Few-Body Syst (2012) 52: 11–18
DOI 10.1007/s00601-011-0258-1

Chun-Sheng Jia · Xiao-Ping Li · Lie-Hui Zhang

Exact Solutions of the Klein–Gordon Equation with Position-Dependent Mass for Mixed Vector and Scalar Kink-Like Potentials

Received: 6 July 2011 / Accepted: 19 September 2011 / Published online: 30 October 2011

Abstract The relativistic problem of spinless particles with position-dependent mass subject to kink-like potentials ($\sim \tanh \alpha x$) is investigated. By using the basic concepts of the supersymmetric quantum mechanics formalism and the functional analysis method, we solve exactly the position-dependent effective mass Klein–Gordon equation with the vector and scalar kink-like potential coupling, and obtain the bound state solutions in the closed form. It is found that in the presence of position-dependent mass there exists the symmetry that the discrete positive energy spectra and negative energy spectra are symmetric about zero energy for the case of a mixed vector and scalar kink-like potential coupling, and in the presence of constant mass this symmetry only appears for the cases of a pure scalar kink-like potential coupling or massless particles.

1 Introduction

The problem of the non-relativistic and relativistic wave equations with spatially dependent masses has been attracting much intention in the literature. Systems with position-dependent mass have been found to be very useful in studying the physical properties of various microstructures, such as semiconductor heterostructure [1], quantum liquids [2], quantum wells and quantum dots [3], ^{3}He clusters [4], compositionally graded crystals [5], etc. The ordering ambiguity of the mass and momentum operators exists in the non-relativistic case [6]. However, it is usually expected that this ordering ambiguity should disappears in the relativistic ambiance. In this regard, some authors investigated the exact solutions of the position-dependent effective mass Dirac equations [7–21] and spatially dependent mass Klein–Gordon equations for various potential models [22–31]. In Ref. [22], the authors investigated the exact solution of the one-dimensional spatially dependent mass Klein–Gordon equation with the inversely linear scalar potential. Dai and Cheng [23] studied the bound state solutions of the Klein–Gordon equation with position-dependent mass for inversely linear scalar and vector potentials that are equal in magnitude. Ikhdair and Sever [24] investigated the solutions of the Klein–Gordon equations with position-dependent mass for the scalar and vector Hulthén potentials and unequal scalar-vector Coulomb-like potentials [25]. Arda et al. [26] studied the solutions of the spatially dependent mass Klein–Gordon equations for the Hulthén potential, modified Woods–Saxon potential [27], q-parameter Pöschl–Teller potential [28] and non-Hermitian generalized Morse potential [29]. With the framework of a D-dimensional spatially dependent mass Klein–Gordon equation, Hassanabadi et al. investigated the behavior of spin-zero particles for a general exponential form of scalar and vector fields [30] and for both Coulomb and Cornell interactions [31].

Recently, de Castro and Hott [32] investigated the relativistic problem of neutral fermions subject to a pseudoscalar kink-like potential ($\sim \tanh \alpha x$). In Ref. [33], de Castro investigated the relativistic problem of

C.-S. Jia (✉) · X.-P. Li · L.-H. Zhang
State Key Laboratory of Oil and Gas Reservoir Geology and Exploitation,
Southwest Petroleum University, Chengdu 610500, People's Republic of China
E-mail: chshjia@263.net

spinless particles subject to a general mixing of vector and scalar kink-like potentials in the case of a constant mass. The parity-conserving pseudoscalar potential is of interest in quantum field theory where a classically stable and a finite localized energy solution of the motion equation can be in topologically stable sectors. Kink models are obtained in quantum field theory as the continuum limit of linear polymer models [34–36]. For this kink-like potential, there exists no bound state in a non-relativistic quantum theory because it gives rise to a ubiquitous repulsive potential. However, bound states of this kink-like potential exist in (1+1)-dimensional Dirac equation with a pseudoscalar potential coupling [32] and in (1+1)-dimensional Klein–Gordon equation with a mixed vector-scalar potential coupling [33]. The PT-symmetric version of the kink-like potential has also been investigated within the framework of the Dirac equation with a vector potential coupling [12,37]. As far as we know, one has not reported the investigation of solving the Klein–Gordon equation with position-dependent mass for the kink-like potential. In the present work, we investigated the relativistic problem of spinless particles subject to the scalar and vector kink-like potentials in the presence of position-dependent mass.

