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Effective Field Theory and the Gamow Shell Model

The ${}^6\text{He}$ Halo Nucleus

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Abstract We combine halo/cluster effective field theory (H/CEFT) and the Gamow shell model (GSM) to describe the 0^+ ground state of ${}^6\text{He}$ as a three-body halo system. We use two-body interactions for the neutron-alpha particle and two-neutron pairs obtained from H/CEFT at leading order, with parameters determined from scattering in the $p_{3/2}$ and s_0 channels, respectively. The three-body dynamics of the system is solved using the GSM formalism, where the continuum states are incorporated in the shell model valence space. We find that in the absence of three-body forces the system collapses, since the binding energy of the ground state diverges as cutoffs are increased. We show that addition at leading order of a three-body force with a single parameter is sufficient for proper renormalization and to fix the binding energy to its experimental value.

1 Introduction

Nuclei located far away from the valley of β -stability display peculiar features that do not occur for well bound nuclei. The strong coupling to the continuum manifests itself in the existence of halo configurations, where some nucleons orbit far away from a core of more tightly bound nucleons, and of Borromean systems, where removal of one nucleon is accompanied by at least one more nucleon. The neutron-rich Helium isotopes ${}^6\text{He}$ and ${}^8\text{He}$ offer two examples of such nuclei: both are Borromean halos that have no bound excited states. They also exhibit the “binding-energy anomaly”, i.e., higher one- and two-neutron emission thresholds in ${}^8\text{He}$ than in ${}^6\text{He}$.

Halo configurations are characterized by a large nuclear radius compared to the size of the tightly bound core or, equivalently, by a small nucleon separation energy compared to the core binding energy. The physics of halo nuclei is a perfect arena for the application of effective field theory (EFT). EFTs provide a powerful framework to exploit separation of scales in physical systems in order to perform systematic, model-independent calculations. If, for example, the relative momentum k of two particles is much smaller than the inverse range of their interaction, $1/R$, using contact interactions observables can be expanded in powers of kR [1].

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The application of EFT to halo and cluster systems, halo/cluster EFT (H/CEFT), was first exemplified in low-energy neutron-alpha particle ($n\alpha$) scattering [2,3]. Even though there is no bound state, the $n\alpha$ T matrix has a resonance pole at an energy $E_{5gs} \simeq 0.8$ MeV much smaller than the $E_{4ex} \simeq 20$ MeV it takes to excite the alpha particle. The physics in the p wave is closely related to halo dynamics, because the ground state in ${}^5\text{He}$ can in a first approximation be described as an $n\alpha$ system in the $p_{3/2}$ -wave configuration, which has the qualitative characteristics of a two-body halo nucleus. H/CEFT captures these features at leading order (LO), and provides at next-to-leading order a good description of $n\alpha$ scattering [2,3]. H/CEFT has also been successfully applied to other dilute two-body systems such as low-energy $\alpha\alpha$ [4] and proton- ${}^7\text{Li}$ [5] scattering, radiative neutron capture on ${}^7\text{Li}$ [6,7], and the electric properties of ${}^{11}\text{Be}$ [8].

The natural follow-up to Refs. [2,3] is to consider the next halo system within the Helium isotope family, that is, ${}^6\text{He}$. This is the aim of the present paper. Since the ${}^6\text{He}$ ground state is bound by only $E_{6gs} \simeq -1.0$ MeV, it is appropriate to assume that it can be described as the three-body system $n+n+\alpha$. The $n\alpha$ interaction is that studied in Refs. [2,3], while the nn force is a contact interaction determined by the low-energy nn scattering parameters [1,9,10]. Although certain aspects of ${}^6\text{He}$ have already been studied with greater resolution as a six-body system, as stressed in Ref. [11] its long-range properties are much more efficiently captured as a three-body system. Moreover, existing ab initio calculations are based on potentials that produce amplitudes which do not satisfy renormalization-group (RG) invariance: the ultraviolet cutoff, originally introduced as an arbitrary separation between physics kept explicit in the theory and physics treated as short ranged, becomes a fit parameter.

It is well understood that the physics of three-body systems can be much richer than the physics of its two-body subsystems. A famous example is the Efimov effect which occurs for a system of non-relativistic particles (bosons or three-or-more-component fermions) with short-range interactions: if the s-wave scattering lengths of its subsystems are tuned to infinity (unitarity limit), there can be an infinite sequence of three-body bound states that has an accumulation point at the three-body threshold [12]. A closely related phenomenon is the Thomas effect [13], where a finite-range two-body potential that is only attractive enough to support a single two-body bound state can produce three-body bound states with arbitrarily large binding energies as the range goes to zero. In EFT at a given order, the Thomas effect is a consequence of an inappropriate omission of a three-body force. Since range effects are small at low energies, they are not present at LO and the Thomas effect appears as the collapse of the three-body ground state when the cutoff is increased. This unacceptable cutoff dependence arises from the large-momentum behavior of the LO two-body interaction. A three-body force is then necessary and sufficient for RG invariance at LO [14–16], allowing three-body energies to be independent of the ultraviolet regulator. The parameter associated with this force provides a scale for the remaining discrete scale invariance, which reflects itself in the Efimov spectrum. The structure of ${}^{22}\text{C}$ and other possible two-neutron halos with s-wave neutron-core interactions was discussed using H/CEFT in Refs. [17,18].

