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Approximate Solutions of D-Dimensional Klein-Gordon Equation with modified Hylleraas Potential

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Abstract We study the D-dimensional Klein-Gordon equation for the modified Hylleraas potential with position dependent mass. We obtain the energy eigenvalues and the corresponding eigenfunctions for any arbitrary l-state using the parametric Nikiforov-Uvarov method. New elegant approximation method is used to deal with the centrifugal term. We also discuss the two limiting cases of this potential, i.e. the Woods-Saxon and Rosen-Morse potentials.

1 Introduction

The Klein-Gordon equation (KGE) with a position dependent mass (PDM) has attracted a great attention in recent years because of its applications in particle, nuclear, semiconductor, condensed matter physics [1–10]. In theoretical researches, many researchers have devoted their attention to finding exact or approximate solutions of the KGE with PDM by using various techniques including the Nikiforov-Uvarov (NU) [11], factorization [12], Lie algebraic [13], super symmetric quantum mechanics [14,15], canonical transformation methods [16] and some others [17]. Furthermore, the eigenfunctions obtained with these methods are usually expressed in terms of the Jacobi, Hermite or associated Laguerre polynomials which are all hypergeometric-type polynomials.On the other hand, the study of many physical systems corresponds to a D-dimensional problem in reality and consequently some authors have investigated the arbitrary-dimension case in their studies [18–20]. In this paper, we solve the KGE in D-dimensions with a PDM interacting with a Hylleraas potential [21,22], which, as will be seen later, is a generalized potential yielding three well-known potentials under certain limits.The paper is organized as follows: In Sect. 2, the KGE D-dimensions is presented. Section 3, is devoted to the review of the NU method. The solution of the KGE is given in Sect. 4. Discussions of the result are given in Sect. 5. Finally, we give a brief conclusion in Sect. 6.

2 Klein-Gordon Equation in D-Dimensions

The KGE for a spherically symmetric potential in D-dimension is [23]

$$-\Delta_D \psi_{nlm}(r, \Omega_N) = \left\{ \left[E_{n,l} - V(r) \right]^2 - [m(r) + S(r)]^2 \right\} \psi_{n,l,m}(r, \Omega_D), \tag{1}$$

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where

$$\Delta_D = \nabla_D^2 = \frac{1}{r^{D-1}}\frac{\partial}{\partial r}\left(r^{D-1}\frac{\partial}{\partial r}\right) - \frac{\Lambda_D^2(\Omega_D)}{r^2}, \tag{2}$$

and $E_{n,l}$, $V(r)$, $m(r)$, $S(r)$ are the energy eigenvalues,vector potential, PDM and scalar potential, respectively. The hyperspherical harmonics $Y_l^m(\Omega_D)$ are the eigen functions of the operator $\Lambda_N^2(\Omega_N)$:

$$\psi_{n,l,m}(r, \Omega_D) = R_{nl}(r)Y_l^m(\Omega_D) \tag{3}$$

and $R_{nl}(r)$ is the hyperradial wave function. It is well known that $\frac{\Lambda_D^2(\Omega_D)}{r^2}$ is a generalization of the centrifugal barrier for the D-dimensional space and involves the angular coordinate Ω_D and

$$\Lambda_D^2(\Omega_D)Y_l^m(\Omega_D) = l(l+D-2)Y_l^m(\Omega_D), \quad D > 1 \tag{4}$$

where l is the angular momentum quantum number. By choosing a common ansatz for the wave function in the form

$$R(r) = r^{-\left(\frac{D-1}{2}\right)}U_{nl}(r), \tag{5}$$

Eq. (1) reduces to [23]

$$\left\{\frac{d^2}{dr^2}+E_{n,l}^2+V^2(r)-2E_{nl}V(r)-m^2(r)-S^2(r)-2m(r)S(r)-\frac{(D+2l-1)(D+2l-3)}{4r^2}\right\}U_{nl}(r)=0 \tag{6}$$

Let us now introduce the parametric form of the NU method.