2 Position-Dependent Mass Effective Klein–Gordon Equation with Vector and Scalar Potentials

The one-dimensional time-independent Klein–Gordon equation for a spinless particle coupled to a scalar potential $S(x)$ and a vector potential $V(x)$ reads [22]

$$-\hbar^2c^2\frac{d^2\Psi(x)}{dx^2}+(M(x)c^2+S(x))^2\Psi(x)=(E-V(x))^2\Psi(x), \tag{1}$$

where E is the energy and $M(x)$ denotes the position-dependent effective mass. Equation (1) can also be written as

$$-\frac{d^2\Psi}{dx^2}+\frac{1}{\hbar^2c^2}\left[S^2(x)-V^2(x)+2M(x)c^2S(x)+2EV(x)+M^2(x)c^4\right]\Psi=\frac{E^2}{\hbar^2c^2}\Psi. \tag{2}$$

We consider the one-dimensional kink-like scalar and vector potentials in the forms

$$S(x)=\alpha\beta_s\tanh\alpha x,\quad V(x)=\alpha\beta_v\tanh\alpha x \tag{3}$$

where the skew parameter, α, and the coupling constants, β_s and β_v, are real numbers. We take the mass function $M(x)$ as the smooth step mass [9]

$$M(x)=M_0(1+\eta\tanh\alpha x), \tag{4}$$

where η is a very small non-negative parameter. The mass increases from the value $M=M_0(1-\eta)$ for $x=-\infty$ to the value $M=M_0(1+\eta)$ for $x=+\infty$. The significant variations occur in the range of $-\frac{1}{\alpha}<x<\frac{1}{\alpha}$, i.e., $M(-1/\alpha)\cong M_0(1-0.762\eta)$, $M(1/\alpha)\cong M_0(1+0.762\eta)$. The smooth step mass has been studied by Peng et al. [9] in the Dirac equation with spatially dependent mass for the generalized Hulthén potential. Substituting Eqs. (3) and (4) into Eq. (2), we obtain a Schrödinger-like equation

$$-\frac{d^2\Psi}{dx^2}+V_{\rm eff}(x)\Psi=\tilde{E}\Psi, \tag{5}$$

where the effective potential $V_{\rm eff}(x)$can be recognized as the exactly solvable Rosen–Morse-like well [38,39]. The effective potential $V_{\rm eff}(x)$ and the effective energy $\tilde{E}$ are given by

$$V_{\rm eff}=-V_1\,\mathrm{sech}^2\alpha x+V_2\tanh\alpha x, \tag{6}$$

$$\tilde{E}=\frac{1}{\hbar^2c^2}(E^2-M_0^2c^4(1+\eta^2)+\alpha^2\beta_s^2-\alpha^2\beta_v^2-2M_0\,c^2\alpha\beta_s\eta), \tag{7}$$

where the parameters V_1 and V_2 are defined as

$$V_1=\frac{1}{\hbar^2c^2}(M_0^2c^4\eta^2+\alpha^2\beta_s^2-\alpha^2\beta_v^2+2M_0c^2\alpha\beta_s\eta), \tag{8}$$

$$V_2=\frac{1}{\hbar^2c^2}(2M_0^2c^4\eta^2+2M_0c^2\alpha\beta_s+2\alpha\beta_vE), \tag{9}$$

The Rosen–Morse-like well is a binding potential well only if $V_1 > 0$ and $|V_2| < 2V_1$, and it possesses the possible discrete effective energies in the range of $\tilde{E} \leq -|V_2|$ [33]. We solve Eq. (5) by employing the supersymmetric quantum mechanics approach [40–42]. The supersymmetric approach has been extensively used to solving the Schrödinger equation [42–47], Klein–Gordon [48–50], Dirac equation [51–54] and DKP equation [55] in the presence of a constant mass. By making the proper replacements of the parameters in the energy spectrum expression and eigenfunctions of the corresponding Rosen–Morse problem reported in Ref. [42], one can easily obtain solutions of the Rosen–Morse-like problem expressed in Schrödinger-like equation (5). However, considering that Schrödinger-like equation (5) is reduced from the spatially dependent mass Klein–Gordon equation (1), and not from a Klein–Gordon equation with a constant mass, we solve Eq. (5) in terms of the supersymmetric approach step by step. Writing the ground-state wave function $\Psi_0(x)$ in the form of $\Psi_0(x) = \exp(-\int W(x)dx)$ and substituting it into Eq. (5), we arrive at the following non-linear Riccati equation for $W(x)$,

$$W^2(x) - \frac{dW(x)}{dx} = -V_1 \text{sech}^2\alpha x + V_2 \tanh \alpha x - \tilde{E}_0, \tag{10}$$

