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Calculation of the Oscillator Strength for the Klein–Gordon Equation with Tietz Potential

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Abstract The Klein–Gordon equation under equal scalar and vector potentials is solved for the Tietz potential in D-dimensions by using supersymmetric quantum mechanics. The spectrum of the system is numerically calculated and the oscillator strength is determined and discussed in terms of parameters of the system.

1 Introduction

The solution of nonrelativistic and relativistic wave equations is very important for many physical systems. Particularly, it has been of great interest to solve the Klein–Gordon equation (KGE) which describes relativistic spin-zero bosons. Different analytical techniques have been assumed by different authors to obtain the exact or approximate solutions of the KGE with various interactions. Frequently used analytical methodologies include supersymmetric quantum mechanics [1–3], asymptotic iteration method (AIM) [4] and the Nikiforov–Uvarov (NU) technique [5]. In addition to all analytical approaches, there exist numerical techniques as well with each one having its own failures and advantages. The latter, despite their notable advantage of reliability, are often time-consuming and unobvious in comparison with their analytical counterparts. In using the former class, however, the approximate schemes are often inevitable to obtain a general solution. In particular, the KGE has been solved for various potentials including the Hulthén [6], Rosen–Morse [7], Poschl–Teller ([8,9] and references therein), etc. In this work, we have solved the radial KGE for the Tietz potential. The outline of our work is as follows. In Sect. 2, we first review the KGE under the Tietz potential. We next apply a physical approximation to obtain the radial equation and thereby calculate the eigen functions and the eigen values of the system via SUSYQM. In Sect. 3 we obtain the corresponding oscillator strengths of the system. Our conclusions are given in Sect. 4 and, illustrative figures and some numerical data are included as well.

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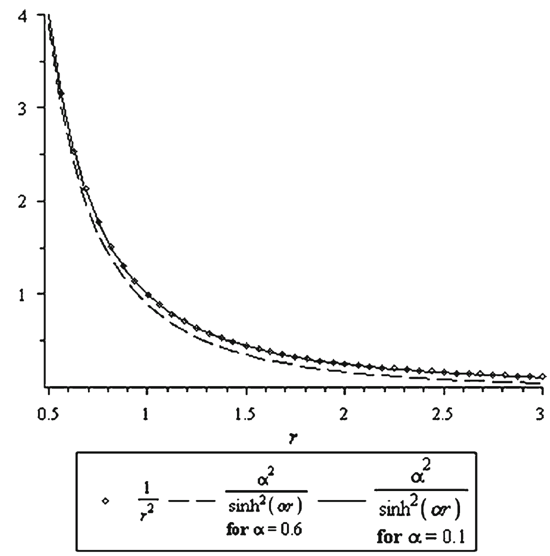

Fig. 1 $\frac{1}{r^2}$ and its approximation $\frac{\alpha^2}{\sinh^2(\alpha r)}$

2 Radial Part of KGE in D-Dimensions

The radial KGE in the presence of vector and scalar potentials in the D-dimensional space is written as [10,11]

$$\left[\frac{d^2}{dr^2} + (E_{n,l} - V(r))^2 - (m + S(r))^2 - \frac{(D+2l-1)(D+2l-3)}{4r^2}\right] u_{n,l}(r) = 0. \tag{1}$$

Here, we consider the so-called Tietz potential which is the more general case of Eckhart and Manning-Rosen potentials written as [12,13]

$$V(r) = S(r) = V_0 \left(\frac{\sinh(\frac{r-r_0}{a})}{\sinh(\frac{r}{a})}\right)^2 = S_1 \coth^2(\alpha r) + S_2 \coth(\alpha r) + S_3, \tag{2}$$

where

$$S_1 = V_0 \sinh^2(\alpha r_0), \quad S_2 = -V_0 \sinh(2\alpha r_0), \quad S_3 = V_0 \cosh^2(\alpha r_0) \text{ and } \alpha = \frac{1}{a}. \tag{3}$$