The situation with p-wave two-body interactions is less clear. There is debate over whether the Efimov effect can be realized in this case—see Refs. [19,20] and references therein. The main issue to be addressed below is whether a three-body force is needed at LO so that the EFT description of ${}^6\text{He}$ is properly renormalized. We emphasize that from an EFT perspective there is no question about the existence of few-body forces: since they are not forbidden by any symmetry they arise at some level. In few-body models of ${}^6\text{He}$ and other Borromean nuclei, it is typically found that a description via two-body forces adjusted to in-vacuum two-body data has (sometimes severe) phenomenological shortcomings, which are ameliorated by the inclusion of three-body forces or, equivalently, modifications of two-body forces due to the presence of a third body (see, for example, Refs. [21–23]). In EFT, the phenomenological importance (or lack thereof) of three-body forces is a consequence of the order they appear. The issue here is whether they appear already at LO in ${}^6\text{He}$. The tool we use to assess the presence of three-body forces at LO is, as always, RG invariance. However, the fact that three-body forces have been shown to be needed for s-wave two-body interactions near the unitary limit does not imply that they are LO when the two-body interactions have different asymptotic behavior, for example when they are p-wave. Contrary to a phenomenological assessment, to which it is complementary, an RG analysis concerns the internal consistency of the theory, and does not depend on the exact values of ${}^6\text{He}$ observables.

A microscopic description of weakly bound/unbound nuclei requires taking into account the interplay between bound states, scattering states, and resonances. In other words, these systems have to be described as open quantum systems (OQSs), in contradistinction with well-bound nuclei, which are nearly isolated from the environment of scattering states and decay channels (“closed quantum systems”). A recent realization of the shell model for OQSs is the so-called Gamow shell model (GSM) [24–30]. The GSM is based on the

Berggren basis [31], which consists of bound, resonant and scattering single-particle wave functions generated by a finite-depth potential, and it provides the mathematical foundation for unifying bound and resonant states—the poles of the T matrix—in the context of the Schrödinger equation. The GSM has been used [26–30] to study the properties of the Helium isotope family using a phenomenological $n\alpha$ potential and a residual two-neutron interaction adjusted to few-body energies.

In this paper we use the formalism of the GSM to solve the Schrödinger equation describing the dynamics of ${}^6\text{He}$ with the contact interactions of H/CEFT, including a possible three-body force. Because the two-body interaction here is energy dependent, some care must be taken with the Berggren relation. In this first application of the GSM to H/CEFT, we focus on the binding energy of the 0^+ ground state.

The paper is organized as follows. In Sect. 2 we review the potentials derived with EFT and used for the study of ${}^6\text{He}$. We introduce the GSM formalism in Sect. 3, and in Sect. 4 we show results for the ground-state energy of ${}^6\text{He}$. We shall see that without a three-body force, the system is not properly renormalized at LO¹. We conclude and summarize in Sect. 5.

2 Two-Body Potentials

Being weakly bound compared to the first excited state of the alpha particle, the $J^\pi = 0^+$ ground state in ${}^6\text{He}$ can be described as a three-body system $n + n + \alpha$. The neutrons in the halo interact with the alpha particle via a two-body interaction $V_{n\alpha}$ and with each other via a potential V_{nn} . We denote the neutron (core) mass by m_n (M_c) and the neutron-core reduced mass by $\mu = m_n M_c / (m_n + M_c)$.

The potential between the α core and a neutron is constructed with EFT as described in Refs. [2, 3]. The small relative momentum means that neutron and alpha particle see each other, in a first approximation, as elementary objects. At LO there is only one contribution, which is in the $p_{3/2}$ channel, and the “dimeron” potential projected onto this channel can be written as

$$V_{n\alpha}(k', k, k_0) = \frac{k'k}{A + Bk_0^2}, \quad (1)$$

where \mathbf{k} (\mathbf{k}') is the incoming (outgoing) relative momentum and $k_0 = \sqrt{2\mu E_{n\alpha}}$ in terms of the total energy $E_{n\alpha}$ of the $n\alpha$ subsystem. A and B are parameters. Since this interaction is singular, a regularization procedure is introduced in form of an ultraviolet cutoff $\Lambda_{n\alpha}$. The cutoff separates the short-distance physics, which is not included explicitly in the dynamics at low energies, and the long-distance physics, which is. This is here achieved by introducing a smooth regulator function

$$F(x) = \exp(-x), \quad (2)$$

whose role is to suppress the high-energy contributions of the potential. We thus replace the potential (1) by

$$V_{n\alpha}(k', k, k_0; \Lambda_{n\alpha}) = \frac{k'k}{A(\Lambda_{n\alpha}) + B(\Lambda_{n\alpha})k_0^2} F(k'^2/\Lambda_{n\alpha}^2) F(k^2/\Lambda_{n\alpha}^2). \quad (3)$$

In order for observables to be RG invariant, i.e., independent of the arbitrary cutoff, the parameters $A(\Lambda_{n\alpha})$ and $B(\Lambda_{n\alpha})$ must depend on $\Lambda_{n\alpha}$.

More precisely, at LO, $A(\Lambda_{n\alpha})$ and $B(\Lambda_{n\alpha})$ are fixed such that the phase shifts at low energies obtained with the potential (3) reproduce the effective range expansion (ERE) in the $p_{3/2}$ channel truncated at the level of the effective “range”:

$$k^3 \cot \delta_{n\alpha}(k) = -\frac{1}{a_{n\alpha}} + \frac{r_{n\alpha}}{2} k^2, \quad (4)$$

with the scattering volume $a_{n\alpha} = -62.951 \text{ fm}^3$ and the effective momentum $r_{n\alpha} = -0.8819 \text{ fm}^{-1}$ [34]. The position k_{res} of the $p_{3/2}$ resonance is obtained from

$$\cot \delta_{n\alpha}(k_{res}) = i, \quad (5)$$

¹ Our first results were presented in Ref. [32]. Similar results have been obtained independently by Ji et al. [33].