where $\tilde{E}_0$ is the effective ground-state energy, and $W(x)$ is a superpotential. Letting the superpotential $W(x)$ as

$$W(x) = A + B \tanh \alpha x, \tag{11}$$

and substituting it into the expression $\Psi_0(x) = \exp(-\int W(x)dx)$, one gets

$$\Psi_0(x) = e^{-Ax}(\cosh \alpha x)^{-B/\alpha}. \tag{12}$$

Considering the bound state boundary conditions, $\Psi_0(\pm\infty) = 0$, we obtain the restriction conditions: $B/\alpha > 0$ and $|A| < B$. Substituting Eq. (11) into Eq. (10) and comparing equal powers of two sides in Eq. (10), we get a set of equations

$$A^2 + B^2 = -\tilde{E}_0, \quad B^2 + \alpha B = V_1, \quad 2AB = V_2. \tag{13}$$

Solving these equations, we obtain

$$A = \frac{V_2}{2B}, \quad B = \frac{\alpha}{2}\left(-1 + \sqrt{1 + \frac{4V_1}{\alpha^2}}\right), \quad \tilde{E}_0 = -\left(\frac{V_2^2}{4B^2} + B^2\right). \tag{14}$$

In terms of the superpotential $W(x)$ given in Eq. (11) and the expression $A = \frac{V_2}{2B}$, one can construct the following two supersymmetric partner potentials

$$V_{\text{eff}+}(x) = W^2(x) + \frac{dW(x)}{dx} = -(B^2 - \alpha B)\text{sech}^2\alpha x + V_2 \tanh \alpha x + \frac{V_2^2}{4B^2} + B^2, \tag{15}$$

$$V_{\text{eff}-}(x) = W^2(x) - \frac{dW(x)}{dx} = -(B^2 + \alpha B)\text{sech}^2\alpha x + V_2 \tanh \alpha x + \frac{V_2^2}{4B^2} + B^2, \tag{16}$$

These two supersymmetric partner potentials satisfy the following relationship

$$V_{\text{eff}+}(x, a_0) = V_{\text{eff}-}(x, a_1) + R(a_1), \tag{17}$$

where $a_0 = B$, a_1 is a function of a_0, i.e., $a_1 = f(a_0) = a_0 - \alpha$, and the remainder $R(a_1)$ is independent of x, $R(a_1) = \left(\frac{V_2^2}{4a_0^2} + a_0^2\right) - \left(\frac{V_2^2}{4a_1^2} + a_1^2\right)$. Equation (17) tells us that the two partner potentials $V_{\text{eff}+}(x)$ and $V_{\text{eff}-}(x)$ possess similar shapes and they are shape-invariant in the senses of Ref. [41]. For the shape-invariant-like potential $V_{\text{eff}-}(x)$, we use the shape invariance approach [41] to determine the exact energy spectra, which are given by

$$\tilde{E}_0^{(-)} = 0, \tag{18}$$

$$\begin{aligned}\tilde{E}_n^{(-)} &= \sum_{k=1}^{n} R(a_k) = R(a_1) + R(a_2) + \cdots + R(a_n)\\ &= \left(\frac{V_2^2}{4a_0^2}+a_0^2\right)-\left(\frac{V_2^2}{4a_1^2}+a_1^2\right)+\left(\frac{V_2^2}{4a_1^2}+a_1^2\right)-\left(\frac{V_2^2}{4a_2^2}+a_2^2\right)\cdots+\left(\frac{V_2^2}{4a_{n-1}^2}+a_{n-1}^2\right)-\left(\frac{V_2^2}{4a_n^2}+a_n^2\right)\\ &= \left(\frac{V_2^2}{4a_0^2}+a_0^2\right)-\left(\frac{V_2^2}{4a_n^2}+a_n^2\right)=\left(\frac{V_2^2}{4B^2}+B^2\right)-\left(\frac{V_2^2}{4a_n^2}+a_n^2\right),\end{aligned} \tag{19}$$

where the quantum number $n = 0, 1, 2, \ldots$. Combining Eqs. (10) and (16) and using Eq. (14), we have the following relation

$$V_{\mathrm{eff}}(x) = -V_1 \mathrm{sech}^2\alpha x + V_2 \tanh \alpha x = V_{\mathrm{eff}-}(x) + \tilde{E}_0. \tag{20}$$

It is obvious that the effective energy $\tilde{E}$ in Eq. (5) can be written as

$$\tilde{E} = \tilde{E}_n^{(-)} + \tilde{E}_0 = -\left(\frac{V_2^2}{4a_n^2} + a_n^2\right). \tag{21}$$

