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Projecting the Bethe–Salpeter Equation onto the Light-Front and Back: A Short Review

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Abstract The technique of projecting the four-dimensional two-body Bethe–Salpeter equation onto the three-dimensional Light-Front hypersurface, combined with the quasi-potential approach, is briefly illustrated, by placing a particular emphasis on the relation between the projection method and the effective dynamics of the valence component of the Light-Front wave function. Some details on how to construct the Fock expansion of both (a) the Light-Front effective interaction and (b) the electromagnetic current operator, satisfying the proper Ward–Takahashi identity, will be presented, addressing the relevance of the Fock content in the operators living onto the Light-Front hypersurface. Finally, the generalization of the formalism to the three-particle case will be outlined.

1 Introduction

In his 1949 seminal paper [1], P.A.M. Dirac proposed three peculiar representations of the Poincaré group, in strict relation with the choice of possible space-time hypersurfaces without a time-like direction, (see also, e.g., [2]): the Instant form, the Light-Front (LF) form and the Point form (cf [3] for a recent review). Each hypersurface leads to a specific, but equivalent, description of the dynamics of a relativistic interacting system. As a matter of fact, the choice of an initial hypersurface, in which the points are separated by space-like distances and therefore no causal connections are allowed, suggests the set of dynamically independent variables, suited for implementing a description of the initial state of an interacting system. In conclusion, Dirac indicated the potentially fruitful role of the Hamiltonian approach for relativistically describing interacting systems, within a field theoretical framework, as well (see, e.g., [2]).

In the Instant form, corresponding to the choice of a constant-time hypersurface in the Minkowski space ($x_0 = 0$), translations and rotations, being the generators in the stability set of the chosen hypersurface, commute with the Hamiltonian. The eigenstates of the Hamiltonian, constructed in terms of the Fock-space basis (in this basis the free Hamiltonian is diagonal), have both the eigenvalues of the total three-momentum and

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the total angular moment as good quantum numbers. Translations and rotations do not mix different Fock-components of the wave function and therefore a truncation in the Fock-space, that is required for practical applications, is stable under those transformations. Differently, the Instant-form boosts have dynamical nature and therefore they mix the Fock components. The shortcoming of a truncated Instant-form Fock basis is made manifest by the lack of Lorentz invariance, e.g. when we evaluate the expectation value of observables involving initial and final rest-frames, having a relative non zero velocity.

In the LF form, where the hypersurface $x^+ = t + z = 0$ is chosen for quantizing the theory, one has a kinematical subgroup of the Poincaré group, built by seven generators (see e.g. [2,4–6]). Such operators, keeping invariant the LF hypersurface, do not contain the interaction and they correspond to (a) three LF translations, (b) a longitudinal LF boost along the z-direction, (c) a rotation around z, and (d) two transverse LF boosts, suitable linear combinations of transverse Instant-form boosts and rotations. The remaining three generators have a dynamical nature: (a) two LF transverse rotations, like the LF boosts but with different sign combinations, and (b) the Hamiltonian $P^- = P^0 + P^3$, i.e., the generator of LF-time x^+ translations. From this classification, immediately an important feature stems out: the stability under LF boosts of any truncated Fock expansion of the physical states (eigenstates of the squared mass Fock-space operator, see, e.g., [6]), since the LF boosts are diagonal in the Fock space [7]. This property is fundamental for a consistent treatment of the boosts when a truncated Fock basis is adopted. For example, if one calculates electromagnetic form factors for momentum transfers along the z-direction ($q^+ > 0$ and $\mathbf{q}_\perp = 0$ [8]), the initial and final rest-frames are related simply by a kinematical transformation, and therefore one could use a truncated description of the initial and final states. As a last remark, it is worth noting that the change to LF variables allows one to linearize the dependence upon the dynamical variable (conjugated to the LF time), with a simplification in the analysis of the poles in the propagators (i.e., $k_0^2 - (m^2 + |\mathbf{k}|^2) + i\epsilon \to k^+ [k^- - (m^2 + |\mathbf{k}_\perp|^2)/k^+] + i\varepsilon$).

The physical outcome of a quantum theory does not depend on the particular space-time hypersurface adopted for quantizing, and therefore the theory has to have an equivalent four-dimensional (4D) formulation, where explicit covariance is built in. In this respect, the Bethe–Salpeter equation (BSE) is a 4D field-theoretical tool [9] to explore the non perturbative physics of bound and scattering states of few-particle systems. Within a LF framework, it can be easily shown that the projection onto the LF hyperplane of the BS amplitude is proportional to the *valence component* of the physical state (see, e.g., [5]). This suggests to take the BSE as the starting point for an investigation of the internal dynamics, alternative to the one where a coupled-equation system, generated by the Hamiltonian acting on the full Fock space, is considered. In order to pave the way from the BSE to the eigenequation for the 3D valence component and back, another important suggestion is offered by the idea of "iterated resolvents" of Refs. [10,11], where it is proposed that the complexity of the Fock-space LF Hamiltonian builds an effective dynamics for the 3D valence state. Indeed, it should be pointed out that the mass operator for the 3D valence component can be *exactly* obtained from the BSE by a quasi-potential (QP) expansion [12], as investigated in Refs. [13–15] (where the associated set of coupled equations for LF Green's functions was derived).

The QP approach makes feasible to single out the "trivial" global propagation of the interacting system by means of a well-defined auxiliary Green's function (see Eqs. (9) and (13), below). This fundamental goal, in turn, allows one to reconstruct the BS amplitude from the 3D valence component, with a one-to-one correspondence, and moreover, to obtain the expansion of the 3D effective interaction, entering the eigenequation for the valence component. Each contribution to the expansion immediately acquaints a transparent physical interpretation (given the strong analogy between the LF evolution and the non relativistic one) that produces a straightforward guidance in evaluating the relative importance of the associated diagrams. The same analysis seems more involved in the studies of the Hamiltonian in a truncated Fock-space, i.e. in the analysis of the generated system of coupled equations. Noteworthy, the convergence of the expansion, that has a direct impact on the lost of covariance with respect to the subset of the dynamical transformations, has been investigated in a simple bosonic Yukawa model in Ref. [13], where the Fock content of each contribution has been recognized as the ordering "parameter". In closing, the QP technique appears so appealing within the LF framework, that it could be very interesting to implement comparisons between actual calculations performed in other relativistic QP approaches, like the covariant spectator theory (see e.g. [24]) applied in many relevant few-nucleon problems.