so at this order $k_{res} = (0.1748 - 0.0313i) \text{ fm}^{-1}$. By solving the Lippman–Schwinger equation with the potential (3), one obtains

$$A(\Lambda_{n\alpha}) = 2\mu \left[\frac{1}{a_{n\alpha}} - \frac{\Lambda_{n\alpha}^3}{4\sqrt{2\pi}} \right], \quad (6)$$

$$B(\Lambda_{n\alpha}) = -\mu \left[r_{n\alpha} + \frac{4}{a_{n\alpha}\Lambda_{n\alpha}^2} + \sqrt{\frac{2}{\pi}}\Lambda_{n\alpha} \right]. \quad (7)$$

The two neutrons in the halo have sufficiently low relative momentum that meson exchange can be considered a short-range force. The neutron–neutron potential is thus taken from the pionless EFT [1, 9, 10]. At LO, the potential is entirely in the 1s_0 channel; in momentum space it is simply a constant C . As before, the potential requires regularization, for which we continue to use the function $F(x)$, but now in terms of the relative momentum between the two neutrons and of an nn cutoff Λ_{nn} :

$$V_{nn}(k', k; \Lambda_{nn}) = C(\Lambda_{nn}) F(k'/\Lambda_{nn}^2) F(k^2/\Lambda_{nn}^2). \quad (8)$$

As previously, we fix the coupling constant $C(\Lambda_{nn})$ with the ERE for nn scattering, but now truncated at the level of the scattering length,

$$k \cot \delta_{nn}(k) = -\frac{1}{a_{nn}}, \quad (9)$$

with $a_{nn} = -18.7 \text{ fm}$ [35]. Again solving the Lippman–Schwinger equation,

$$C(\Lambda_{nn}) = \frac{1}{m_n} \left[\frac{1}{a_{nn}} - \frac{\Lambda_{nn}}{\sqrt{2\pi}} \right]^{-1}. \quad (10)$$

Note that we do not modify the nn potential in ^6He to account for the presence of the α core, as frequently done [26–29]. This modification is a three-body effect that in EFT is represented by three-body forces, which are present starting at some order, since they are not forbidden by any symmetry. We want to determine whether such a force is needed at LO to renormalize the $n + n + \alpha$ system.

3 Schrödinger Equation with the Gamow Shell Model

We now consider the solution of the Schrödinger equation for the $n + n + \alpha$ system with the Gamow Shell Model. We use coordinates inspired by the Cluster Orbital Shell Model [29, 36]: \mathbf{r}_i is the position of neutron $i = 1, 2$ relative to the α core, and \mathbf{p}_i the corresponding momentum. The Hamiltonian of the $n + n + \alpha$ system with the two-body interactions $V_{n\alpha}$ and V_{nn} is written as

$$H = \sum_{i=1}^2 \left[\frac{\mathbf{p}_i^2}{2\mu} + V_{n\alpha}(k_{0i}; \Lambda_{n\alpha}) \right] + V_{nn}(\Lambda_{nn}) + \frac{\mathbf{p}_1 \cdot \mathbf{p}_2}{M_c}. \quad (11)$$

This Hamiltonian is translationally invariant, the recoil term $\mathbf{p}_1 \cdot \mathbf{p}_2/M_c$ stemming from the choice of coordinates.

We work within the framework of the Gamow shell model formalism [24–30] to solve the dynamics generated by the Hamiltonian (11). The three-body equation is solved using a single-particle (sp) basis. The set of sp states that define the one-body valence space is taken as the set of eigenstates of the LO potential $V_{n\alpha}(k_0; \Lambda_{n\alpha})$. They are solutions of the one-body Schrödinger equation

$$H^{sp}|\Psi\rangle = \left[\frac{\mathbf{p}^2}{2\mu} + V_{n\alpha}(k_0; \Lambda_{n\alpha}) \right] |\Psi\rangle = E_{n\alpha}|\Psi\rangle. \quad (12)$$

By inserting the completeness relation projected on the $p_{3/2}$ partial wave,

$$\int_c dk k^2 |k\rangle\langle k| = 1, \quad (13)$$

along a contour \mathcal{C} in the fourth quadrant of the complex momentum plane, Eq. (12) can be written as an equation for the momentum-space wave function $\Psi(k) = \langle k|\Psi\rangle$,

$$\int_{\mathcal{C}} dk' k'^2 \langle k| \left[\frac{p^2}{2\mu} + V_{n\alpha}(k_0; \Lambda_{n\alpha}) \right] |k'\rangle \Psi(k') = E_{n\alpha} \Psi(k). \quad (14)$$

In this paper the contour \mathcal{C} is chosen to be made out of three straight-line segments $\mathcal{C}_{1,2,3}$, $\mathcal{C} = \mathcal{C}_1 + \mathcal{C}_2 + \mathcal{C}_3$, as indicated in Fig. 1. Segment \mathcal{C}_1 extends from $k_0 = 0$ to $k_1 = k_{1r} + ik_{1i}$, segment \mathcal{C}_2 from k_1 to $k_2 = k_{2r}$, and segment \mathcal{C}_3 from k_2 to $k_3 = k_{max}$, where $k_{max} \geq k_{2r} \geq k_{1r} \geq 0 \geq k_{1i} > -k_{1r}$ are real numbers. Since the sp set must be finite, the contour integral along \mathcal{C} is performed up to a cutoff k_{max} and discretized with a quadrature method. In this case, k_{max} must be chosen large enough such that all low-energy physics below $\Lambda_{n\alpha}$ is taken into account. Here, we typically choose $k_{max} \sim 3\Lambda_{n\alpha}$. Had we chosen a sharp regulator for $F(x)$, k_{max} would have been such that $k_{max} = \Lambda_{n\alpha}$ since in that case $V_{n\alpha}(k_0, \Lambda_{n\alpha})$ would have vanished for momentum above $\Lambda_{n\alpha}$. In practice, the contour \mathcal{C} is discretized using a Gauss–Legendre quadrature using N_i points for the segment \mathcal{C}_i , for a total number $N_{sh} = N_1 + N_2 + N_3$ of discretization points.