3 Concept of Parametric Nikiforov-Uvarov Method

The NU method [11] was proposed to solve a second-order linear differential equation by reducing it to a generalized equation of hypergeometric-type with the form

$$\psi''(s) + \frac{\tilde{\tau}(s)}{\sigma(s)}\psi'(s) + \frac{\tilde{\sigma}(s)}{\sigma^2(s)}\psi(s) = 0 \tag{7}$$

where the prime denote the differentiationwith respect to s, $\sigma(s)$ and $\tilde{\sigma}(s)$ are polynomials at most of second degree and $\tilde{\tau}(s)$ is a first-degree polynomial. The particular solution of Eq. (7) is obtained by using the common ansatz for the wave function as

$$\psi(s) = \varphi(s)\, y_n(s) \tag{8}$$

which reduces Eq. (7) into a hypergeometric-type equation:

$$\sigma(s)\, y_n''(s) + \tau(s)\, y_n'(s) + \lambda y_n(s) = 0 \tag{9}$$

where $\varphi(s)$ is defined as the logarithmic derivative

$$\frac{\varphi'(s)}{\varphi(s)} = \frac{\pi(s)}{\sigma(s)} \tag{10}$$

and the other wave function $y_n(s)$ is the hypergeometric-type function whose polynomial solution satisfies the Rodriques relation,

$$y_n(s) = \frac{C_n}{\rho(s)}\frac{d^n}{ds^n}\left[\sigma^n(s)\rho(s)\right] \tag{11}$$

where C_n is the normalization constant and the weight function $\rho(s)$ satisfies the condition

$$(\sigma(s)\rho(s))' = \tau(s)\rho(s) \tag{12}$$

The required $\pi(s)$ and λ for the NU method are defined as

$$\pi(s) = \frac{\sigma' - \tilde{\tau}}{2} \pm \sqrt{\left(\frac{\sigma' - \tilde{\tau}}{2}\right)^2 - \tilde{\sigma}(s) + k\sigma(s)} \tag{13}$$

and

$$\lambda = k + \pi'(s) \tag{14}$$

Therefore, the determination of k in Eq. (13) is the necessary step in the calculation of $\pi(s)$ for which the discrimination of the square root in Eq. (13) is set to zero. The eigenvalue equation defined in Eq. (14) takes the form

$$\lambda = \lambda_n = -n\tau' - \frac{n(n-1)}{2}\sigma'', \quad n = 0, 1, 2, \ldots\ldots \tag{15}$$

where

$$\tau(s) = \tilde{\tau}(s) + 2\pi(s) \tag{16}$$

has a negative derivative to ensure the existence of bound-state solutions. The energy eigenvalues are obtained by comparing Eqs. (14) with (15).

The parametric generalization of the NU method that is valid for both central and non-central exponential type potential [24] can be derived by comparing the generalized hypergeometric-type equation

$$\psi''(s) + \frac{(c_1 - c_2 s)}{s(1 - c_3 s)}\psi'(s) + \frac{1}{s^2(1 - c_3 s)^2}\left[-\xi_1 s^2 + \xi_2 s - \xi_3\right]\psi(s) = 0, \tag{17}$$

with Eq. (7). By a simple comparison, we have the correspondence

$$\tilde{\tau}(s) = c_1 - c_2 s \tag{18}$$

$$\sigma(s) = s(1 - c_3 s) \tag{19}$$

$$\tilde{\sigma}(s) = -\xi_1 s^2 + \xi_2 s - \xi_3 \tag{20}$$

Substituting Eqs. (18–20) into Eq. (13), we find

$$\pi(s) = c_4 - c_5 s \pm \left[(c_6 - c_3 k_\pm)s^2 + (c_7 + k_\pm)s + c_8\right]^{\frac{1}{2}}, \tag{21}$$

where

$$c_4 = \frac{1}{2}(1 - c_1), \quad c_5 = \frac{1}{2}(c_2 - 2c_3), \quad c_6 = c_5^2 + \xi_1$$
$$c_7 = 2c_4 c_5 - \xi_2, \quad c_8 = c_4^2 + \xi_3 \tag{22}$$

we obtain the parametric $k_\pm$from the condition that the function under the square root should be square of a polynomial