The LF projection of the BSE for few-particle systems, and the consequent truncation of the physical state in the Fock space, can be useful if there is a dominant valence state, or if the normalization can be saturated at large extent by including only few Fock components. In Nuclear Physics, the nucleonic component is largely dominant, as in the deuteron case, while in applications to hadrons, Fock components beyond the valence one have been recognized to be relevant, in particular in the description of physical quantities pertaining to inelastic channels, like structure functions and generalized parton distributions (from deeply virtual Compton scattering [16]). Recently, a signature of the the relevance of the components beyond the valence one has been singled out in the evaluation of the nucleon electromagnetic form factors by using constituent quark degrees of freedom in the $\mathbf{q}_\perp = 0$ frame [8]. As a matter of fact, a zero in the proton electric form factor has been associated with a cancellation between valence and nonvalence contributions to the electromagnetic current (see e.g. [17,18] for the experimental overview).

Aim of the present review is to yield some insights on (a) the technique for projecting the 4D BSE onto the three-dimensional (3D) LF hypersurface using the quasi-potential approach and (b) the construction of the effective dynamics of the valence component of the LF wave function for both two- [13,14] and three-particle systems [19,20]. Moreover, some details on how to obtain an effective electromagnetic current operator (a) acting on the valence component of composite two-boson and two-fermion systems, and (b) fulfilling the Ward–Takahashi identity (WTI) [21–23], are given.

The review is organized as follows. In Sect. 2 the QP reduction is presented as a tool to eliminate the relative LF-time. In Sect. 3, the relation one-to-one between the BS amplitude and the LF valence wave-function is illustrated. In Sect. 4, a hierarchy of coupled equations for LF Green's functions is discussed in order to show the Fock content of the projection of the BS equation onto the LF hyperplane. In Sect. 5 the projection technique for three-body BS equations is briefly discussed, along with the role of induced three-body forces. In Sect. 6, the electromagnetic current operator, acting on the valence component and fulfilling the LF Ward–Takahashi identity is introduced, with some remarks on both structure and covariance of the truncated (in the Fock space) current. In Sect. 7, a summary and some perspectives are presented.

2 The Quasi-Potential Reduction: A Tool for Eliminating the Relative LF-Time

The 4D BSE (see, e.g. [9]) is the starting point of many studies of few-body systems which aim to account for relativity. The BSE requires a relativistic field-theoretical approach, based on an interacting Lagrangian able to model the system under investigation. In the particular case of a two-body scattering, the 4D scattering equation for the transition matrix, $T(K)$, with total four-momentum K is written as follows

$$T(K) = V(K) + V(K)\mathcal{G}_0(K)T(K)\ , \tag{1}$$

where the interaction $V(K)$ contains, in principle, all the possible two-body irreducible diagrams. The two-body disconnected Green's function, $\mathcal{G}_0(K)$, should include self-energy terms, but they are neglected in the approach developed so far. Therefore, in the case of two bosons, $\mathcal{G}_0(K)$ becomes

$$G_0(K) = \frac{\imath^2}{2\pi}\frac{1}{\widehat{k}_1^2 - m_1^2 + i\varepsilon}\frac{1}{\widehat{k}_2^2 - m_2^2 + i\varepsilon}, \tag{2}$$

where $\widehat{k}_i^\mu$ is the four-momentum operator and the factor 2π is introduced for convenience. For two fermions one has $\mathcal{G}_0(K) \to G_0^F(K) = \left(\widehat{\not k}_1 + m_1\right)\left(\widehat{\not k}_2 + m_2\right) G_0(K)$.

The two-particle bound-state with total 4-momentum K_B, and $K_B^2 = M_B^2$, corresponds to a T-matrix pole. The residue is associated with the vertex function, namely the nontrivial part of the BS amplitude, Ψ_B, solution of the following homogeneous equation

$$|\Psi_B\rangle = \mathcal{G}_0(K_B)V(K_B)\,|\Psi_B\rangle\,. \tag{3}$$

The normalization condition [9] has to be satisfied in order to fully determine $|\Psi_B\rangle$. For scattering states with total 4-momentum K, the BS amplitude is a solution of the following inhomogeneous equation

$$\left|\Psi^+\right\rangle = |\Psi_0\rangle + \mathcal{G}_0(K)V(K)\left|\Psi^+\right\rangle\ , \tag{4}$$

while the T-matrix, solution of Eq. (1), corresponds to the connected four-point function which brings information on both the scattering and bound states. In Eqs. (3) and (4) the four-momentum conserving δ-function is factorized out.

Since in the LF projection method a central role is played by the on-minus-shell propagation, as it will be emphasized below, let us write the relevant matrix elements of the free two-boson Green's function, viz

$$\langle k_1'^{-} | \, G_0(K) \, | k_1^- \rangle = \frac{\imath^2}{2\pi} \frac{\delta\left(k_1'^{-} - k_1^-\right)}{\widehat{k}_1^+ (K^+ - \widehat{k}_1^+) \left(k_1^- - \widehat{k}_{1on}^- + \frac{i\varepsilon}{\widehat{k}_1^+}\right) \left(K^- - k_1^- - \widehat{k}_{2on}^- + \frac{i\varepsilon}{K^+ - \widehat{k}_1^+}\right)}, \tag{5}$$

where the LF four-momenta are $k_i = (k_i^- := k_i^0 - k_i^3 \; , \; k_i^+ := k_i^0 + k_i^3 \; , \; \mathbf{k}_{i\perp})$, $\widehat{k}_{ion}^- = (\widehat{\mathbf{k}}_{i\perp}^2 + m^2)/\widehat{k}_i^+$ $(i = 1, 2)$ is the on-minus-shell momentum operator, with eigenfunctions given by the LF plane waves, $\langle x_i^- \mathbf{x}_{i\perp} | k_i^+ \mathbf{k}_{i\perp} \rangle = \mathcal{N} \, e^{-\imath (\frac{1}{2} k_i^+ x_i^- - \mathbf{k}_{i\perp} . \mathbf{x}_{i\perp})}$. The completeness relation and the normalization are

$$\int \frac{dk^+ d^2 k_\perp}{2(2\pi)^3} \; |k^+ \mathbf{k}_\perp\rangle\langle k^+ \mathbf{k}_\perp \; | = \mathbf{1}, \tag{6}$$

and $\langle k'^+ \mathbf{k}'_\perp | k^+ \mathbf{k}_\perp \rangle = 2(2\pi)^3 \delta(k'^+ - k^+) \delta(\mathbf{k}'_\perp - \mathbf{k}_\perp)$, respectively.