Inserting the expression $\tilde{E} = \frac{1}{\hbar^2 c^2}(E^2 - M_0^2 c^4(1+\eta^2) + \alpha^2\beta_s^2 - \alpha^2\beta_v^2 - 2M_0c^2\alpha\beta_s\eta)$ into Eq. (21) and using expression (9), we obtain a second-degree algebraic equation for the Klein–Gordon energies

$$\left(1 + \frac{\alpha^2\beta_v^2}{\hbar^2c^2a_n^2}\right)E_n^2 + \frac{\alpha\beta_v\delta}{\hbar^2c^2a_n^2}E_n + \frac{\delta^2}{4\hbar^2c^2a_n^2} + \hbar^2c^2a_n^2 - \varepsilon = 0, \tag{22}$$

where $\delta = 2(M_0^2c^4\eta + M_0c^2\alpha\beta_s)$ and $\varepsilon = (1+\eta^2)M_0^2c^4 + 2M_0c^2\alpha\beta_s\eta + \alpha^2\beta_s^2 - \alpha^2\beta_v^2$. By solving Eq. (22), we find the following relativistic energy spectra for a spinless particle in the context of the spatially dependent mass Klein–Gordon equation with the vector and scalar kink-like potentials,

$$E_n = -\frac{\alpha\beta_v\delta}{2(\hbar^2c^2a_n^2 + \alpha^2\beta_v^2)} \pm \frac{\sqrt{\alpha^2\beta_v^2\delta^2 - 4(\hbar^2c^2a_n^2+\alpha^2\beta_v^2)(\delta^2/4 + \hbar^4c^4a_n^4 - \hbar^2c^2a_n^2\varepsilon)}}{2(\hbar^2c^2a_n^2+\alpha^2\beta_v^2)}. \tag{23}$$

Substituting Eq. (21) into Eq. (5), we obtain the following equation

$$-\frac{d^2\Psi(x)}{dx^2} + (-V_1\mathrm{sech}^2\alpha x + V_2\tanh\alpha x)\Psi(x) = -\left(\frac{V_2^2}{4a_n^2} + a_n^2\right)\Psi(x). \tag{24}$$

Introducing the new variable $z = -\tanh\alpha x$ and writing the wave function $\Psi(x)$ as $\Psi(x) = (\frac{1-z}{2})^{-p}(\frac{1+z}{2})^{-w}$ $P(z)$, Eq. (24) can be reduced to the following equation satisfied by $P(z)$,

$$(1-z^2)\frac{d^2P}{dz^2} + [-2w + 2p - (2 - 2p - 2w)z]\frac{dP}{dz} + n(n - 2p - 2w + 1)P = 0, \tag{25}$$

where p and w are defined as $p = \frac{V_2}{2\alpha a_n} - \frac{a_n}{\alpha}$ and $w = -\frac{V_2}{2\alpha a_n} - \frac{a_n}{\alpha}$, respectively. Equation (25) is the well-known differential equation satisfied by the Jacobi polynomials $P_n^{-2p,-2w}(z)$. Thus, the wave function $\Psi(x)$ can be written as,

$$\Psi_n(x) = \left(\frac{1+\tanh\alpha x}{2}\right)^{-p}\left(\frac{1-\tanh\alpha x}{2}\right)^{-w} P_n^{-2p,-2w}(-\tanh\alpha x). \tag{26}$$

In order to compare our results with those reported by de Castro in Ref. [33], we write β_s and β_v in the forms [33]

$$\beta_s = \hbar cg, \quad \beta_v = \hbar cg\sin\left(\frac{\pi\xi}{2}\right), \tag{27}$$

where the variable ξ is defined into the interval $(-1, 1)$. Making the replacements of $\delta = 2(M_0^2 c^4 \eta + M_0 c^2 \alpha \beta_s)$, $\varepsilon = (1+\eta^2) M_0^2 c^4 + 2M_0 c^2 \alpha \beta_s \eta + \alpha^2 \beta_s^2 - \alpha^2 \beta_v^2$ and $a_n = B - n\alpha = \frac{\alpha}{2}\left(-1 + \sqrt{1 + \frac{4V_1}{\alpha^2}}\right) - n\alpha$ in Eq. (23), and employing the expressions for V_1, V_2, β_s and β_v given in Eqs. (7), (8) and (27), the Klein–Gordon energy formula (23) can be written in the form of