If $k_{1i} = 0$, the contour is along the real axis, and the solutions of Eq. (14) consist of bound states and scattering states. If $k_{1i} \neq 0$, solutions consist instead of bound states, resonant states located above the contour, and complex-scattering scattering states along the contour [31]. In order to include the $p_{3/2}$ resonance, we take $k_{1r} = 0.18 \text{ fm}^{-1}$, $k_{1i} = -0.08 \text{ fm}^{-1}$, and $k_{2r} = 0.5 \text{ fm}^{-1}$. For each value of $\Lambda_{n\alpha}$ the Schrödinger equation (14) is solved with the LO potential along the complex contour \mathcal{C} . Results for the energy of the resonance are independent of the choice of the contour as long as it goes below the resonance and as long as the discretization is precise enough. To illustrate this point, we show in Table 1 the position of the resonance $k_{p_{3/2}}$ at $\Lambda_{n\alpha} = 3.1 \text{ fm}^{-1}$ as a function of N_{sh} . We start from $N_1 = 4$, $N_2 = 4$, and $N_3 = 9$, that is $N_{sh} = 17$, and add one shell in each part of the contour up to $N_{sh} = 35$. We can see that, for this last value of N_{sh} , convergence to four significant figures has been reached. The position of the $p_{3/2}$ resonance as a function of $\Lambda_{n\alpha}$ is shown in Fig. 2.

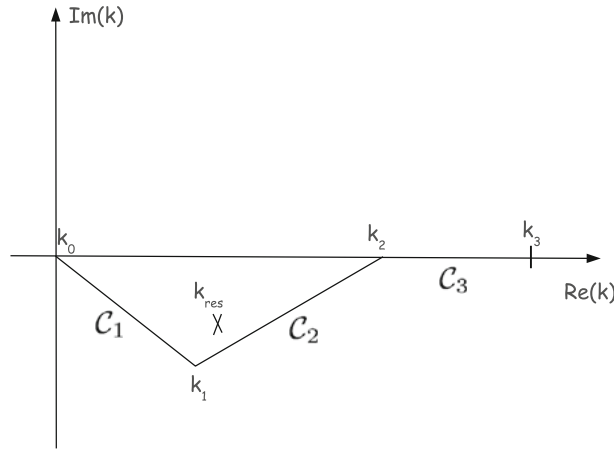


Fig. 1 Contour $\mathcal{C} = \mathcal{C}_1 + \mathcal{C}_2 + \mathcal{C}_3$ in the complex-momentum plane considered in the text. The points $k_0 = 0$, $k_1 = k_{1r} + ik_{1i}$, $k_2 = k_{2r}$, and $k_3 = k_{max}$ delimiting the segments $\mathcal{C}_{1,2,3}$ are indicated. We also show the position k_{res} of a resonance appearing in the Berggren relation (15)

Table 1 Position of the resonance $k_{p_{3/2}}$ at $\Lambda_{n\alpha} = 3.1 \text{ fm}^{-1}$ as a function of N_{sh} , the total number of discretized points on the complex contour defined with $k_{1r} = 0.18 \text{ fm}^{-1}$, $k_{1i} = -0.08 \text{ fm}^{-1}$, $k_{2r} = 0.5 \text{ fm}^{-1}$, and $k_{max} = 10.0 \text{ fm}^{-1}$

N_{sh}	$\text{Re}(k_{p_{3/2}}) [\text{fm}^{-1}]$	$\text{Im}(k_{p_{3/2}}) [\text{fm}^{-1}]$
17	0.17595	-0.03344
20	0.17615	-0.03354
23	0.17594	-0.03327
26	0.17590	-0.03324
29	0.17589	-0.03322
32	0.17589	-0.03323
35	0.17589	-0.03323

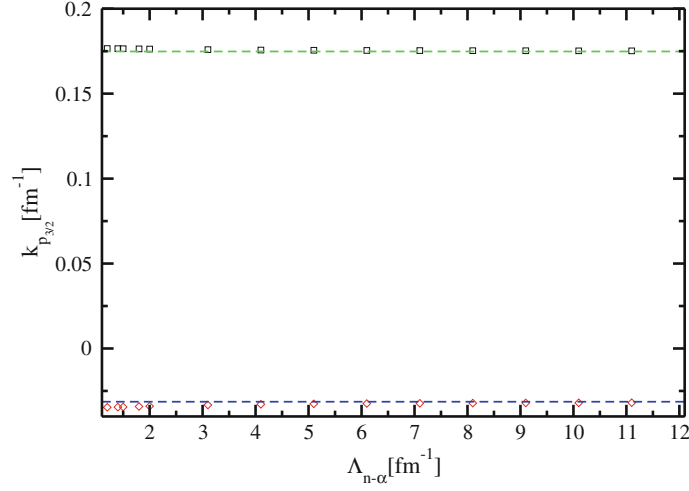


Fig. 2 Position of the lowest $p_{3/2}$ $n\alpha$ resonance as a function of the cutoff $\Lambda_{n\alpha}$. Points at the *top* (*bottom*) are the results for the real (imaginary) momentum of the resonance coming from the numerical solution of the Schrödinger equation described in the text. The *dashed lines* are the corresponding momenta obtained from the effective range expansion with empirical values for the scattering volume and effective momentum

We also show, for comparison, the value k_{res} obtained directly from the ERE. As it can be seen from the figure, as the cutoff $\Lambda_{n\alpha}$ increases the position of the resonance quickly converges to k_{res} .