The free two-fermion propagator, $G_0^F(K)$, can be decomposed in an on-minus-shell term, $\overline{G}_0(K)$, and a part that contains the so-called instantaneous (in the LF-time) contribution, since the Dirac propagator can be separated in two terms as follows

$$\frac{\not{k} + m}{k^2 - m^2 + i\varepsilon} = \frac{\not{k}_{on} + m}{k^+ (k^- - k_{on}^- + \frac{i\varepsilon}{k^+})} + \frac{\gamma^+}{2k^+}. \tag{7}$$

where the first term yields the on-minus-shell propagation, while the second one the LF-time instantaneous term of the Dirac propagator. For the fermion case, we will be interested in the analogous of Eq. (5), written in terms of $\overline{G}_0(K)$, namely

$$\langle k_1'^{-} | \, \overline{G}_0(K) \, | k_1^- \rangle = \left(\widehat{\not{k}}_{1on} + m_1\right) \left(\widehat{\not{k}}_{2on} + m_2\right) \langle k_1'^{-} | \, G_0(K) \, | k_1^- \rangle, \tag{8}$$

For the sake of a unified formal treatment of two-boson and two-fermion systems, *in what follows we put* $\overline{G}_0(K) \equiv G_0(K)$.

One could easily extend the present analysis to systems composed by particle–antiparticle or by other mixtures, like a fermion and a boson (see e.g. [25]).

The first step for projecting the BSE onto the LF surface is the introduction of the free-resolvent, i.e., the Fourier transform in K^- of the global x^+-time free propagator of the two-particle system. This amounts to integrate the matrix elements, Eq. (5) (or Eq. (8)), of the 4D $G_0(K)$ over k_1^- and $k_1'^-$, so that the relative LF time between the particles is eliminated, and one remains with a dependence upon K^-, i.e.

$$|G_0(K)| := \int dk_1'^- dk_1^- \, \langle k_1'^- | \, G_0(K) \, | k_1^- \rangle \equiv g_0(K) \tag{9}$$

where $g_0(K)$, called the global LF-time free propagator, is a 3D operator depending upon the LF momenta $(k_i^+, \mathbf{k}_{i\perp})$ only, and it is explicitly given by

$$g_0(K) = \frac{\widehat{P}}{\widehat{k}_1^+ (K^+ - \widehat{k}_1^+) \left(K^- - \widehat{k}_{1on}^- - \widehat{k}_{2on}^- + i\varepsilon\right)}, \tag{10}$$

where $\widehat{P} = i\theta(K^+ - \widehat{k}_1^+)\theta(\widehat{k}_1^+)$ for two-bosons and

$$\widehat{P} = i\theta(K^+ - \widehat{k}_1^+)\theta(\widehat{k}_1^+)(2m_1)(2m_2)\Lambda_+(\widehat{k}_{1on})\Lambda_+(\widehat{k}_{2on})$$

for two-fermions. The projector $\Lambda_+(\widehat{k}_{on}) = \left(\widehat{\not{k}}_{on} + m\right)/2m$ is the positive energy spinor projector. A positive value for K^+ is used without any loss of generality.

In Eq. (9), the vertical bars "|" on the right side and on the left one indicate that the minus components in $|k^-\rangle$ and $\langle k'^-|$ have to be integrated out [13,14], respectively. Notice that, for two fermions, the inverse of $g_0(K)$ exists only in the valence sector, since the projectors, Λ_+, single out only positive energy states.

Within the QP approach [12], where an auxiliary interaction, $W(K)$, is introduced, the full T-matrix is solution of the following coupled equations

$$T(K) = W(K) + W(K)\widetilde{G}_0(K)T(K), \tag{11}$$
$$W(K) = V(K) + V(K)\Delta_0(K)W(K), \tag{12}$$

where $\Delta_0(K) = G_0(K) - \widetilde{G}_0(K)$ for bosons and $\Delta_0(K) = G_0^F(K) - \widetilde{G}_0(K)$ for fermions. The auxiliary Green's function $\widetilde{G}_0(K)$ is a 4D operator, depending upon the four-momenta of the two constituents, and it represents the key quantity of the LF projection. It is the 4D image of the 3D dimensional $g_0(K)$, that, we strongly stress, does not contain the relative-time propagation of the system. Therefore $\Delta_0(K)$ just takes into account such a propagation in the 4D space, namely it will be an essential ingredient in the description of the internal dynamics of the systems. The 4D operator $\widetilde{G}_0(K)$ is defined by

$$\widetilde{G}_0(K) = \overline{\Pi}_0(K)g_0(K)\Pi_0(K), \tag{13}$$

where

$$\overline{\Pi}_0(K) = G_0(K)|\, g_0^{-1}(K), \qquad \Pi_0(K) = g_0^{-1}(K)\, |G_0(K). \tag{14}$$

These operators, *the free reverse LF-projection operators*, connect three and four dimensional quantities. In the next section the corresponding interacting operators will be introduced. It is worth noting that the choice of $\widetilde{G}_0(K)$ and the corresponding integral equation for $W(K)$, Eq. (12), allows only for LF two-body irreducible terms.

The solution by iteration of Eq. (12) is given by

$$W(K) = \sum_{n=1}^{\infty} W_n(K), \tag{15}$$

with $W_n(K) = V(K)\ (\Delta_0(K)\ V(K))^{n-1}$. The diagrammatic analysis of the series (15) shows that for each term one has a specific Fock content, associated to the propagation of the virtual intermediate states, as discussed to some extent in Sect. 4. Of course, a truncation of the sum in Eq. (15) puts a bound on the number of the Fock components involved in the actual calculation. Moreover, the same holds for the 3D effective interaction $w(K) := \Pi_0(K)W(K)\overline{\Pi}_0(K)$ that determines the valence component, as shown in the following section. In particular, the 3D effective interaction, since it is non diagonal in the Fock space, makes possible the coupling of the valence sector to the higher Fock components of the wave function (formal details are given in Sect. 4 and [15]).