$$k_\pm = -(c_7 + 2c_3 c_8) \pm 2\sqrt{c_8 c_9} \tag{23}$$

where

$$c_9 = c_3 c_7 + c_3^2 c_8 + c_6 \tag{24}$$

Hence, the function $\pi(s)$ in Eq. (21) becomes

$$\pi(s) = c_4 + c_5 s - \left[\left(\sqrt{c_9} + c_3\sqrt{c_8}\right)s - \sqrt{c_8}\right] \tag{25}$$

and, for the negative k_- values

$$k_- = -(c_7 + 2c_3 c_8) - 2\sqrt{c_8 c_9} \tag{26}$$

Thus, from the relation$\tau(s) = \tilde{\tau}(s) + 2\pi(s)$, we have

$$\tau(s) = c_1 + 2c_4 - (c_2 - 2c_5)s - 2\left[\left(\sqrt{c_9} + c_3\sqrt{c_8}\right)s - \sqrt{c_8}\right], \tag{27}$$

whose derivative must be negative:

$$\tau'(s) = -2c_3 - 2\left[\sqrt{c_9} + c_3\sqrt{c_8}\right] < 0 \tag{28}$$

Solving Eqs. (14) and (16), we obtain the parametric energy equation as

$$(c_2 - c_3)n + c_3 n^2 - (2n+1)c_5 + (2n+1)\left[\sqrt{c_9} + c_3\sqrt{c_8}\right] + c_8 + 2c_3c_8 + 2\sqrt{c_8c_9} = 0 \tag{29}$$

The weight function $\rho(s)$ is obtained as

$$\rho(s) = s^{c_{10}}(1 - c_3 s)^{c_{11}} \tag{30}$$

and together with Eq. (11), we have

$$y_n(s) = P_{n(c_{10}, c_{11})}(1 - 2c_3 s), \tag{31}$$

where

$$c_{10} = c_1 + 2c_4 + 2\sqrt{c_8} \tag{32}$$

$$c_{11} = 1 - c_1 - 2c_4 + \frac{2}{c_3}\sqrt{c_9} \tag{33}$$

and $P_n^{(\alpha,\beta)}(s)$ are the Jacobi polynomials. The other part of the wave function is obtained from Eq. (10) as

$$\varphi(s) = s^{c_{12}}(1 - c_{3s})^{c_{13}}, \tag{34}$$

where

$$c_{12} = c_4 + \sqrt{c_8}, \quad c_{13} = -c_4 + \frac{1}{c_3}\left(\sqrt{c_9} - c_5\right) \tag{35}$$

Thus, the total wave function becomes

$$\psi(s) = N_{nl} s^{c_{12}}(1 - c_3 s)^{c_{13}} P_n^{(c_{10}, c_{11})}(1 - 2c_3 s). \tag{36}$$

whereN_{nl} is the normalization constant.

4 Solutions of Klein-Gordon Equation

In the present investigation, we consider a modified deformed Hylleraas potential of the form [21,22]

$$V(r) = \frac{V_0}{b}\frac{\left[a + g e^{\left(\frac{r - r_c}{\alpha}\right)}\right]}{\left(1 + e^{\left(\frac{r - r_c}{\alpha}\right)}\right)} \tag{37}$$

where V_0 is the potential depth, α denotes the adjustable parameter, r_c shows the distance from the equilibrium position and,a, b and g are the Hylleraas parameters. The behavior of the Hylleraas potential as a function of r is displayed in Figs.1, 2, 3 for various values of r_c and α for a = 1,b = 5,$V_0 = 2.0\ MeV$. We also consider a PDM of the form [23]

$$m(r) = m_0 + \frac{m_1}{\left(1 + e^{\left(\frac{r - r_c}{\alpha}\right)}\right)} \tag{38}$$

The behavior of the mass function vs. r is shown in Fig. (4)for $r_c = 0.1, \alpha = 0.1\ and\ 0.2$, respectively.