Since the ${}^6\text{He}$ ground state is bound, it is equivalent to use for the sp states either a set of shells located along the real continuum axis, or a complex-continuum set of states along a complex contour \mathcal{C} together with the $p_{3/2}$ resonance. In the three-body calculations presented below we have used shells located along the real energy axis, that is, we have taken $k_{1i} = 0$. For instance, for $\Lambda_{n\alpha} = 7.1 \text{ fm}^{-1}$, we use $k_{1r} = 5.0 \text{ fm}^{-1}$, $k_{2r} = 12.0 \text{ fm}^{-1}$, and $k_{max} = 21.0 \text{ fm}^{-1}$, with $N_1 = N_2 = N_3 = 30$. We could certainly decrease the number of shells to reach the same precision in the value of the ${}^6\text{He}$ ground state. We have not studied in detail what the smallest admissible number of points would be. Indeed, for a system made of three particles this is not of a great importance, since the diagonalization of the non-symmetric Hamiltonian matrix can be performed rather fast.

From the set of eigenstates of the Schrödinger equation (12) a sp basis is generated. For an energy-independent potential, a resonant state $|\Psi_{res}\rangle$ above, and scattering states along, \mathcal{C} (see Fig. 1) satisfy the usual Berggren relation [31],

$$|\Psi_{res}\rangle\langle\tilde{\Psi}_{res}| + \int_{\mathcal{C}} dk k^2 |\Psi(k)\rangle\langle\tilde{\Psi}(k)| = 1, \quad (15)$$

where the bra $\langle\tilde{\Psi}|$ conjugate to the ket $|\Psi\rangle$ is such that $\langle\tilde{\Psi}|r\rangle = \langle r|\Psi\rangle$. However, the potential $V_{n\alpha}(k_0; \Lambda_{n\alpha})$ being energy-dependent, the eigenstates of Eq. (14) are not orthogonal and Eq. (15) does *not* hold. One then has to consider an extra step to generate a basis. This is achieved, after having discretized the contour \mathcal{C} , by solving the matrix equation

$$\sum_{i=1}^{N_{sh}} |\Psi_i\rangle\langle\Psi_i^{boc}| = 1, \quad (16)$$

where $|\Psi_i\rangle$ is one of the N_{sh} discrete sp eigenstates of the potential, Eq. (12), and $\langle\Psi_i^{boc}|$ its bi-orthogonal complement. By construction,

$$\langle\Psi_i^{boc}|\Psi_j\rangle = \delta_{ij}. \quad (17)$$

A complication is that for sufficiently large values of the cutoff, $\Lambda_{n\alpha} \geq \Lambda_b \simeq 1.8 \text{ fm}^{-1}$, the potential $V_{n\alpha}(k_0; \Lambda_{n\alpha})$ supports a bound state $|\Psi_b\rangle$. At $\Lambda_{n\alpha} = 1.8 \text{ fm}^{-1}$ the energy of this bound state is $E_b = -20.941 \text{ MeV}$, that is, outside the range of validity of our EFT approach. As a consequence, we do not want to include it in the valence space. From the practical point of view, we first tried to construct the bi-orthogonal

basis by including the bound state in the bi-orthogonalization procedure, and then omitting it when constructing the many-body basis to solve the three-body problem. This procedure turned out to give rather peculiar results in the sense that the value obtained for the ${}^6\text{He}$ ground state displayed a discontinuous behavior as $\Lambda_{n\alpha}$ varied from values below to above Λ_b . We suspect that despite the fact that the bound state is not included in the basis, it has an indirect effect, for it is present during the phase of construction of the sp basis according to Eq. (16). One has then to figure out another way to generate the sp shells when a deep bound state is present.

For $\Lambda_{n\alpha} \geq \Lambda_b$, we generate the sp basis by converting the energy dependence of $V_{n\alpha}(k_0; \Lambda_{n\alpha})$ into momentum dependence by introducing an energy-independent potential $V'_{n\alpha}(\Lambda_{n\alpha})$ that reproduces the half-on-shell T matrix [37],

$$\langle k' | V'_{n\alpha}(\Lambda_{n\alpha}) | \Psi \rangle = \langle k' | V_{n\alpha}(k_0; \Lambda_{n\alpha}) | \Psi \rangle, \quad (18)$$

where k_0 is obtained from the energy of the H^{sp} eigenstate $|\Psi\rangle$. For each discretized value k'_i along the contour \mathcal{C} , we solve Eq. (18) in order to generate $V'_{n\alpha}(k'_i, k_j; \Lambda_{n\alpha})$ without considering the bound state $|\Psi_b\rangle$. For each k'_i we have N_{sh} unknowns, $V'_{n\alpha}(k'_i, k_j; \Lambda_{n\alpha})$ with $j = 1, \dots, N_{sh}$, and $N_{sh} - 1$ equations,

$$\langle k'_i | V'_{n\alpha}(\Lambda_{n\alpha}) | \Psi_j \rangle = \langle k'_i | V_{n\alpha}(k_0; \Lambda_{n\alpha}) | \Psi_j \rangle, \quad j = 1, \dots, N_{sh}, \quad j \neq b. \quad (19)$$

In order to solve this linear system we impose the condition

$$\langle k'_i | V'_{n\alpha}(\Lambda_{n\alpha}) | k_{N_{sh}} \rangle = 0, \quad (20)$$

with $|k_{N_{sh}}\rangle$ being the state with the largest momentum on the contour \mathcal{C} . This leads to a small error, since at such high momentum, $k_{N_{sh}} \sim k_{max} \sim 3\Lambda_{n\alpha}$, the influence of the potential is negligible. Moreover this error can be made arbitrarily small by increasing k_{max} . The potential $V'_{n\alpha}(\Lambda_{n\alpha})$ is non-Hermitian and has right eigenvectors $|\Psi_i\rangle$ and left eigenvectors $\langle \Psi_i^{left} |$. The right eigenvectors are by construction the eigenvectors of the original energy-dependent potential $V_{n\alpha}(k_0; \Lambda_{n\alpha})$, and we now have the following completeness relation:

$$\sum |\Psi_i\rangle \langle \Psi_i^{left} | = 1. \quad (21)$$

For $\Lambda_{n\alpha} < \Lambda_b$, the two previous procedures for constructing the sp basis are completely identical, the left eigenvectors $\langle \Psi_i^{left} |$ of $V'_{n\alpha}(\Lambda_{n\alpha})$ obtained with the second method being equal to the bi-orthogonal complement states $\langle \Psi_i^{boc} |$ obtained in the first method by solving Eq. (16).