It is worth noting that, in model studies (see [13,26–30]), the expansion (15) is rapidly converging, since the probability of higher Fock states is quickly decreasing [31,32].

In fermionic models, with point-like couplings, the LF BSE was investigated by retaining the lowest order kernel and subtle problems, related to the divergences produced by the dependence upon the transverse momentum, were found [33–35]. Part of the difficulties can be ascribed to the absence of the instantaneous terms, as analyzed in Refs. [14,36].

Following Refs. [13,14,21–23], one can construct a 3D LF T-matrix, $t(K)$, from the 4D one. In particular, one has

$$t(K) = \Pi_0(K)T(K)\overline{\Pi}_0(K) = w(K) + w(K)g_0(K)t(K) = w(K) + w(K)g(K)w(K) \tag{16}$$

where $g(K)$ is the interacting LF Green's function, fulfilling the integral equation

$$g(K) = g_0(K) + g_0(K)w(K)g(K), \tag{17}$$

Notice that it also holds $g(k) = g_0(k) + g_0(k)t(K)g_0(k)$. The 3D operator, $g(K)$, is the Fourier transform in K^- of the global LF-time propagator, viz

$$g(K) = |G_0(K)| + |G_0(K)T(K)G_0(K)|, \tag{18}$$

and evolves the system from an initial state, defined on a given LF hypersurface, to another one, after a LF-time interval $x^+_f - x^+_i > 0$. By iterating once the integral Eq. (11), and using Eqs. (13) and (16), one has

$$\begin{aligned} T(K) &= W(K) + W(K)\left[\widetilde{G}_0(K) + \widetilde{G}_0(K)T(K)\widetilde{G}_0(K)\right]W(K) \\ &= W(K) + W(K)\bar{\Pi}_0(K)g(K)\Pi_0(K)W(K), \end{aligned} \tag{19}$$

This relation allows one to map the 3D dynamics into the 4D space.

It turns out that the on-mass-shell matrix elements of $T(K)$, which define the two-constituent scattering amplitude, are identical to the ones obtained from the on-minus-energy-shell matrix elements of $t(K)$ (see discussion in [13,14]). Unless otherwise indicated, the operators $g_0(K)$ and $w(K)$ have to be evaluated with a "$+\imath\varepsilon$" prescription.

3 The BS Amplitude and the LF Valence Component

The relation between the BS amplitude $|\Psi_B\rangle$ and the valence component of the LF wave function, $|\Phi_B\rangle$, for a bound state with total momentum K_B is given by [14] (let us recall that we formally put $G_0(K) \equiv \overline{G}_0(K)$)

$$|\Psi_B\rangle = \mathcal{G}_0(K_B)W(K_B)G_0(K_B)|\ g_0^{-1}(K_B)\,|\Phi_B\rangle\ , \tag{20}$$

where the valence component is the solution of the following eigenequation

$$|\Phi_B\rangle = g_0(K_B)w(K_B)\,|\Phi_B\rangle\,, \tag{21}$$

It should be pointed out that, in the case of fermions, the instantaneous terms from the Dirac propagators appear in (a) $\mathcal{G}_0 \equiv G_0^F$, (b) W and (c) the effective interaction w.

The identity $G_0(K_B)|\ \left(g_0^{-1}(K_B) - w(K_B)\right)|\Phi_B\rangle = 0$ can be added to Eq. (20) in order to get

$$|\Psi_B\rangle = [1 + \Delta_0(K_B)W(K_B)]\,G_0(K_B)|\ g_0^{-1}(K_B)\,|\Phi_B\rangle\,, \tag{22}$$

This expression holds not only for bound states but also for scattering states (see e.g. [23]), viz

$$\left|\Psi^+\right\rangle = \Pi(K)\ \left|\Phi^+\right\rangle \tag{23}$$

where $\Pi(K)$, *the interacting reverse LF-projection operator*, is given by

$$\Pi(K) := [1 + \Delta_0(K)W(K)]\,G_0(K)|\ g_0^{-1}(K) = G_R(K)|\ g^{-1}(K). \tag{24}$$

with

$$G_R(K) := G_0(K) + \mathcal{G}(K)V(K)G_0(K) = \mathcal{G}(K)\mathcal{G}_0^{-1}(K)G_0(K), \tag{25}$$

Notice that in $G_R(K)$ the on-minus-shell-Green's function $G_0(K)$ appears on the rightmost position, and this leads to apply the "|" operation on the right.

The operator $\Pi(K)$ acts on the valence component of the LF wave function in Eqs. (22) and (23) and it allows one to fully reconstruct the 4D BS amplitude for both bound and scattering states, starting from the valence wave function. The LF-conjugated operator, $\overline{\Pi}(K)$ is given by

$$\overline{\Pi}(K) := g^{-1}(K)\,|G_L(K) \tag{26}$$

with $G_L(K) := G_0(K) + G_0(K)V(K)\mathcal{G}(K) = G_0(K)\mathcal{G}_0^{-1}(K)\mathcal{G}(K)$, that allows the "|" operation on the left. The reverse LF projectors make compact the relations between operators and states living onto a 3D hypersurface and the full 4D counterparts. For instance the relation between the BS amplitude and the valence component can be written as $|\Psi\rangle = \Pi(K)|\Phi\rangle$ and $\langle\Psi| = \langle\Phi|\,\overline{\Pi}(K)$. These relations can be applied to both two-boson [13,14] and two-fermion systems [21–23] with the proper choice of $\mathcal{G}$, $\mathcal{G}_0$ and the on-minus-shell $G_0(K)$. Reversely, the valence component of the LF wave function can be obtained directly from the BS amplitude by using Eqs. (24) and (25) [13,14,21–23]

$$|G_0(K)\mathcal{G}_0^{-1}(K)\,|\Psi\rangle =\ |G_0(K)\mathcal{G}_0^{-1}(K)\mathcal{G}(K)\mathcal{G}_0^{-1}(K)G_0(K)|\ g^{-1}(K)\,|\Phi\rangle = |\Phi\rangle\,. \tag{27}$$

To conclude this section, it is worth noting that the 3D valence LF wave functions are solutions of the squared mass eigenvalue equation:

$$g^{-1}(K)\,|\Phi\rangle = 0, \tag{28}$$

with suitable boundary conditions for bound and scattering states, respectively. In particular, the LF scattering state is the solution of the inhomogeneous equation,

$$\left|\Phi^{+}\right\rangle = |\Phi_0\rangle + g_0(K)w(K)\left|\Phi^{+}\right\rangle, \tag{29}$$

with outgoing boundary condition.