$$E_n = -\frac{M_0 c^2 \left(\frac{M_0 c}{\hbar\alpha}\eta + g\right) g \sin\left(\frac{\pi\xi}{2}\right)}{\tilde{a}_n^2 + g^2 \sin^2\left(\frac{\pi\xi}{2}\right)} \pm \frac{c\sqrt{\left(M_0 c \left(\frac{M_0 c}{\hbar\alpha}\eta + g\right) g \sin\left(\frac{\pi\xi}{2}\right)\right)^2 - \left(\tilde{a}_n^2 + g^2 \sin^2\left(\frac{\pi\xi}{2}\right)\right)\left(M_0^2 c^2 \left(\frac{M_0 c}{\hbar\alpha}\eta + g\right)^2 + \hbar^2\alpha^2\tilde{a}_n^4 - \tilde{a}_n^2\tilde{\varepsilon}\right)}}{\tilde{a}_n^2 + g^2 \sin^2\left(\frac{\pi\xi}{2}\right)} \tag{28}$$

where $\tilde{a}_n$ and $\tilde{\varepsilon}$ are defined as

$$\tilde{a}_n = \alpha\left[-n - \frac{1}{2} + \frac{1}{2}\sqrt{1 + \frac{4}{\hbar^2 c^2 \alpha^2}\left(M_0^2 c^4 \eta^2 + 2M_0 c^3 \hbar\alpha g\eta + \hbar^2 c^2 \alpha^2 g^2 \cos^2\left(\frac{\pi\xi}{2}\right)\right)}\right], \tag{29}$$

$$\tilde{\varepsilon} = (1+\eta^2) M_0^2 c^2 + 2M_0 \hbar c\alpha g\eta + \hbar^2 \alpha^2 g^2 \cos^2\left(\frac{\pi\xi}{2}\right). \tag{30}$$

For the bound states, we can obtain the constraint condition satisfied by the quantum number n from $\tilde{E} \le -|V_2|$ and in terms of Eqs. (21), (29) and (30),

$$n = 0, 1, 2, \ldots \le \frac{1}{2}\left(-1 + \sqrt{1 + \frac{4}{\hbar^2 c^2 \alpha^2}\left(c^2\tilde{\varepsilon} - M_0^2 c^4 \eta^2\right)}\right) - \frac{\sqrt{|V_2|}}{2\sqrt{\alpha}}. \tag{31}$$

In the case of constant mass, $\eta = 0$, expression (28) becomes into

$$E_n = -\frac{M_0 c^2 g^2 \sin\left(\frac{\pi\xi}{2}\right)}{\tilde{a}_n^2 + g^2 \sin^2\left(\frac{\pi\xi}{2}\right)} \pm \frac{c a_n \sqrt{M_0^2 c^2 \left(\tilde{a}_n^2 - g^2 \cos^2\left(\frac{\pi\xi}{2}\right)\right) + \hbar^2\alpha^2\left(\frac{g^2}{4}\sin^2(\pi\xi) + \tilde{a}_n^2 g^2 \cos(\pi\xi) - \tilde{a}_n^4\right)}}{\tilde{a}_n^2 + g^2 \sin^2\left(\frac{\pi\xi}{2}\right)}. \tag{32}$$

This is just consistent with expression (17) of Ref. [33].

From Eq. (28), we can see that when $M_0 = 0$ (massless particle), the discrete positive energy spectra and negative energy spectra are symmetric about $E_n = 0$. This symmetry also happens for the case of a pure scalar kink-like potential coupling ($\xi = 0$). When $g = -\frac{M_0 c}{\hbar\alpha}\eta$ and $\xi \ne 0$, the same symmetry occurs for a mixed scalar and vector kink-like potential coupling. However, in the presence of a constant mass, this symmetry does not occur for a mixed scalar and vector coupling [33]. When $g = -\frac{M_0 c}{\hbar\alpha}\eta$, the Klein–Gordon eigenenergies are plotted in Fig. 1 for the two lowest bound states as a function of ξ. The parameter are chosen for furnishing the two bounded solutions, which are taken as $\hbar = c = 1$, $M_0 = 3$, $\alpha = 1/2$, and $\eta = 0.15$. Figure 1 shows that the positive energy levels and negative energy levels are symmetric about $E_n = 0$.