From the sp basis, we construct the antisymmetrized three-body basis states coupled to good total angular momentum J , $|(\Psi_i, \Psi_j)_{i \leq j}^J\rangle$, which are eigenstates of the Hamiltonian $H_1^{sp} + H_2^{sp}$ with eigenvalues $E_i + E_j$:

$$(H_1^{sp} + H_2^{sp}) |(\Psi_i, \Psi_j)_{i \leq j}^J\rangle = (E_i + E_j) |(\Psi_i, \Psi_j)_{i \leq j}^J\rangle. \quad (22)$$

The corresponding bi-orthogonal complement is $\langle (\Psi_i^{left}, \Psi_j^{left})_{i \leq j}^J |$.

The interaction $V_{nn}(\Lambda_{nn})$ is defined in terms of relative coordinates between the two neutrons. Since our Hamiltonian was written in terms of $n\alpha$ coordinates, a transformation is necessary to express the matrix elements $V_{nn}(\Lambda_{nn})$ in the shell model basis $|(\Psi_i, \Psi_j)_{i \leq j}^J\rangle$. For this purpose we use an expansion on a set of harmonic-oscillator (HO) wave functions, as in Ref. [30]. That is, we project $V_{nn}(\Lambda_{nn})$ on a HO set $|ab\rangle$, where a and b label sp states of HOs in the $n\alpha$ coordinate, and consider the nn interaction

$$V_{nn}^{osc}(\Lambda_{nn}) = \sum_{a < b} \sum_{c < d} |ab\rangle \langle ab | V_{nn}(\Lambda_{nn}) | cd\rangle \langle cd|, \quad (23)$$

where the restriction in the sum is due to the antisymmetry of the two-neutron state. Using Moshinsky transformations [38], one can easily calculate $\langle ab | V | cd\rangle$. Results for the three-body energy are independent of the values of the HO frequency, as long as enough HO states are included in the expansion.

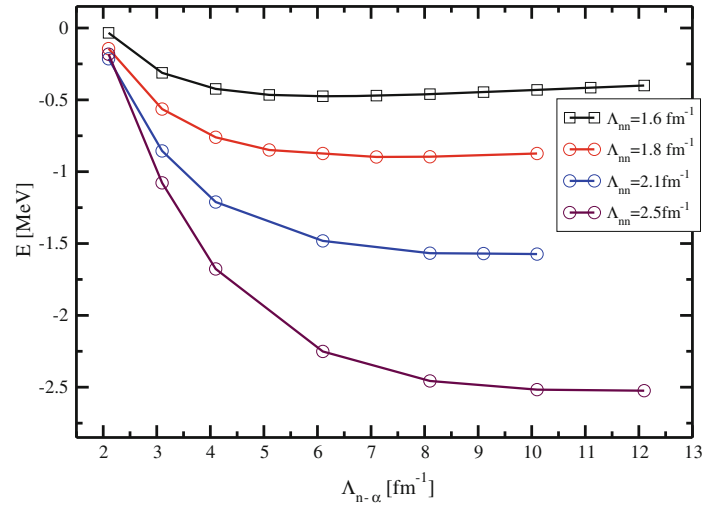


Fig. 3 Ground-state energy of ${}^6\text{He}$ from the LO two-body potentials $V_{n\alpha}(k_0; \Lambda_{n\alpha})$ and $V_{nn}(\Lambda_{nn})$. For each value of Λ_{nn} the cutoff $\Lambda_{n\alpha}$ is varied

4 Results for the ${}^6\text{He}$ Ground State

The ground state of ${}^6\text{He}$ is coupled to $J^\pi = 0^+$ and the three-body basis states are constructed from the sp states of the $V_{n\alpha}(k_0; \Lambda_{n\alpha})$ potential as described in the previous section. At LO, only $p_{3/2}$ shells are included in the valence space and, as a consequence, all matrix elements of the recoil term in Eq. (11) vanish. For each value of $\Lambda_{n\alpha}$ the coupling constants $A(\Lambda_{n\alpha})$ and $B(\Lambda_{n\alpha})$ are fixed such that the ERE in the $p_{3/2}$ channel truncated at the level of the effective “range” is reproduced. Similarly, $C_0(\Lambda_{nn})$ is fixed such that the 1s_0 nn scattering length is reproduced.

Figure 3 shows the energy $E_{nn\alpha}$ of the ground state in ${}^6\text{He}$ for different values of $\Lambda_{n\alpha}$ and Λ_{nn} . For each value of Λ_{nn} the cutoff $\Lambda_{n\alpha}$ is increased. One can see that the energy initially quickly decreases, then slowly rises. For $\Lambda_{nn} = 1.6 \text{ fm}^{-1}$, for example, $E_{nn\alpha}$ goes from -0.034 MeV for $\Lambda_{n\alpha} = 2.1 \text{ fm}^{-1}$ to -0.475 MeV for $\Lambda_{n\alpha} = 6.1 \text{ fm}^{-1}$, then to -0.400 MeV for $\Lambda_{n\alpha} = 12.1 \text{ fm}^{-1}$. As Λ_{nn} increases, the initial decrease becomes steeper, and the increase is postponed to higher values of $\Lambda_{n\alpha}$. For instance, at $\Lambda_{nn} = 2.5 \text{ fm}^{-1}$, the energy goes from -0.182 to -2.251 to -2.524 MeV in the same range of $\Lambda_{n\alpha}$ values.