4 Hierarchy Equations for LF Green's Functions

In order to gain deep insights in the Fock content of the dynamical quantities involved in the approach presented in the previous sections, one can combine the investigation of the Fock structure of a Hamiltonian (actually a hadronic Hamiltonian) performed in Refs. [10,11], with the LF projection technique [15]. For that purpose, the free resolvent is rewritten as

$$g_0(K) = |G_0(K)| := \int dk_1'^{-}dk_1^{-}\left\langle k_1'^{-}\right| G_0(K)\left|k_1^{-}\right\rangle = i\widehat{\Omega}^{-1}g_0^{(2)}(K)\widehat{\Omega}^{-1}, \tag{30}$$

where the phase space operator is conveniently defined by $\widehat{\Omega} := \sqrt{\widehat{k}_1^{+}(K^{+}-\widehat{k}_1^{+})}$. It should be pointed out that $\widehat{\Omega}$ makes the LF projection "|" invariant with respect to the kinematical subgroup of the Poincaré group. For spinless particles, the free two-body LF Green's function is a particular case of the N-body LF Green's function given by

$$g_0^{(N)}(K) = \left[\prod_{j=1}^{N}\theta(\widehat{k}_j^{+})\theta(K^{+}-\widehat{k}_j^{+})\right]\left(K^{-}-\widehat{K}_0^{(N)-}+i\varepsilon\right)^{-1}, \tag{31}$$

where $\widehat{K}_0^{(N)-} = \sum_{j=1}^{N}\widehat{k}_{jon}^{-}$ is the free LF Hamiltonian.

Let us consider a two-boson system. By introducing the operator $\widehat{\Omega}$, the interacting LF Green's function, Eq. (17), can be rewritten as follows

$$g^{(2)}(K) = g_0^{(2)}(K) + g_0^{(2)}(K)\nu(K)g^{(2)}(K), \tag{32}$$

where $g^{(2)} \equiv -i\widehat{\Omega}g(K)\widehat{\Omega},\ \nu(K) = i\widehat{\Omega}^{-1}w(K)\widehat{\Omega}^{-1}$. From Eq. (15), the leading and next-to-leading order contributions to $\nu(K)$ are given by

$$\nu^{(2)}(K) = i\left[\widehat{\Omega}g_0(K)\right]^{-1}|G_0(K)V(K)G_0(K)|\left[g_0(K)\widehat{\Omega}\right]^{-1}, \tag{33}$$

$$\begin{aligned}\nu^{(4)}(K) &= i\left[\widehat{\Omega}g_0(K)\right]^{-1}|G_0(K)V(K)G_0(K)V(K)G_0(K)|\left[g_0(K)\widehat{\Omega}\right]^{-1}\\ &-i\left[\widehat{\Omega}g_0(K)\right]^{-1}|G_0(K)V(K)\widetilde{G}_0(K)V(K)G_0(K)|\left[g_0(K)\widehat{\Omega}\right]^{-1}.\end{aligned} \tag{34}$$

Notice that $\nu^{(4)}(K)$ is two-body irreducible, due to the subtraction of the last term in Eq. (34). The content of the operator $\nu^{(n)}$ in the LF Fock-space can be investigated, within a ladder approximation, in a Yukawa bosonic Lagrangian model, $\mathcal{L}_I^B = g_S\phi_1^{\dagger}\phi_1\sigma + g_S\phi_2^{\dagger}\phi_2\sigma$ with $\phi_1,\ \phi_2$ and σ bosonic fields. Then, the interaction vertex operator, acting between Fock states differing by one quantum σ, has matrix element given by e.g.,

$$\langle qk_\sigma|v|k\rangle = -2(2\pi)^3\delta^3(\tilde{q}+\tilde{k}_\sigma-\tilde{k})\frac{g_S}{\sqrt{q^{+}k_\sigma^{+}k^{+}}}\theta(k_\sigma^{+}), \tag{35}$$

where the LF momenta are indicated by the convention: $\tilde{q} \equiv \{q^{+}, \mathbf{q}_\perp\}$. The states are normalized according to (6). The effective interaction $\nu(K)$, up to next-to-leading order in v can be obtained from the suitable LF-time ordered diagrams, since the LF projections, "|", allows one to play a game analogous to the case of the non relativistic perturbation theory in the Fock space. Then, one gets [10,11,15]

$$\nu(K) \approx \nu^{(2)}(K) + \nu^{(4)}(K) = v g_0^{(3)}(K) v + v g_0^{(3)}(K) v g_0^{(4)}(K) v g_0^{(3)}(K) v. \tag{36}$$

By reminding that in the Yukawa model, the BSE kernel $V(K)$ contains two interaction vertexes, v, according to Eqs. (33) and (34) one can perform the following identifications: (a) $\nu^{(2)} \equiv v g_0^{(3)}(K) v$ and (b) $\nu^{(4)} \equiv v g_0^{(3)}(K) v g_0^{(4)}(K) v g_0^{(3)}(K) v$. Such expressions straightforwardly show the Fock content of each term. In particular, the presence of $g_0^{(3)}$ and $g_0^{(4)}$ points to an intermediate propagation of three and four bosons, respectively. Once the previous analysis is performed at any order, it turns out that in general

$$\nu(K) = v g^{(3)}(K) v.$$

In turn, from Eq. (36), one sees that $g^{(3)}$ is coupled to the four-body Green's function, which should be coupled to the five-body one, and so on. By an obvious generalization, one can construct a hierarchy of coupled equations for the interacting bosonic Green's functions, viz

$$\begin{aligned} g^{(2)}(K) &= g_0^{(2)}(K) + g_0^{(2)}(K) v g^{(3)}(K) v g^{(2)}(K) \ldots \\ g^{(N)}(K) &= g_0^{(N)}(K) + g_0^{(N)}(K) v g^{(N+1)}(K) v g^{(N)}(K) \ldots \end{aligned} \tag{37}$$

Those coupled equations encode the full Fock-space content of the QP expansion, within the LF projection framework.