By substituting Eq. (2) in Eq. (1), we have

$$\left[\frac{d^2}{dr^2} + E_{n,l}^2 - m^2 + (-2E_{n,l} - 2m)(S_1 \coth^2(\alpha r) + S_2 \coth(\alpha r) + S_3) - \frac{(D+2l-1)(D+2l-3)}{4r^2}\right] u_{n,l}(r) = 0. \tag{4}$$

For the centrifugal term, we make use of the approximation [14,15]

$$\frac{1}{r^2} \sim \frac{\alpha^2}{\sinh^2(\alpha r)}, \tag{5}$$

which is a quite logical alternative for $\alpha < 0.1$ (See Fig. 1). From Eq. (5), Eq. (4) is written as

$$\left[\frac{d^2}{dr^2} + \left(-2S_1(E_{n,l} + m) - \frac{\alpha^2(D+2l-1)(D+2l-3)}{4}\right) \operatorname{csc} h^2(\alpha r) - 2S_2(E_{n,l} + m)\coth(\alpha r) + \left(E_{n,l}^2 - m^2 - 2(E_{n,l} + m)(S_1 + S_3)\right)\right] u_{n,l}(r) = 0, \tag{6}$$

or

$$-\frac{d^2u_{n,l}(r)}{dr^2}+V_{\text{eff}}(r)u_{n,l}(r)=\widetilde{E}_{n,l}u_{n,l}(r), \tag{7}$$

where

$$V_{\text{eff}}(r)=V_1\operatorname{csc}h^2(\alpha r)+V_2\coth(\alpha r), \tag{8}$$

$$\widetilde{E}_{n,l}=E_{n,l}^2-m^2-2(E_{n,l}+m)(S_1+S_3), \tag{9}$$

and

$$V_1=2S_1(E_{n,l}+m)+\frac{\alpha^2(D+2l-1)(D+2l-3)}{4}, \tag{10-a}$$

$$V_2=2S_2(E_{n,l}+m). \tag{10-b}$$

Bearing in mind Eq. (A-1), we search for the Riccati equation

$$\phi^2-\phi'=V_{\text{eff}}-\widetilde{E}_{0,l}, \tag{11}$$

which is

$$\phi(r)=A+B\coth(\alpha r). \tag{12}$$

Substituting Eq. (12) into Eq. (11) and comparing equal powers we find

$$2BA=V_2, \tag{13-a}$$

$$B^2+\alpha B=V_1, \tag{13-b}$$

$$A^2+B^2=-\widetilde{E}_{0,l}, \tag{13-c}$$

or equivalently

$$A=\frac{V_2}{2B}, \tag{14-a}$$

$$B=\frac{1}{2}\left[-\alpha\pm\sqrt{\alpha^2+4V_1}\right], \tag{14-b}$$

$$\widetilde{E}_{0,l}=-\left(\frac{V_2^2}{4B^2}+B^2\right). \tag{14-c}$$

Therefore, our partner potentials are

$$V_{\text{eff}^+}(r)=\phi^2+\frac{d\phi}{dr}=\operatorname{csc}h^2(\alpha r)[B^2-\alpha B]+B^2+\frac{V_2^2}{4B^2}+V_2\coth(\alpha r), \tag{15-a}$$

$$V_{\text{eff}^-}(r)=\phi^2-\frac{d\phi}{dr}=\operatorname{csc}h^2(\alpha r)[B^2+\alpha B]+B^2+\frac{V_2^2}{4B^2}+V_2\coth(\alpha r), \tag{15-b}$$

which are shape invariant via a mapping of the for $B\to B-\alpha$. Thus, from Eq. (A-2),

$$R(a_n)=\left(a_{n-1}^2+\frac{V_2^2}{4a_{n-1}^2}\right)-\left(a_n^2+\frac{V_2^2}{4a_n^2}\right),$$

$$\widetilde{E}_{n,l}^-=\sum_{k=1}^{n}R(a_k)=\left(a_0^2+\frac{V_2^2}{4a_0^2}\right)-\left(a_n^2+\frac{V_2^2}{4a_n^2}\right), \tag{16}$$

where $n = 0, 1, 2, \ldots$ and

$$a_n = a_0 - n\alpha, \quad a_0 = B. \tag{17}$$

From Eqs. (14-c) and (16) the eigen values are

$$\widetilde{E}_{n,l} = \widetilde{E}_{n,l}^{-} + \widetilde{E}_{0,l} = -\left((B - n\alpha)^2 + \frac{V_2^2}{4(B - n\alpha)^2}\right). \tag{18}$$