This behavior can be understood from the qualitative renormalization features of the system. As $\Lambda_{n\alpha}$ increases, the phase space of the three-body system increases, the attractive nn interaction is better resolved, and the binding energy increases. This is consistent with the pattern observed in Ref. [39] for the energy of a three-fermion system interacting via a two-body force constructed with EFT at LO. In that case, for a fixed cutoff of the two-body interaction, the total energy of the system decreases as the size of the model space increased. As Λ_{nn} increases, presumably more correlations are cut off for too small a value of $\Lambda_{n\alpha}$, generating the faster decrease. However, there is also a residual dependence on $\Lambda_{n\alpha}$ from $V_{n\alpha}(k_0; \Lambda_{n\alpha})$. Even though the potential has been properly renormalized, that is, the coupling constants $A(\Lambda_{n\alpha})$ and $B(\Lambda_{n\alpha})$ have been fixed so that the truncated ERE is reproduced, there still is a dependence for finite values of the cutoff, as seen in Fig. 2. The energy of the $p_{3/2}$ resonance goes from $k = 0.7714 - 0.2947i$ to $k = 0.7696 - 0.2896i \text{ MeV}$ when $\Lambda_{n\alpha}$ goes from 6.1 fm^{-1} to 12.1 fm^{-1} . This means that, as $\Lambda_{n\alpha}$ is varied within this range, there is a variation $\simeq 0.005 \text{ MeV}$, or about 7 %, in the norm of the energies of the $p_{3/2}$ resonance, which is consistent with a variation of about 15 % in the three-body energy in the same range—for example a variation of $\simeq 0.075 \text{ MeV}$ for $\Lambda_{nn} = 1.6 \text{ fm}^{-1}$.

One can clearly see from Fig. 3 that as the cutoffs Λ_{nn} and $\Lambda_{n\alpha}$ are increased, the energy decreases without reaching a stabilized value. To stress this fact, in Fig. 4 we plot the ${}^6\text{He}$ ground-state energy as function of $\Lambda_{nn} = \Lambda_{n\alpha}$. We have checked that the results are similar if other relations are assumed between Λ_{nn} and $\Lambda_{n\alpha}$, for example, if we take the minimum energy for each Λ_{nn} , which is equivalent to choosing $\Lambda_{n\alpha}$ large enough so that all correlations of the nn interaction have been resolved by the three-body system.

The nearly linear dive of the ground state seen in Fig. 4 is reminiscent of the behavior observed with LO two-body forces in EFTs for systems of three bosons or three-or-more-component fermions [14–16]. There, the

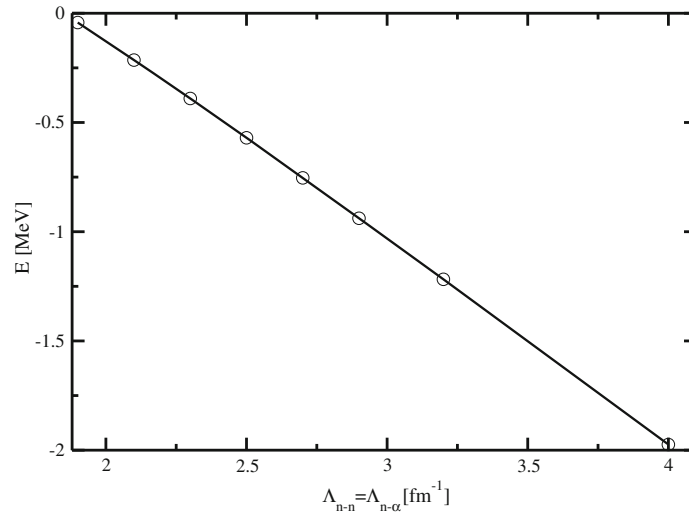


Fig. 4 Ground-state energy of ${}^6\text{He}$ from the LO two-body potentials $V_{n\alpha}(k_0; \Lambda_{n\alpha})$ and $V_{nn}(\Lambda_{nn})$, for $\Lambda_{nn} = \Lambda_{n\alpha}$

diverge is even faster, more like quadratic in the cutoff, stemming from the strong s-wave interactions among the three pairs. In either case, what we see is a collapse of the ground state under short-range two-body interactions similar to the one first observed by Thomas [13]. It is an indication that the three-body problem has not been properly renormalized with only two-body interactions [14–16]. The cutoff dependence is not decreasing as the cutoffs increase, as one would expect from residual cutoff dependence in a renormalized system that has been truncated correctly, but is instead increasing with positive powers of the cutoffs.

The solution to this problem has to be found outside the two-body subsystems, which are perfectly well defined and well described by the EFT. We thus add to LO a three-body force to renormalize the three-body problem. An s-wave three-body force does not have any impact on the structure of ${}^6\text{He}$ at this order since the $n\alpha$ subsystems are in a relative $p_{3/2}$ wave. The lowest-derivative three-body force that does not vanish in the channel of interest can be written, in the coordinates we are using, as

$$V_{nn\alpha}(k'_1, k'_2, k_1, k_2) = D(\Lambda_{nn\alpha}) k'_1 k_1 k'_2 k_2 F(k_1^2/\Lambda_{nn\alpha}^2) F(k_2^2/\Lambda_{nn\alpha}^2) F(k_1^2/\Lambda_{nn\alpha}^2) F(k_2^2/\Lambda_{nn\alpha}^2), \quad (24)$$

with k'_i (k_i) the outgoing (incoming) momentum for the i th $n\alpha$ subsystem. Here $\Lambda_{nn\alpha}$ is a three-body cutoff and D is a low-energy coupling constant with dimensions of mass⁻⁹, whose dependence on $\Lambda_{nn\alpha}$ is adjusted so that three-body observables be (nearly) cutoff independent.