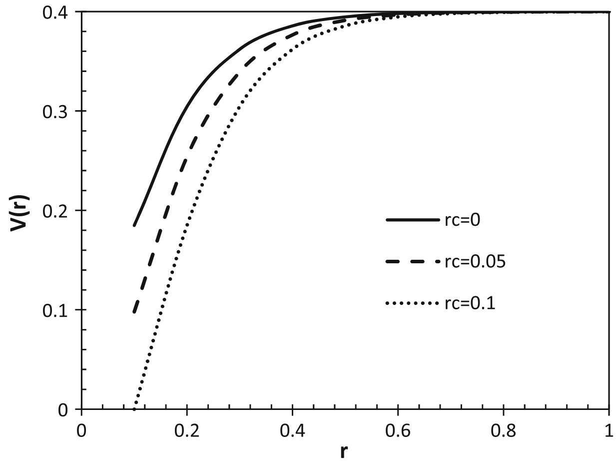

Fig. 1 Plot of the V as a function of r with $\alpha = 0.1$, b = 5, a = −1, $V_o = 2.0$ Mev for various values of $r_c = 0, 0.05$ and 0.1

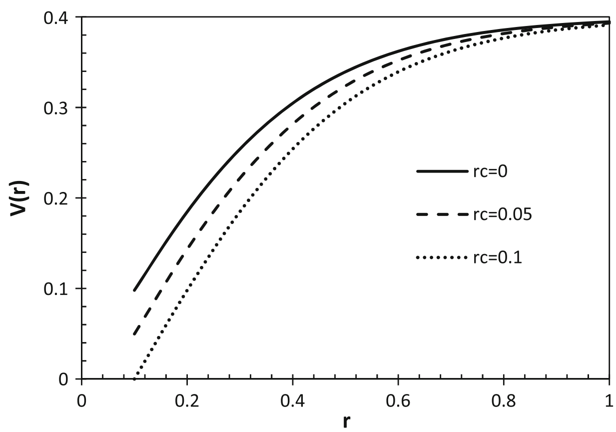

Fig. 2 Plot of the V as a function of r with $\alpha = 0.2$, b = 5, a = −1, $V_o = 2.0$ Mev for various values of $r_c = 0, 0.05$ and 0.1

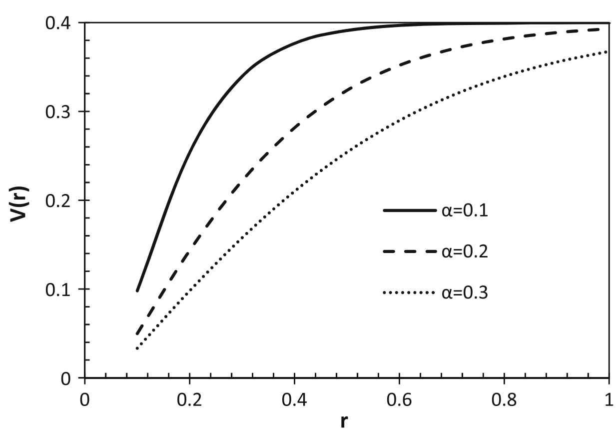

Fig. 3 Plot of the V as a function of r with $r_c = 0.05$, b = 5, a = −1, $V_o = 2.0$ Mev for various values of $\alpha = 0.1, 0.2$ and 0.3

We now introduce the elegant approximation for the centrifugal term as follows [25]

$$\frac{1}{r^2} \approx a_1 + \frac{a_2}{\left(1 + e^{\left(\frac{r-r_c}{\alpha}\right)}\right)} + \frac{a_3}{\left(1 + e^{\left(\frac{r-r_c}{\alpha}\right)}\right)^2}, \tag{39}$$

where

$$a_1 = \frac{1}{r_c^2}\left(1 - \frac{3\alpha}{r_c} + \frac{3\alpha^2}{r_c^2}\right) \tag{40}$$

$$a_2 = \frac{1}{r_c^2}\left(\frac{4\alpha}{r_c} + \frac{6\alpha^2}{r_c^2}\right) \tag{41}$$

$$a_3 = \frac{1}{r_c^2}\left(\frac{-\alpha}{r_c} + \frac{3\alpha^2}{r_c^2}\right) \tag{42}$$

We compared the approximation of Eq. (39) for $\alpha = 0.1\, and\, 0.2$ denoted as $f_1\, and\, f_2$ with the centrifugal term $f = \frac{1}{r^2}$ in Fig. 5. This shows that the approximation is in good agreement with the centrifugal term.