Finally, from Eq. (9) we have

$$E_{n,l}^2 - m^2 - 2(E_{n,l} + m)(S_1 + S_3) = -\left((B - n\alpha)^2 + \frac{V_2^2}{4(B - n\alpha)^2}\right), \tag{19}$$

which determines the spectrum of the system. In order to extract the wavefunction of the system we start from

$$\left[\frac{d^2}{dr^2} + \left(-2S_1(E_{n,l} + m) - \frac{\alpha^2(D + 2l - 1)(D + 2l - 3)}{4}\right)\csc h^2(\alpha r) - 2S_2(E_{n,l} + m)\coth(\alpha r) + \left(E_{n,l}^2 - m^2 - 2(E_{n,l} + m)(S_1 + S_3)\right)\right]u_{n,l}(r) = 0, \tag{20}$$

which we write as

$$\left[z^2\frac{d^2u_{n,l}(z)}{dz^2} + z\frac{du_{n,l}(z)}{dz} + \left(-\frac{V_1'}{\alpha^2} + \frac{V_2'}{4\alpha^2}\right)\frac{1}{1 - z} + \frac{V_1'}{\alpha^2}\frac{1}{(1 - z)^2} + \frac{V_2'}{4\alpha^2}\frac{z}{1 - z} + \frac{V_3'}{4\alpha^2}\right]u_{n,l}(z) = 0. \tag{21}$$

where

$$V_1' = -V_1, \quad V_2' = -V_2, \quad V_3' = \widetilde{E}_{n,l}, \quad z = \exp(-2\alpha r). \tag{22}$$

Now, we analyze the asymptotic behavior of Eq. (21) [16]. When $r \longrightarrow 0 (z \longrightarrow 1)$, we have a solution $u_{n,l}(z) = (1 - z)^{\lambda}$ for Eq. (21), where the parameter λ is given by

$$\lambda = l' + 1, \tag{23-a}$$

$$l' = \frac{1}{2}\left(-1 + \sqrt{1 - \frac{4V_1'}{\alpha^2}}\right). \tag{23-b}$$

In the other limit, i.e. $r \longrightarrow \infty (z \longrightarrow 0)$, we get the solution $u_{n,l}(z) = z^{\eta}$ with

$$\eta = \sqrt{-\frac{V_2' + V_3'}{4\alpha^2}}, \quad \xi = \sqrt{\frac{V_2' - V_3'}{4\alpha^2}}. \tag{24}$$

Applying the transformation $u_{n,l}(z) = z^{\eta}(1 - z)^{\lambda} f_{n,l}(z)$, Eq. (21) reduces to

$$z(1 - z)\frac{d^2 f_{n,l}(z)}{dz^2} + [1 + 2\eta - (1 + 2\lambda + 2\eta)z]\frac{df_{n,l}(z)}{dz} - \left(2\eta\lambda + \eta^2 + \lambda^2 + \frac{V_3' - V_2'}{4\alpha^2}\right) f_{n,l}(z) = 0. \tag{25}$$

The latter is just a hypergeometric equation with the solution

$$f_{n,l}(z) = {}_2F_1(a, b; 1 + 2\eta; z), \tag{26}$$

where the parameters a and b are

$$a = \lambda + \eta + \xi, \tag{27-a}$$

$$b = \lambda + \eta - \xi. \tag{27-b}$$

Table 1 $E_{n,l}$ for some values of n, l for $\alpha = 0.1$, $m = 1$, $S_1 = 1.5$, $S_2 = 3$, $S_3 = 5$