Here for simplicity we take $\Lambda_{nn\alpha} = \Lambda_{nn} = \Lambda_{n\alpha}$. We find that we can then keep the ${}^6\text{He}$ ground-state energy $E_{nn\alpha}$ constant as the cutoff is varied. We show in Fig. 5 the resulting running of the coupling constant $D(\Lambda_{nn\alpha})$ when the ${}^6\text{He}$ ground-state energy is fixed to its experimental value $E_{6gs} = -0.98$ MeV [40]. At low cutoffs, D is negative. From Fig. 4 we see that at a cutoff $\Lambda_0 \simeq 2.9$ fm⁻¹ the energy calculated with only two-body forces agrees with the experimental value, so $D(\Lambda_0) = 0$. Above Λ_0 , $D(\Lambda_{nn\alpha})\Lambda_{nn\alpha}^2$ is positive and approximately constant in the region of cutoffs we could probe. We cannot, however, exclude a limit-cycle-like behavior at higher cutoffs, as observed for s-wave systems in Refs. [14–16].

Again like for three bosons or three-or-more-component fermions [14–16], RG invariance requires the three-body force to appear at LO. Naturalness together with naive dimensional analysis suggests that D would scale as M^{-9} , with M a large mass scale such as the alpha-particle binding momentum or the pion mass. If that were the case after renormalization, the three-body force (24) would be a very high-order effect. Instead here, as for the three-nucleon system [14–16], a certain amount of fine-tuning is present: the low-energy scale responsible for the existence of the shallow two-nucleon 1s_0 virtual bound state and the shallow ${}^5\text{He}$ $p_{3/2}$ resonance must appear in the renormalized three-body force as well. The infrared enhancement of the LO two-body interactions dominates the running of the LO three-body force, making its effects much larger than the naturalness expectation. While in the pure s-wave case the enhancement is proportional to the square of the large scattering length [14–16], here it must be roughly the square of the large scattering volume.

With the three-body force so determined, we have looked for other 0^+ bound states and found none within the cutoff range we investigated. This is perhaps not surprising. It has been argued that the Efimov effect [12] is present if both the scattering volume and the effective momentum in a system with pairwise p-wave

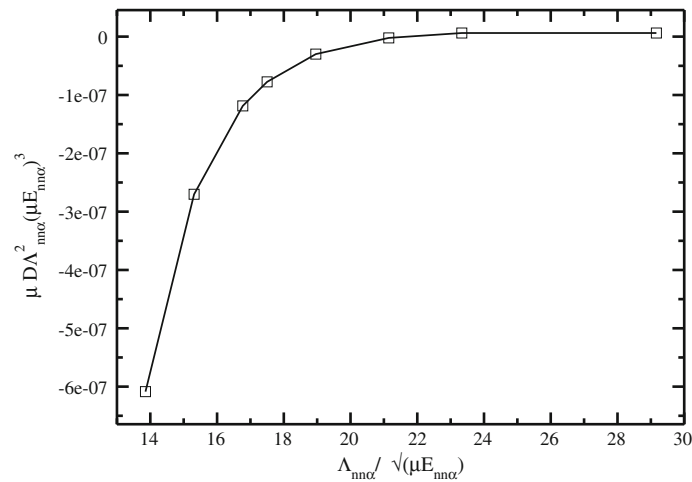


Fig. 5 Dimensionless three-body coupling constant $\mu^4 E_{nn\alpha}^3 \Lambda_{nn\alpha}^2 D$ as function of $\Lambda_{nn\alpha} / \sqrt{\mu E_{nn\alpha}}$, when the ${}^6\text{He}$ ground-state energy $E_{nn\alpha}$ is fixed at its experimental value, for $\Lambda_{nn\alpha} = \Lambda_{nn} = \Lambda_{n\alpha}$

interactions are large, although there is debate about whether this can be realized [19,20]. Since $r_{n\alpha}$ is not particularly large, we would not expect here an Efimov tower of shallow three-body states anyway.

5 Conclusions and Outlook

In this paper we have described for the first time the ground state of ${}^6\text{He}$ using interactions derived from Halo/Cluster Effective Field Theory, where the alpha-particle core is treated as an explicit field [2,3]. The two-body $n\alpha$ and nn interactions are of the contact type, with parameters determined from two-body scattering data. The three-body dynamics of the system was solved using the formalism of the Gamow Shell Model [24,25], where the set of single-particle states (resonant and continuum) is given by the $n\alpha$ potential. We had to adapt the formalism to accommodate the energy dependence of the LO $n\alpha$ EFT potential. This is also the first time the GSM has been applied to the solution of EFT.

We have seen that, at leading order, two-body forces are not sufficient to properly renormalize the three-body system, even though they provide a systematic expansion for two-body scattering [2,3]. Indeed, as the cutoffs are increased the energy of the three-body ground state does not stabilize and would collapse for an arbitrarily large cutoff. The addition of a single three-body counterterm is enough to achieve renormalization-group invariance². We have obtained the RG running of the corresponding coupling constant by demanding that the binding energy be fixed at its experimental value. These results are similar to what happens in systems with s-wave interactions [14–16]. However, because the two-body interactions are different such similarity was not granted a priori, and indeed the results differ in detail. In any case, we have shown here that three-body forces are necessary at LO for consistency of the H/CEFT.

Our work thus provides a unique RG approach to ${}^6\text{He}$ and paves the way for more comprehensive studies of halo nuclei with H/CEFT. For the future, we plan to carry out a more extensive investigation of ${}^6\text{He}$, including higher-order corrections and calculation of other observables (such as the ground-state radius and the first excited-state energy). At the cost of more computational resources, other members of the He isotope family could be investigated as well, along the lines of Refs. [26–29]. More generally, we hope that the combination of EFT and GSM will prove to be a valuable tool in the study of other three-body resonant states, such as the Hoyle state in ${}^{12}\text{C}$.

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² It is our understanding that the same conclusion was reached by Ji et al. [33] (Phillips, D.R., personal communication).

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