3 Conclusion

In this work we have investigated the relativistic problem of spinless particles with position-dependent mass subject to kink-like potentials. The position-dependent effective mass Klein–Gordon equation with a mixed

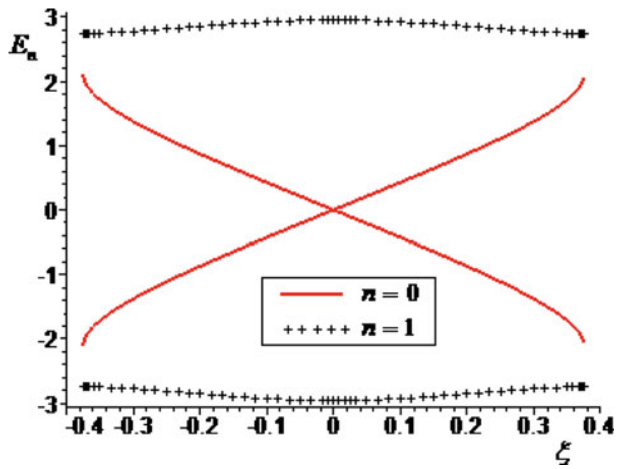

Fig. 1 A plot of the Klein-Gordon energy levels for the kink-like potential as a function of ξ in the two cases: $n = 0$ (red solid line) and $n = 1$ (black cross line)

vector and scalar kink-like potential coupling can been solved exactly by using the basic concepts of the supersymmetric quantum mechanics formalism and the functional analysis method. We give the exact bound state solutions in the closed form. The kink-like potential is absent of bound states in the context of the non-relativistic Schrödinger equation with a constant mass, but it possesses discrete relativistic energy spectra in the context of the Klein–Gordon theory with the vector and scalar coupling scheme in the presence of position-dependent mass. Under the condition of constant mass, there exists the symmetry that the discrete positive energy spectra and negative energy spectra are symmetric about zero energy only for the cases of a pure scalar coupling or massless particles. However, in the presence of position-dependent mass, this symmetry also appears for the case of a mixed vector and scalar kink-like potential coupling.

Acknowledgments We would like to thank the kind referees for their helpful comments and suggestions which have greatly improved the manuscript. This work was supported by the National Natural Science Foundation of China under Grant No. 10675097, the Doctorate Foundation of the State Education Ministry of China under Grant No. 20040615004, the Science Foundation of Southwest Petroleum University under Grant No. 2010XJZ198 and the Science Foundation of State Key Laboratory of Oil and Gas Reservoir Geology and Exploitation of China under Grant No. PLN-ZL001.

References

1. Von Roos, O.: Position-dependent effective masses in semiconductor theory. Phys. Rev. B **27**, 7547 (1983)
2. Arias de Saavedra, F., Boronat, F., Polls, A., Fabrocini, A.: Effective mass of one ^{4}He atom in liquid ^{3}He. Phys. Rev. B **50**, 4248 (1994)
3. Serra, L., Lipparini, E.: Spin response of unpolarized quantum dots. Europhys. Lett. **40**, 667 (1997)
4. Barranco, M., Pi, M., Gatica, S.M., Hernandez, E.S., Navarro, J.: Structure and energetics of mixed ^{4}He–^{3}He drops. Phys. Rev. B **56**, 8997 (1997)
5. Geller, M.R., Kohn, W.: Quantum mechanics of electrons in crystals with graded composition. Phys. Rev. Lett. **70**, 3103 (1993)
6. De Souza Dutra, A., Almeida, C.A.S.: Exact solvability of potentials with spatially dependent effective masses. Phys. Lett. A **275**, 25 (2000)
7. Alhaidari, A.D.: Solution of the Dirac equation with position-dependent mass in the Coulomb field. Phys. Lett. A **322**, 72 (2004)
8. Vakarchuk, I.O.: The Kepler problem in Dirac theory for a particle with position-dependent mass. J. Phys. A Math. Gen. **38**, 4727 (2005)
9. Peng, X.L., Liu, J.Y., Jia, C.S.: Approximation solution of the Dirac equation with position-dependent mass for the generalized Hulthén potential. Phys. Lett. A **352**, 478 (2006)
10. Jia, C.S., de Souza Dutra, A.: Position-dependent effective mass Dirac equations with PT-symmetric and non-PT-symmetric potentials. J. Phys. A Math. Gen. **39**, 11877 (2006)
11. Jia, C.S., Liu, J.Y., Wang, P.Q., Che, C.S.: Trapping neutral fermions with a PT-symmetric trigonometric potential in the presence of position-dependent mass. Phys. Lett. A **369**, 274 (2007)
12. Jia, C.S., de Souza Dutra, A.: Extension of PT-symmetric quantum mechanics to the Dirac theory with position-dependent mass. Ann. Phys. (N. Y.) **323**, 566 (2008)