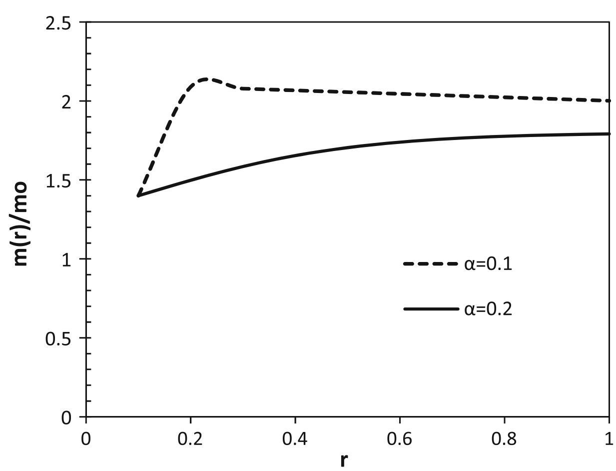

Fig. 4 Variation of the mass as a function of position for various values of α = 0.1 and 0.2 with r_c = 0.1

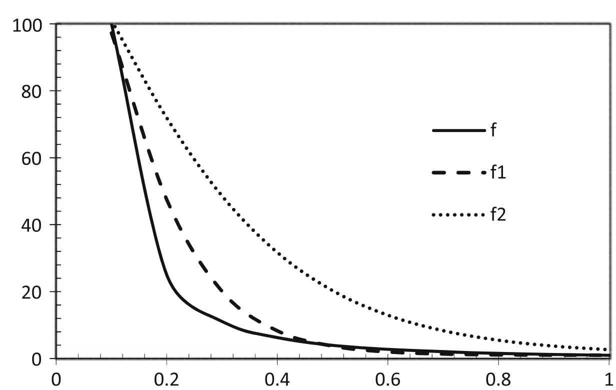

Fig. 5 Comparison of the centrifugal term $f = 1/r^2$ with the elegant approximation f_1 for $\alpha = 0.1$ with $a_1 = 1$, $a_2 = 150$, $a_3 = 85$, $r_c = 0.1$ and f_2 for $\alpha = 0.2$ with $a_1 = 1$, $a_2 = 150$, $a_3 = 100$ and $r_c = 0.1$

Substituting Eqs. (37), (38) and (39) into Eq. (6), we have

$$\left\{\frac{d^2}{dr^2}+E_{n,l}^2+\frac{V_0^2}{b^2}\left(\frac{ae^{-\left(\frac{r-r_c}{\alpha}\right)}+g}{1+e^{-\left(\frac{r-r_c}{\alpha}\right)}}\right)^2-\frac{2E_{nl}V_0}{b}\left(\frac{g+ae^{-\left(\frac{r-r_c}{\alpha}\right)}}{1+e^{-\left(\frac{r-r_c}{\alpha}\right)}}\right)-\left[m_0+\frac{m_1e-^{\left(\frac{r-r_c}{\alpha}\right)}}{\left(1+e^{-\left(\frac{r-r_c}{\alpha}\right)}\right)}\right]^2\right.$$
$$-\frac{S_0^2}{b^2}\left(\frac{ae^{\left(\frac{r-r_c}{\alpha}\right)}+g}{1+e^{-\left(\frac{r-r_c}{\alpha}\right)}}\right)^2-\frac{2S_0}{b}\left(\frac{ae^{-\left(\frac{r-r_c}{\alpha}\right)+g}}{1+e^{-\left(\frac{r-r_c}{\alpha}\right)}}\right)\left(m_0+\frac{m_1e^{-\left(\frac{r-r_c}{\alpha}\right)}}{\left(1+e^{-\left(\frac{r-r_c}{\alpha}\right)}\right)}\right)$$
$$\left.-\frac{(D+2l-1)\,(D+2l-3)}{4r^2}\left(a_1+\frac{a_2e^{-\left(\frac{r-r_c}{\alpha}\right)}}{1+e^{-\left(\frac{r-r_c}{\alpha}\right)}}+\frac{a_3e^{-\left(\frac{r-r_c}{\alpha}\right)}}{\left(1+e^{-\left(\frac{r-r_c}{\alpha}\right)}\right)^2}\right)\right\}U_{nl}(r)=0, \tag{43}$$