$\|n,l\rangle$	$E_{n,l}$	$\|n,l\rangle$	$E_{n,l}$
$\|0,0\rangle$	−0.9988887517	$\|2,0\rangle$	−0.9845457758
$\|0,1\rangle$	−0.9961804418	$\|2,1\rangle$	−0.9798217698
$\|0,2\rangle$	−0.9916785742	$\|2,2\rangle$	−0.9721359550
$\|0,3\rangle$	−0.9854050687	$\|2,3\rangle$	−0.9622605359
$\|0,4\rangle$	−0.9773569553	$\|2,4\rangle$	−0.9504218067
$\|0,5\rangle$	−0.9675271518	$\|2,5\rangle$	−0.9366920572
$\|1,0\rangle$	−0.9940306855	$\|3,0\rangle$	−0.9703029262
$\|1,1\rangle$	−0.9899888641	$\|3,1\rangle$	−0.9652218988
$\|1,2\rangle$	−0.9836986243	$\|3,2\rangle$	−0.9565294616
$\|1,3\rangle$	−0.9755209929	$\|3,3\rangle$	−0.9452221542
$\|1,4\rangle$	−0.9655192492	$\|3,4\rangle$	−0.9317168126
$\|1,5\rangle$	−0.9537051313	$\|3,5\rangle$	−0.9161810201

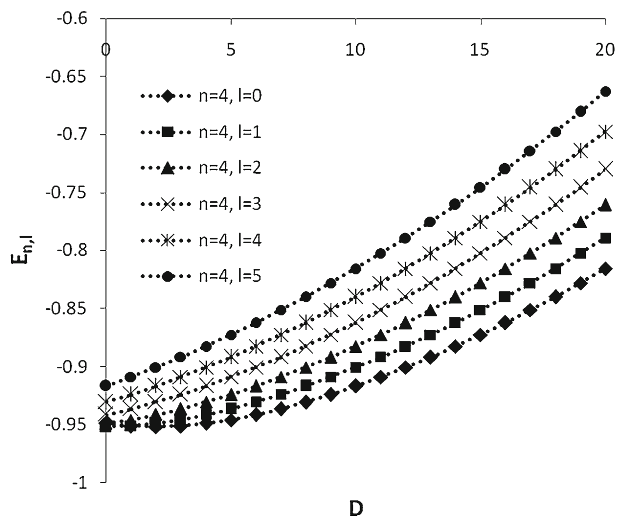

Fig. 2 $E_{n,l}$ versus D for some values of n, l, $\alpha = 0.1$, $m = 1$, $S_1 = 1.5$, $S_2 = 3$, $S_3 = 5$

When either a or b is a negative integer $-n$, the hypergeometric function $f_{n,l}(z)$ can be reduced to a polynomial of degree n. Therefore

$$u_{n,l}(r) = N_{n,l}\exp(-2\alpha\eta r)(1-\exp(-2\alpha r))^{l'+1}{}_2F_1(-n, n+2(l'+1)+2\eta; 1+2\eta; \exp(-2\alpha r)), \quad (28)$$

where

$$N_{n,l} = \frac{\Gamma(n+1+2\eta)}{n!\Gamma(1+2\eta)}. \quad (29)$$

In Table 1, we have reported some numerical results for some values of n, l in the case of $D = 3$. In Fig. 2 we have presented $E_{n,l}$ versus D for some values of n, l. The behavior of $E_{n,l}$ versus S_1, S_2, S_3 and α for some values of n, l is shown in Figs. 3, 4, 5, and 6, respectively.

3 Oscillator Strengths

Let us now use the results to calculate the so-called oscillator strength. We know that the absorption of light yields a transition from a quantum state to another one. The transition probabilities play a significant role in many physical systems from stellar to subatomic ones [17]. In particular, the oscillator strength gives additional information on the fine structure and selection rules of the optical absorption [18]. In transition from a lower state ψ_i to an upper one, ψ_j , the equivalent length and the velocity forms for the absorption oscillator strength is