An analogous study can be carried out for the two-fermion system, by adopting the Yukawa model given by $\mathcal{L}_I^F = g_S \overline{\Psi_1} \Psi_1 \sigma + g_S \overline{\Psi_2} \Psi_2 \sigma$, with Ψ_1 and Ψ_2 being the fermionic fields. The interaction vertex operator, acting between Fock states differing by zero, one and two σ's, has matrix elements given by

$$\langle (q, s') k_\sigma | v | (k, s) \rangle = -2m (2\pi)^3 \delta^3(\tilde{q} + \tilde{k}_\sigma - \tilde{k}) \frac{g_S}{\sqrt{q^+ k_\sigma^+ k^+}} \theta(k_\sigma^+) \overline{u}(q, s') u(k, s) \tag{38}$$

$$\langle \ (q, s') k'_\sigma | v | (k, s) k_\sigma \rangle = -2 (2\pi)^3 \delta^3(\tilde{q} + \tilde{k}'_\sigma - \tilde{k} - \tilde{k}_\sigma) \delta_{s's} \frac{g_S^2}{\sqrt{k'^+_\sigma k^+_\sigma}} \frac{\theta(k'^+_\sigma)\theta(k^+_\sigma)}{k^+ + k^+_\sigma} \tag{39}$$

$$\langle (q, s') k'_\sigma k_\sigma | v | (k, s) \rangle = -2 (2\pi)^3 \delta^3(\tilde{q} + \tilde{k}'_\sigma + \tilde{k}_\sigma - \tilde{k}) \delta_{s's} \frac{g_S^2}{\sqrt{k'^+_\sigma k^+_\sigma}} \frac{\theta(k'^+_\sigma)\theta(k^+_\sigma)}{k^+ - k^+_\sigma} \tag{40}$$

for fermion states properly normalized. The instantaneous terms in the two-fermion propagator give origin to Eqs. (39) and (40). Since $\nu(K)$ has terms that couple sectors of the Fock space that differ at most by two sigma's [14], then one gets the following expression for the coupled set of Green's functions

$$\begin{aligned} g^{(2)}(K) &= g_0^{(2)}(K) + g_0^{(2)}(K) v \Big[g^{(3)}(K) + g^{(4)}(K) + g^{(3)}(K) v g^{(4)}(K) \\ &\quad + g^{(4)}(K) v g^{(3)}(K) \Big] v g^{(2)}(K), \ldots \\ g^{(N)}(K) &= g_0^{(N)}(K) + g_0^{(N)}(K) v \Big[g^{(N+1)}(K) + g^{(N+2)}(K) + g^{(N+1)}(K) v g^{(N+2)}(K) \\ &\quad + g^{(N+2)}(K) v g^{(N+1)}(K) \Big] v g^{(N)}(K), \ldots \end{aligned} \tag{41}$$

It is important to notice that truncating in the Fock space the effective interaction ν is different from truncating the coupled set of Eqs. (38) or (41). This can be easily understood by considering the two-boson case and restricting the intermediate-state propagation up to four-particles, namely retaining up to $g_0^{(4)}(K)$. Then, one gets $g^{(2)}(K) \simeq g_0^{(2)}(K) + g_0^{(2)}(K) v g^{(3)}(K) v g^{(2)}(K)$ with $g^{(3)}(K) \simeq g_0^{(3)}(K) + g_0^{(3)}(K) v g_0^{(4)}(K) v g^{(3)}(K)$, where one has propagations up to four particles, given the presence of $g^{(3)}(K)$. On the other side, from Eq. (36), one has $v g_0^{(3)}(K) v$, without higher Fock propagations.

5 LF Projection of Three-Body BS Equations

The approach briefly revised in Sects. 2 and 3 has been extended to three-particle systems in Ref. [19,20]. In the last years, within a LF framework, 3-body systems have been (a) investigated within zero-range models [37,38], (b) applied to the description of the nucleon [39] and (c) adopted for analyzing the quark mass effects in heavy baryons [40]. It is very important to notice that the QP expansion allows one to systematically deal with higher Fock-state contributions to the dynamics of the three-body valence component, in fully analogy with the two-body case.

The starting point of the investigation performed in Ref. [19,20] is the three-body Bethe–Salpeter equation for the transition-matrix, within the ladder approximation, viz

$$T = V + V\mathcal{G}_0 T; \quad V = \sum V_i; \quad V_i = V^{(2)}_{jk} S_i^{-1}, \tag{42}$$

where $V^{(2)}_{jk}$ stands for an interaction between particles j and k corresponding to 2-body irreducible diagrams. S_i is the individual particle propagator and $\mathcal{G}_0 = S_1 S_2 S_3$. *The complexities produced by the spin degrees of freedom are omitted in what follows*.

The same QP formalism shown in Eqs. (11) and (12) can be applied to the three-body BSE, but with the obvious extension to the three-body case of the free Green's function, namely $\mathcal{G}_0 = S_1 S_2 \to \mathcal{G}_0 = S_1 S_2 S_3$. In order to get the three-body LF Green's function $g_0(K)$, one has to project onto the LF hyperplane by integrating over two independent minus momentum components, i.e. $\tilde{G}_0 := G_0 || g_0^{-1} || G_0$ with $g_0 := ||G_0||$ and G_0 the generalization to the three-particle case of the on-minus-shell two-body propagators (see Eq. (5) for bosons and Eq. (8) for fermions). The double bar "|| "operation on the right or on the left means

$$||G_0 := \int dk_1^- dk_2^- \left\langle k_1^-, k_2^- \right| G_0, \quad G_0|| := \int dp_1^- dp_2^- \; G_0 \left| p_1^-, p_2^- \right\rangle. \tag{43}$$

The Faddeev decomposition of the QP as $W_i = V_i + V_i \Delta_0 \sum_j W_j$, leads to the components of the effective interaction $w_i := \Pi_0 W_i \overline{\Pi}_0 := g_0^{-1} || G_0 W_i G_0 || g_0^{-1}$. The free reverse LF time operators for the three-body system, Π_0 and $\overline{\Pi}_0$, are introduced in analogy to the two-body case, with the difference that they now contain a double integration. Finally three-body LF transition matrix reads as follows

$$t := \Pi_0 T \overline{\Pi}_0 = \sum_{i=1}^{3} w_i + \sum_{i=1}^{3} w_i g_0 t, \tag{44}$$

where the standard Faddeev decomposition, $t = \sum_{i=1}^3 t_i$, leads to a couple set of equations $t_i = \bar{t}_i + \bar{t}_i g_0 (t_j + t_k)$ with $\bar{t}_i = (1 - w_i g_0)^{-1} w_i$.