If we take the transformation, $s=-e^{-\left(\frac{r-r_c}{\alpha}\right)}$, Eq. (43) becomes

$$\frac{d^2U_{nl}}{ds^2}+\frac{(1-s)}{s\,(1-s)}\frac{dU_{nl}}{ds}+\frac{1}{s^2\,(1-s)^2}\left[-As^2+Bs-C\right]U_{nl}\,(s)=0, \tag{44}$$

where

$$A=-\varepsilon^2+\alpha^2\left[\begin{array}{l}-\frac{a^2V_0^2}{b^2}+\frac{2E_{nl}V_0a}{b}+m_1^2+2m_0m_1+\frac{2m_0S_0a}{b}\\+\frac{2m_1S_0a}{b}+\frac{a^2S_0^2}{b^2}+\frac{(D+2l-1)(D+2l-3)[a_1+a_2]}{4}\end{array}\right] \tag{45}$$

$$B=-2\varepsilon^2+\alpha^2\left[\begin{array}{l}-\frac{2V_0^2ag}{b^2}+\frac{2E_{nl}V_0(a+g)}{b}+\frac{2m_0S_0(a+g)}{b}+2m_0m_1+\\\frac{2m_1S_0g}{b}+\frac{2S_0^2ag}{b^2}+\frac{(D+2l-1)(D+2l-3)(a_1+a_2+a_{3})}{2}\end{array}\right] \tag{46}$$

$$C=-\varepsilon^2+\alpha^2\left[\begin{array}{l}\frac{2E_{nl}V_0g}{b}+\frac{2m_0S_0g}{b}+\\m_1^2+\frac{S_0^2g^2}{b^2}-\frac{V_0^2g^2}{b^2}+\frac{(D+2l-1)(D+2l-3)a_1}{4}\end{array}\right] \tag{47}$$

and the dimensionless parameter ε, is defined as follows

$$\varepsilon^2=\alpha^2\left[E_{nl}^2-m_0^2\right] \tag{48}$$

Now, comparing Eq. (44) with Eq. (17), we obtain the following parameters:

$$\begin{aligned}c_1&=c_2=1,c_3=1,\\\xi_1&=A,\xi_2=B,\xi_3=C\\c_4&=0,c_5=-\frac{1}{2},\\c_6&=\frac{1}{4}+A\\c_7&=-B\\c_8&=C\\c_9&=\frac{1}{4}+A-B+C\\c_{10}&=1+2\sqrt{C}\\c_{11}&=2+2\left(\sqrt{\frac{1}{4}+A-B+C}+\sqrt{C}\right),\\c_{12}&=\sqrt{C}\\c_{13}&=-\frac{1}{2}-\left(\sqrt{\frac{1}{4}+A-B+C}+\sqrt{C}\right)\end{aligned} \tag{49}$$

Substituting the values of the parameters given by Eq. (49) into Eq. (29), we find the energy relation as

$$n^2+n+\frac{1}{2}+\sqrt{\frac{1}{4}+A-B+C}\left[(2n+1)+2\sqrt{C}\right]+(2n+1)\sqrt{C}-B+2C=0 \tag{50}$$

To obtain the wave function, we proceed as follows.By using Eq. (30), we find the weight function as

$$\rho(s)=s^{(1+2\mu)}(1-s)^{2\upsilon} \tag{51}$$

Using Eq. (31), we get

$$n(s)=P_n^{(1+2\mu,2\upsilon)}(1-2s) \tag{52}$$

where $\mu=\sqrt{C}$ and $\upsilon=\sqrt{A+C-B+\frac{1}{4}}$.The other part of the wave function is obtained from Eq. (34) as

$$\varphi(s)=s^{\mu}(1-s)^{\upsilon+\frac{1}{2}} \tag{53}$$

Finally, we obtain the corresponding Klein-Gordon wave function as

$$U_{nl}(r)=N_{nl}s^{\mu}(1-s)^{\upsilon+\frac{1}{2}}P_n^{(2\mu,2\upsilon)}(1-2s) \tag{54}$$

where [12].