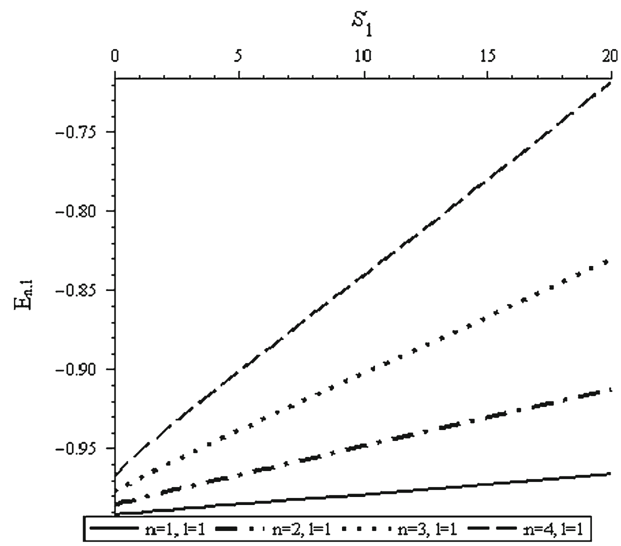

Fig. 3 $E_{n,l}$ versus S_1 for some values of $n, l, \alpha = 0.1, m = 1, D = 3, S_2 = 3, S_3 = 5$

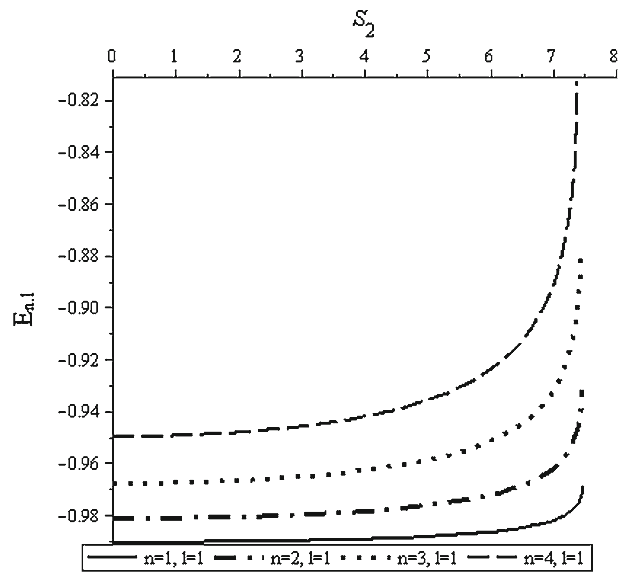

Fig. 4 $E_{n,l}$ versus S_2 for some values of $n, l, \alpha = 0.1, m = 1, S_1 = 1.5, D = 3, S_3 = 5$

$$f_{i,j}^{l} = \frac{2m(E_j - E_i)}{3\hbar^2} |\langle \psi_j | r | \psi_i \rangle|^2, \tag{30-a}$$

$$f_{i,j}^{v} = \frac{2}{3m(E_j - E_i)} |\langle \psi_j | p | \psi_i \rangle|^2. \tag{30-b}$$

In Figs. 7, 8, and 9, we have plotted the variation of the length and velocity oscillator strengths versus S_1, S_2 and S_3 , respectively.

4 Conclusion

The high number of relativistic spin-zero systems in various physical sciences motivated us to study the KGE. For the sake of generality, we considered the problem for arbitrary dimension. Our choice of the interaction

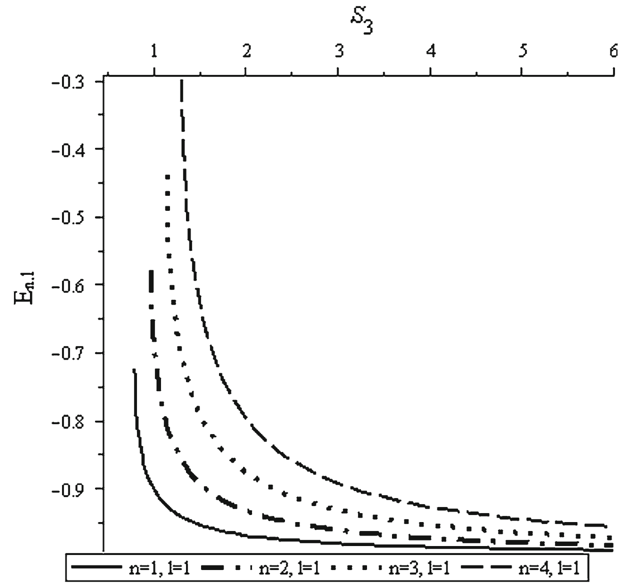

Fig. 5 $E_{n,l}$ versus S_3 for some values of $n, l, \alpha = 0.1, m = 1, S_1 = 1.5, S_2 = 3, D = 3$

