtleties of the residual gauge invariance in the covariant (Landau) gauge modify the mechanism of the dynamical mass generation in the Schwinger model. A finite-volume treatment reveals that the gauge zero mode plays a crucial role here. A derivation of the vacuum structure and of the chiral symmetry breaking mechanism in the Schwinger model within the present formulation based on the exact solution of field equations is underway.

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