One is tempted to identify $\bar{t}_i$ with the two-body subsystem 3D transition matrix, however this is not possible. It contains irreducible three-body terms from the point of view of the LF global propagation. To clarify this point, let us consider an example with the interaction Lagrangian density: $\mathcal{L}_I = \sum_{i=1}^3 g \phi_i^\dagger \phi_i \sigma$, where three different spin zero bosons exchange a scalar quantum σ. Moreover, let us simplify our discussion assuming that V_i corresponds to the 4D ladder one-boson exchange. The expansion of W_i, e.g., up to next-to-leading order is given by

$$w_i = \Pi_0 V_i \overline{\Pi}_0 + \Pi_0 V_i \Delta_0 V_i \overline{\Pi}_0 + \Pi_0 V_i \Delta_0 (V_j + V_k) \overline{\Pi}_0 + \cdots\,, \tag{45}$$

The leading-order $\Pi_0 V_i \overline{\Pi}_0$ corresponds to a LF two-particle interaction in the presence of the ith boson, and it is built through the coupling of three- and four-particle Fock sectors. Pictorially, one has an intermediate propagation of a a four-particle state, i.e. three ϕ_i bosons and an exchanged quantum σ, between an initial and final three-boson free propagations. The term $\Pi_0 V_i \Delta_0 V_i \overline{\Pi}_0$ corresponds to two-body stretched boxes involving two boson out three, with the third one, the ith boson, acts as a spectator. The term $\Pi_0 V_i \Delta_0 V_j \overline{\Pi}_0$ corresponds to an *induced three-body force* due to the elimination of the relative LF-time between the particles. It should be pointed out that the induced three-body forces, from the 4D point of view, are quite different from the intrinsic three-body forces, and play a very important role in the determination of the three-body dynamics, as shown by Karmanov and Maris in [41], where the calculation of the three-boson bound-state mass has been presented.

6 Electromagnetic Current and LF Ward–Takahashi Identity

The electromagnetic current operator plays a central role for the phenomenology, and therefore it deserves a detailed analysis within any relativistic framework. In particular, one should pay attention to the fulfillment of the Ward–Takahashi identity (WTI) [42], as well. For a system of two charged particles 1 and 2, in the Minkowski space, the WTI reads as follows

$$Q_\mu \mathcal{J}^\mu(Q) = \left[\mathcal{G}^{-1}, \widehat{e}_1\right] + (1 \leftrightarrow 2), \tag{46}$$

where the charge operator for particle i has matrix elements given by $\langle k_i|\widehat{e}_i|p_i\rangle = e_i \delta^4\left(k_i - p_i - Q\right)$. The full Green's function of the interacting two-particle system is a solution of $\mathcal{G}(K) = \mathcal{G}_0(K) + \mathcal{G}_0(K)V(K)\mathcal{G}(K)$. The current operator may contain two terms, a free contribution and an interacting part, viz

$$\mathcal{J}^\mu(Q) = \mathcal{J}_0^\mu(Q) + \mathcal{J}_I^\mu(Q), \tag{47}$$

It is worth noting that the free term, $\mathcal{J}_0^\mu(Q)$, leads to the impulse approximation derived by Mandelstam [43], where self-energy insertions/vertex corrections were disregarded.

Once the relation between the BS amplitude and the 3D LF valence component (see Sect. 3) has been established, the LF em current operator can be constructed from the matrix element of the 4D current (see [21–23]). For both scattering and bound states one has

$$\left\langle \Psi_f \right| \mathcal{J}^\mu(Q) \left|\Psi_i\right\rangle = \langle \phi_f| j^\mu(K_f, K_i)|\phi_i\rangle, \tag{48}$$

where $j^\mu(K_f, K_i)$ is the 3D LF current, that acts on the valence wave functions and *includes two-body operators*. It is given by

$$j^\mu(K_f, K_i) := \overline{\Pi}(K_f)\mathcal{J}^\mu(Q)\Pi(K_i). \tag{49}$$

A 3D LF electromagnetic current operator for two-boson interacting systems, acting on the valence state and fulfilling a WTI, has been also obtained by using the method of gauging Eqs. [44]. The achieved result is in agreement with Eq. (49).

In order to implement the WTI for the 3D current one starts with the corresponding relation for the 4D current, Eq. (46), and then one applies the reverse LF projector, as suggested by Eq. (49). The WTI satisfied by $j^\mu(K_f, K_i)$ is given by

$$Q_\mu j^\mu(K_f, K_i) = \overline{\Pi}(K_f)\left[\mathcal{G}^{-1}(K_f)\widehat{e} - \widehat{e}\mathcal{G}^{-1}(K_i)\right]\Pi(K_i), \tag{50}$$

with $\widehat{e} = \widehat{e}_1 + \widehat{e}_2$. After applying some formal manipulations, illustrated in great detail in [23], one gets

$$Q_\mu j^\mu(K_f, K_i) = g^{-1}(K_f)\, \widehat{\mathcal{Q}}^L_{LF} - \widehat{\mathcal{Q}}^R_{LF}\, g^{-1}(K_i) \tag{51}$$

where there are the fully interacting 3D LF Green's functions, labeled by the total initial and final momenta, and the *left* and *right* LF charge operators have been introduced, according to the following definition

$$\widehat{\mathcal{Q}}^L_{LF} = |G_0(K_f)\, \mathcal{G}_0^{-1}(K_f)\, \widehat{e}\, \Pi(K_i) = |G_0(K_f)\, \mathcal{G}_0^{-1}(K_f)\, \widehat{e}\, \Pi_0(K_i), \tag{52}$$