$$P_n^{(a,b)}(s)=\frac{\Gamma(n+a+1)}{n!\Gamma(a+1)}\,{}_2F_1(-n,n+a+b+1;1+a;\frac{1-s}{2}) \tag{55}$$

with $2\mu>-1, 2\nu>-1$. We are able to write Eq. (54) in terms of the hypergeometric polynomials as

$$U_{nl}(r)=N_{nl}s^{\mu}(1-s)^{\upsilon+\frac{1}{2}}\frac{\Gamma(n+2\mu+1)}{n!\Gamma(2\mu+1)}{}_2F_1(-n,n+2\mu+2\upsilon+1;1+2\mu;\frac{1-s}{2}) \tag{56}$$

The normalization constant N_{nl} can be found from normalization condition for in special case D=3

$$\int_0^{\infty}|R(r)|^2r^2dr=\int_0^{\infty}|U_{nl}(r)|^2dr=\alpha\int_0^1\frac{1}{s}|U_{nl}(r)|^2ds=1 \tag{57}$$

by using the following integral formula

$$\begin{aligned}&\int_0^1(1-z)^{2(\delta+1)}z^{2\lambda-1}\{{}_2F_1(-n,n+2(\delta+\lambda+1);1+2\lambda;z)\}^2dz\\&=\frac{(n+\delta+1)\,n!\Gamma(n+2\delta+2)\Gamma(2\lambda)\Gamma(2\lambda+1)}{(n+\delta+\lambda+1)\,\Gamma(n+2\lambda+1)\Gamma(2(\delta+\lambda+1)+n)}\end{aligned} \tag{58}$$

After some calculations, we obtain the normalization constant as

$$N_{nl}=\sqrt{\frac{n!2\mu(n+\nu+\frac{1}{2}+\mu)\Gamma(2\left(\nu+\frac{1}{2}+\mu\right)+n)}{\alpha(n+\nu+\frac{1}{2})\Gamma(n+2\mu+1)\Gamma(n+2\nu+1)}} \tag{59}$$

5 Results and Discussion

In this section, we are going to study three special cases of the Hylleraas potential and their corresponding eigenvalues and eigen functions by choosing some special cases of the parameters in the deformed Hylleraas potential.

5.1 Woods-Saxon Potential

If we put $a = g = 0, b = -1$ *and* $m_1 = 0$, the deformed Hylleraas potential turns into the Woods-Saxon potential [26].

$$V(r) = \frac{-V_0}{1 + e^{\frac{r-r_c}{\alpha}}} \tag{60}$$

The corresponding energy eigenvalues from Eq. (50) are found to be

$$E_{nl}^2 - m_0^2 = -\frac{1}{4\alpha^2}\left[\frac{\delta'}{n+\sigma'} + (n+\sigma')\right]^2 + \beta', \tag{61}$$

where

$$\delta' = \frac{(D+2l-1)(D+2l-3)\,a_2\alpha^2}{4} \tag{62}$$

$$\sigma' = \left(\frac{1}{2} + \sqrt{\frac{1}{4} + \frac{(D+2l-1)(D+2l-3)\,\alpha^2 a_3}{4}}\right), \tag{63}$$

$$\beta' = \frac{(D+2l-1)(D+2l-3)(a_1+a_2+a_3)}{4}. \tag{64}$$

and the wave function is

$$R_{nl}(r) = N_{nl} r^{-\left(\frac{D-1}{2}\right)} \left(-e^{\frac{r-r_c}{\alpha}}\right)^{\mu} \left(1 + e^{\frac{r-r_c}{\alpha}}\right)^{v+\frac{1}{2}} P_n^{(1+2\mu,2v)}\left(1 + 2e^{\frac{r-r_c}{\alpha}}\right), \tag{65}$$

where $\mu = \sqrt{C}$ and $v = \sqrt{A + C - B + \frac{1}{4}}$ with

$$A = -\varepsilon^2 + \frac{(D+2l-1)(D+2l-3)\,a_1}{4}\alpha^2 \tag{66}$$

$$B = -2\varepsilon^2 + \alpha^2\left[\frac{(D+2l-1)(D+2l-3)\,a_1}{2} + \frac{(D+2l-1)(D+2l-3)\,a_2}{4}\right] \tag{67}$$

$$C = -\varepsilon^2 + \alpha^2\left[\frac{(D+2l-1)(D+2l-3)(a_1+a_2+a_3)}{4}\right]. \tag{68}$$

This result is consistent with Ref. [26].