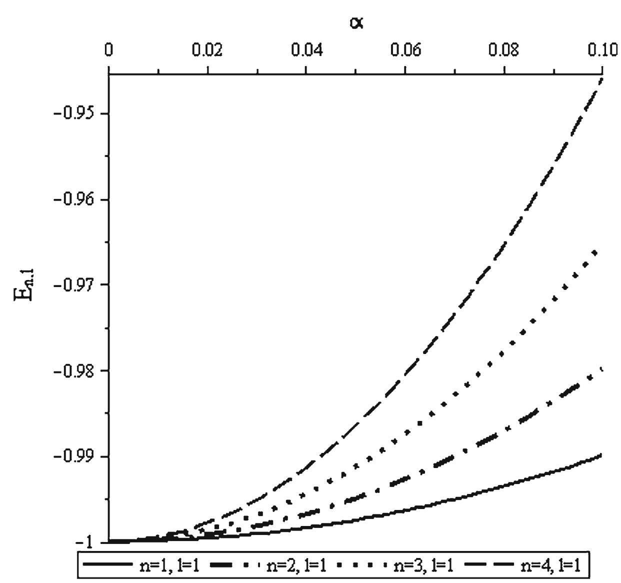

Fig. 6 $E_{n,l}$ versus α for some values of $n, l, D = 3, m = 1, S_1 = 1.5, S_2 = 3, S_3 = 5$

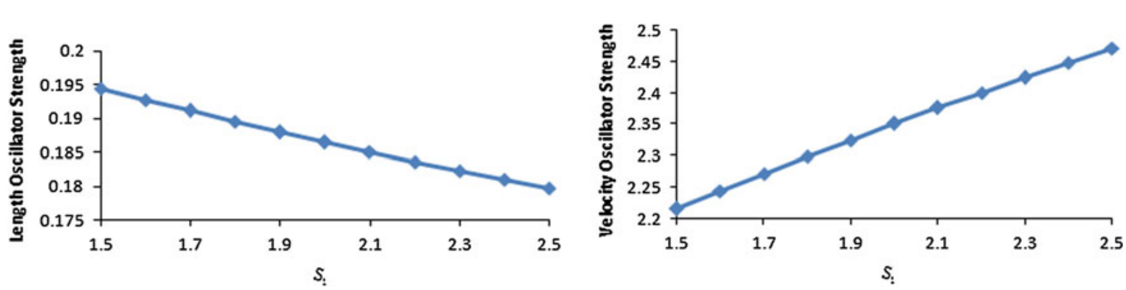

Fig. 7 Length and velocity oscillator strength versus $S_1\, for, m = 1, \alpha = 0.1, S_2 = 3, S_3 = 5$

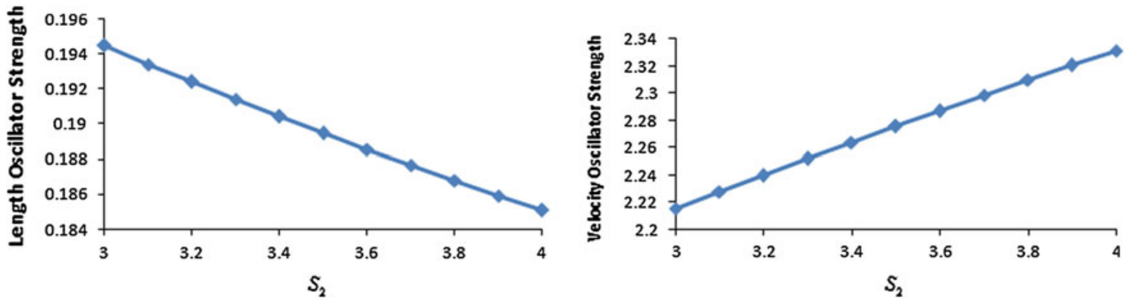

Fig. 8 Length and velocity oscillator strength versus $S_2 for, m = 1, \alpha = 0.1, S_1 = 1.5, S_3 = 5$