$$\widehat{\mathcal{Q}}^R_{LF} = \overline{\Pi}(K_f)\, \widehat{e}\, \mathcal{G}_0^{-1}(K_i)\, G_0(K_i)| = \overline{\Pi}_0(K_f)\, \widehat{e}\, \mathcal{G}_0^{-1}(K_i)\, G_0(K_i)|. \tag{53}$$

with the same formal assumption indicated below Eq. (8). In the case of fermions, the explicit expression, e.g. for particle 1, is given by

$$\widehat{\mathcal{Q}}^L_{1LF} = \Lambda_+(\widehat{k}_{1on})\frac{m_1}{k_1^+}\gamma_1^+\, \widehat{e}_{1LF}\, \Lambda_+(\widehat{k}_{1on})\Lambda_+(\widehat{k}_{2on}), \tag{54}$$

$$\widehat{\mathcal{Q}}^R_{1LF} = \Lambda_+(\widehat{k}_{1on})\, \widehat{e}_{1LF}\frac{m_1}{k_1^+}\gamma_1^+\Lambda_+(\widehat{k}_{1on})\Lambda_+(\widehat{k}_{2on}), \tag{55}$$

where the notation $\widehat{e}_{1LF}$ indicates the 3D LF counterpart of the 4D operator $\hat{e}$, with matrix elements given by

$$\langle k_1'^+, \mathbf{k}'_{1\perp}|\widehat{e}_{1LF}|k_1^+, \mathbf{k}_{1j\perp}\rangle := e_1\delta^3\left(\tilde{k}'_1 - \tilde{k}_1 - \tilde{Q}\right). \tag{56}$$

The normalization condition can be obtained sandwiching the operator $\gamma^+ m/k^+$ between LF spinors.

In the case of a spin-zero boson system, one simply has $\widehat{\mathcal{Q}}^R_{1LF} \equiv \widehat{\mathcal{Q}}^L_{1LF} = \widehat{e}_{1LF}$.

It is very important to stress that the LF charge operator are interaction free, within the approach of Refs. [21–23], where no self-energy corrections were included.

From Eq. (51), current conservation straightforwardly follows by taking the matrix elements between 3D LF valence components, solutions of the wave Eq. (28) and noting that the *left* and *right* charge operators do not contain any $\imath\epsilon$ dependence.

By multiplying both the left and right hand sides of Eq. (51) by $g(K_f)$ and $g(K_i)$, respectively, one gets

$$Q_\mu g(K_f) j^\mu(K_f, K_i) g(K_i) = \widehat{\mathcal{Q}}^L_{LF} g(K_i) - g(K_f) \widehat{\mathcal{Q}}^R_{LF}, \tag{57}$$

which corresponds to the LF projection of the 5-point function without instantaneous terms in the external legs in the case of fermions. For bosons this identity was also derived in [44].

Within a field theoretical approach, the Poincaré covariance of $\mathcal{J}_\mu$ is ensured. Therefore, the covariance properties of the matrix elements of the operator j_μ, are fulfilled, since all the matrix elements of the lhs of Eq. (48) are properly related through the Lorentz transformations. But for a truncated QP expansion, see Eq. (15), it is expected that the full covariance of the description will be lost, at some extent. A quantitative analysis of such a Poincaré covariance breaking can be systematically carried out by considering more and more terms in the QP expansion. In any case, even after introducing a proper truncation, the corresponding LF current fulfills a WTI, where truncated LF Green's functions appear [21–23]. The corresponding current conservation can be retrieved by using the valence wave functions obtained from eigenequations with the inverse of the truncated LF Green's function. Although the explicit expression of the truncated LF current is rather involved, the physical picture that arises is quite sensible: the truncated LF two-body current, that fulfills the proper WTI, is generated by attaching the photon in all the possible ways to the truncated effective interaction operators, that are present in the truncated reverse LF projectors (see Eq. (49)) and are irreducible with respect to LF two-body propagation.

Within a framework kinematically Poincaré invariant, the rich phenomenology of the electromagnetic processes can be addressed by adopting the approach presented in this section. It is interesting just to remind few issues. In the fermionic case, the instantaneous contributions play an essential role at the formal level, and therefore one should be eager to investigate possible signatures of those terms. Furthermore, for both bosonic and fermionic systems, the choice of a frame different from the celebrated Drell–Yan one, namely a frame where the plus component of the momentum transfer is not vanishing, allows one to study the Fock content of the system state, by coupling small components to large components (for an analysis of the physical impact on actual cases in Hadronic Physics see, e.g., Refs. [17,45,46]).

7 Summary and Perspectives

In this review, we have given a short presentation of the LF projection method, based on a combination of the integration on the minus component of the constituent four-momenta and the quasi-potential formalism [12]. This approach allows one to formally establish a one-to-one relation between the Bethe–Salpeter amplitude and the 3D LF valence component of a bosonic or fermionic system, and it makes natural to address the issue of the Fock content of the dynamics governing the system under investigation. In particular, in Sect. 4, it has been shown that by using the quasi-potential approach, in the spirit of the "iterated resolvents" suggested by H.C. Pauli [10,11], one can construct a set of coupled equations for the LF resolvents, that allows one to arrange a sort of tomography in the Fock space of the Bethe–Salpeter equation. Finally, the extension of the LF projection method to the three-particle case has been outlined.

From the phenomenological point of view, the last section contained the most relevant topic. There, we briefly presented the derivation of the conserved 3D LF electromagnetic current operator and the associated Ward–Takahashi identity. The issue of the truncation in the Fock space of the LF current, and the consequent elaboration for obtaining a truncated Ward–Takahashi identity has been discussed. In particular, it has been pointed out that in order to fulfill the current conservation, the valence wave functions have to be solutions of the suitable eigenequation, obtained from the truncated Green's function.

In conclusion, we would mention that, within the framework of the LF projection method, the calculation of the deuteron electromagnetic form factors, with two-body currents obtained from a pion exchange interaction, and the investigation of the relation between the analytic properties of the Nakanishi representation and the Fock decomposition of the BS amplitude (see e.g. [29,30]) are in progress.

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