5.2 Rosen-Morse Potential

If we choose $a = -1, b = -1, g = 1$ *and* $m_1 = 0$, then the Hylleraas potential becomes the Rosen-Morse potential [28].

$$V(r) = -V_0 \tanh\left(\frac{r - r_c}{\alpha'}\right) \tag{69}$$

For this case, the energy eigenvalues and wave function are respectively given as

$$E_{nl}^2 - m_{0^2} = -\frac{1}{4\alpha'^2}\left[\frac{\delta''}{n+\sigma''} + (n+\sigma'')\right]^2 + \beta'', \tag{70}$$

$$\psi_{lnm}(r,\Omega_D) = N_{nl} r^{-\left(\frac{D-1}{2}\right)} \left(-e^{\frac{r-r_c}{\alpha'}}\right)^{\mu} \left(1 + e^{\frac{r-r_c}{\alpha'}}\right)^{v+\frac{1}{2}} P_n^{(1+2\mu,2v)}\left(1 + 2e^{\frac{r-r_c}{\alpha'}}\right) Y_l^m(\Omega_D) \tag{71}$$

where,

$$\alpha' = 2\alpha \tag{72}$$

$$\delta'' = 4E_{nl}V_0\alpha'^2 + 4m_0S_0\alpha'^2 + \frac{(D+2l-1)(D+2l-3)\,a_2\alpha'^2}{4} \tag{73}$$

$$\sigma'' = \left(\frac{1}{2} + \sqrt{-3V_0^2\alpha'^2 + 3S_0^2\alpha'^2 + \alpha'^2\left(S_0^2 - V_0^2\right) + \frac{1}{4} + \frac{(D+2l-1)(D+2l-3)\,\alpha'^2 a_3}{4}}\right), \tag{74}$$

$$\beta'' = 2E_{nl}V_0 - V_0^2 + S_0^2 + 2m_0S_0 + \frac{(D+2l-1)(D+2l-3)(a_1+a_2+a_3)}{4}. \tag{75}$$

$\mu = \sqrt{C}$, $v = \sqrt{A + C - B + \frac{1}{4}}$ with

$$A = -\varepsilon^2 + \alpha'^2\left[-2E_{nl}V_0 - 2m_0S_0 + S_0^2 - V_0^2 + \frac{(D+2l-1)(D+2l-3)\,a_1}{4}\right] \tag{76}$$

$$B = -2\varepsilon^2 + \alpha'^2\left[-2S_0^2 + 2V_0^2 + \frac{(D+2l-1)(D+2l-3)\,a_1}{2} + \frac{(D+2l-1)(D+2l-3)\,a_2}{4}\right] \tag{77}$$

$$C = -\varepsilon^2 + \alpha'^2\left[2E_{nl}V_0 + 2m_0S_0 + S_0^2 - V_0^2 + \frac{(D+2l-1)(D+2l-3)(a_1+a_2+a_3)}{4}\right] \tag{78}$$

This result is consistent with that of Ref. [28].

6 Conclusion

We have obtained the energy eigenvalues and the corresponding wave functions of the D-dimensional Klein-Gordon equation for modified Hylleraas potential with a position dependent mass. With appropriate choiceofthe values of $a$and$b$ in the modified Hylleraas potential, we obtained the energy eigenvalues and the wave functions of the Woods-Saxon and Rosen-Morse potentials. Finally, the eigenvalues and the wave functions of these special cases are in exact agreement with the previously published works of [26–28] for the special case of $D = 3,\ and\, m_1 = 0$.

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