13. Jia, C.S., Wang, P.Q., Liu, J.Y., He, S.: Relativistic confinement of neutral fermions with partially exactly solvable and exactly solvable PT-symmetric potentials in the presence of position-dependent mass. Int. J. Theor. Phys. **47**, 2513 (2008)
14. Mustafa, O., Mazharimousavi, S.H.: (1+1)-Dirac particle with position-dependent mass in complexified Lorentz scalar interactions: effectively $\mathcal{PT}$-symmetric. Int. J. Theor. Phys. **47**, 1112 (2008)
15. Mustafa, O.: Energy-levels crossing and radial Dirac equation: supersymmetry and quasi-parity spectral signatures. Int. J. Theor. Phys. **47**, 1300 (2008)
16. Arda, A., Sever, R., Tezcan, C., Akçay, H.: Analytical solutions to the Klein-Gordon equation with position-dependent mass for q-parameter Pöschl-Teller potential. Chin. Phys. Lett. **27**, 040306 (2010)
17. Ikhdair, S.M., Sever, R.: Solutions of the spatially-dependent mass Dirac equation with the spin and pseudo-spin symmetry for the Coulomb-like potential. Appl. Math. Comput. **216**, 545 (2010)
18. Longhi, S.: Optical realization of relativistic non-Hermitian quantum mechanics. Phys. Rev. Lett. **105**, 013903 (2010)
19. Hamzavi, M., Rajabi, A.A., Hassanabadi, H.: Exact pseudospin symmetry solution of the Dirac equation for spatially-dependent mass Coulomb potential including a Coulomb-like tensor interaction via asymptotic iteration method. Phys. Lett. A **374**, 4303 (2010)
20. Panahi, H., Bakhshi, Z.: Dirac equation with position-dependent effective mass and solvable potentials in the Schrödinger equation. J. Phys. A Math. Theor. **44**, 175304 (2011)
21. Eshghi, M., Mehraban, H.: Solution of the Dirac equation with position-dependent mass for q-parameter modified Pöschl-Teller and Coulomb-like tensor potential. Few-Body Syst. (2011). doi:10.1007/s00601-011-0238-5
22. de Souza Dutra, A., Jia, C.S.: Classes of exact Klein–Gordon equations with spatially dependent masses: regularizing the one-dimensional inversely linear potential. Phys. Lett. **352**, 484 (2006)
23. Dai, T.Q., Cheng, Y.F.: Bound state solutions of the Klein-Gordon equation with position-dependent mass for the inversely linear potential. Phys. Scr. **79**, 015007 (2009)
24. Ikhdair, S.M., Sever, R.: Any l-state improved quasi-exact analytical solutions of the spatially dependent mass Klein-Gordon equation for the scalar and vector Hulthén potentials. Phys. Scr. **79**, 035002 (2009)
25. Ikhdair, S.M.: Exact Klein-Gordon equation with spatially dependent masses for unequal scalar-vector Coulomb-like potentials. Eur. Phys. J. A **40**, 143 (2009)
26. Arda, A., Sever, R., Tezcan, C.: Approximate analytical solutions of the Klein-Gordon equation for the Hulthén potential with the position-dependent mass. Phys. Scr. **79**, 015006 (2009)
27. Arda, A., Sever, R.: Approximate l-state solutions to the Klein-Gordon equation for modified Woods-Saxon potential with position dependent mass. Int. J. Mod. Phys. A **24**, 3985 (2009)
28. Arda, A., Sever, R., Tezcan, C.: Analytical solutions to the Klein-Gordon equation with position-dependent mass for q-parameter Pöschl-Teller potential. Chin. Phys. Lett. **27**, 010306 (2010)
29. Arda, A., Sever, R.: Effective-mass Klein-Gordon equation for non-PT/non-Hermitian generalized Morse potential. Phys. Scr. **82**, 065007 (2010)
30. Hassanabadi, H., Zarrinkamar, S., Hamzavi, M., Rajabi, A.A.: Relativistic spinless bosons in exponential fields. Few-Body Syst. **51**, 69 (2011)
31. Hassanabadi, H., Rahimov, H., Zarrinkamar, S.: Cornell and Coulomb interactions for the D-dimensional Klein-Gordon equation. Ann. Phys. (Berlin) **523**, 566 (2011)
32. De Castro, A.S., Hott, M.: Trapping neutral fermions with kink-like potentials. Phys. Lett. A **351**, 379 (2006)
33. De Castro, A.S.: Effects of a mixed vector-scalar kink-like potential for spinless particles in two-dimensional space-time. Int. J. Mod. Phys. A **22**, 2609 (2007)
34. Goldstone, J., Wilczek, F.: Fractional quantum numbers on solitons. Phys. Rev. Lett. **47**, 986 (1981)
35. Jackiw, J., Semenoff, G.: Continuum quantum field theory for a linearly conjugated diatomic polymer with fermion fractionization. Phys. Rev. Lett. **50**, 439 (1983)
36. Niemi, A.J., Semenoff, G.: Fermion number fractionization in quantum field theory. Phys. Rep. **135**, 99 (1986)
37. Jia, C.S., Diao, Y.F., Liu, J.Y.: Bounded solutions of the Dirac equation with a PT-symmetric kink-like vector potential in two-dimensional space-time. Int. J. Theor. Phys. **47**, 664 (2008)
38. Rosen, N., Morse, P.M.: On the vibrations of polyatomic molecules. Phys. Rev. **42**, 210 (1932)
39. Nieto, M.M.: Exact wave-function normalization constants for the B_0z-$U_0{}^{-2}z$ and Pöschl-Teller potentials. Phys. Rev. A **17**, 1273 (1978)
40. Witten, E.: Dynamical breaking of supersymmetry. Nucl. Phys. B **185**, 513 (1981)
41. Gendenshtein, L.E.: Derivation of exact spectra of the Schrödinger equation by means of supersymmetry. Sov. Phys. JETP Lett. **38**, 356 (1983)
42. Cooper, F., Khare, A., Sukhatme, U.: Supersymmetry and quantum mechanics. Phys. Rep. **251**, 267 (1995)
43. Kostelecky, V.A., Nieto, M.M.: Evidence for a phenomenological supersymmetry in atomic physics. Phys. Rev. Lett. **53**, 2285 (1984)
44. Kostelecky, V.A., Nieto, M.M.: Evidence from alkali-metal-atom transition probabilities for a phenomenological atomic supersymmetry. Phys. Rev. A **32**, 1293 (1985)
45. Kostelecky, V.A., Nieto, M.M., Truax, D.R.: Supersymmetry and the relationship between the Coulomb and oscillator problems in arbitrary dimensions. Phys. Rev. D **32**, 2627 (1985)
46. Lahiri, A., Roy, P.K., Bagchi, B.: Supersymmetry in atomic physics and the radial problem. J. Phys. A Math. Gen. **20**, 3825 (1987)
47. Lahiri, A., Roy, P.K., Bagchi, B.: Supersymmetry and the three-dimensional isotropic oscillator problem. J. Phys. A Math. Gen. **20**, 5403 (1987)
48. Chen, T., Diao, Y.F., Jia, C.S.: Bound state solutions of the Klein-Gordon equation with the generalized Pöschl-Teller potential. Phys. Scr. **79**, 065014 (2009)
49. Xu, Y., He, S., Jia, C.S.: Approximate analytical solutions of the Klein-Gordon equation with the Pöschl-Teller potential including the centrifugal term. Phys. Scr. **81**, 045001 (2010)