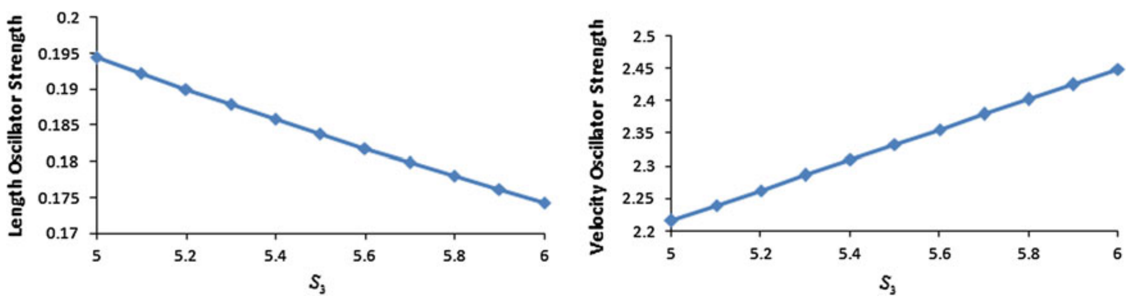

Fig. 9 Length and velocity oscillator strength versus $S_3 for, m = 1, \alpha = 0.1, S_1 = 1.5, S_2 = 3$

was the Tietz potential. The latter is the more general case of Eckhart and Manning-Rosen potentials that yield notable phenomenological consequences. To obtain useful results, the higher states (both eigen function and eigen energies) have to be calculated. Therefore, we first applied a Pekeris-type approximation to be able to obtain the arbitrary-state solution. For the calculations, instead numerical programming, we used the analytical technique of supersymmetry quantum mechanics and thereby obtained the general solutions. We calculated the oscillator strength that required the calculation higher states. The calculation enables us to determine the transition rates for absorption, spontaneous and stimulated emission, lifetimes of electronic levels, concentration of impurities, static polarizabilities and consequently in construction or checking of theoretical and phenomenological models. The energy shows a degenerate behavior when l increases to $l+1$ and D reduces to $D-2$, i.e. $E_{n,l}^{D} = E_{n,l+1}^{D-2}$. In addition, the particle is less bound when l increases. Our last figures clearly show that the length oscillator strength decreases for increasing potential parameters S_1, S_2 and S_3 (the behavior is quite reverse for the velocity oscillator strength).

Appendix A

Supersymmetry quantum mechanics: within this appendix, a through introduction to SUSY quantum mechanics is included. These few lines are form [1,3]. Our first goal in SUSYQM mechanics is finding solution of the Riccati equation

$$V_{\mp} = \phi^2 \mp \phi', \tag{A-1}$$

with V being the potential of Schrödinger equation. If

$$V_{+}(a_0, x) = V_{-}(a_1, x) + R(a_1), \tag{A-2}$$

where a_1 is a new set of parameters uniquely determined from the old set a_0 via the mapping $F : a_0 \mapsto a_1 = F(a_0)$ and the residual term $R(a_1)$ does not include x , the partner potentials are shape invariant and the necessary information of the system is obtained via [1,3]

$$E_n = \sum_{s=1}^{n} R(a_s), \tag{A-3}$$

$$\phi_n^-(a_0, x) = \prod_{s=0}^{(n-1)} \left(\frac{A^\dagger(a_s)}{(E_n - E_s)^{0.5}} \right) \phi_0^-(a_n, x), \tag{A-4}$$

$$\phi_0^-(a_n, x) = C \exp\left(- \int_0^x dz \phi(a_n, z) \right), \tag{A-5}$$

$$A_s^\dagger = -\frac{\partial}{\partial x} + \phi(a_s, z). \tag{A-6}$$

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