50. Setare, M.R., Hatami, O.: Algebraic approach for shape invariant potentials in Klein-Gordon equation. Int. J. Theor. Phys. **48**, 2977 (2009)
51. Zarrinkamar, S., Rajabi, A.A., Hassanabadi, H.: Dirac equation in the presence of Coulomb and linear terms in (1 + 1) dimensions: the supersymmetric approach. Ann. Phys. (N. Y.) **325**, 1720 (2010)
52. Zarrinkamar, S., Rajabi, A.A., Hassanabadi, H.: Dirac equation for the harmonic scalar and vector potentials and linear plus Coulomb-like tensor potential: the SUSY approach. Ann. Phys. (N. Y.) **325**, 2522 (2010)
53. Zarrinkamar, S., Hassanabadi, H., Rajabi, A.A.: Dirac equation for a Coulomb scalar, vector and tensor interaction. Int. J. Mod. Phys. A **26**, 1011 (2011)
54. Zarrinkamar, S., Rajabi, A.A., Hassanabadi, H.: Supersymmetric study of the pseudospin symmetry limit of the Dirac equation for a pseudoharmonic potential. Phys. Scr. **83**, 015009 (2011)
55. Zarrinkamar, S., Rajabi, A.A., Rahimov, H., Hassanabadi, H.: DKP equation under a vector Hulthén-type potential: an approximate solution. Mod. Phys. Lett. A **26**, 1621 (2011)