



A variation of Hilbert’s axioms for euclidean geometry

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Abstract We propose a variation of Hilbert’s axioms for euclidean geometry which appears to us to be more intuitive, and which supports more directly Euclid’s original approach to the criteria for congruence of triangles.

Keywords Hilbert’s axioms · Euclid · Euclidean geometry · Plane geometry

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In 1899, D. Hilbert supplied for the first time a set of axioms which can serve as a rigorous and complete foundation for Euclid’s geometry, see [5, 6]. Thus, finally, the idea originating in Euclid’s “Elements” of a treatise of geometry based uniquely on a few basic assumptions from which the whole wealth of geometrical truths could be obtained uniquely by deduction came true. For long centuries, Euclid’s work itself was regarded as a model of such a theory, so that philosophical treatises using mainly deduction were said to proceed “more geometrico”. From a modern point of view, however, Euclid did not state explicitly all his basic assumptions, for instance properties of order which he used nevertheless. On the basis of work of Pasch, Hilbert cured these defects.

In fact, this was also the first time that any mathematical theory was uniquely founded on a purely axiomatic basis. Furthermore, Hilbert insisted that such a procedure should be independent of any relation to reality, which was totally novel

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at that time, and besides quite astonishing for a theory in geometry, of all things. The significance of Hilbert's axiomatic approach thus cannot be overestimated. For appreciations see for instance [1, 4, 11] and the contribution of M. Toepell in [2, pp. 710–723].

Thus, it is not astonishing that Hilbert's book has had 14 editions in Germany alone, and that modern treatises of euclidean geometry continue to be based on Hilbert's axioms, e.g. [3]. Also, these axioms have been discussed in various ways in the literature throughout the 120 years since their publication.

Here we deal with one of these axioms which seems to us rather counter-intuitive; it postulates more or less the validity of the side-angle-side criterion (SAS) for the congruence of two triangles, whereas this result would commonly be felt to deserve a proof on the basis of more intuitive axioms. The proof by Euclid in his "Elements" in I.4 makes use of the method of superposition; he imagines the first triangle to be moved and superimposed on the second triangle, and then compares the two. This method is not justified by any of Euclid's postulates or axioms, which is probably the reason why Hilbert preferred to assume SAS as an axiom. Also, later on, Hilbert gives a proof of the side-side-side criterion (SSS) for the congruence of two triangles which differs from Euclid's proof in I.8; here, Euclid makes use of the method of superposition, as well.

In this note, we propose to replace the SAS criterion in Hilbert's axiom system by an axiom which justifies Euclid's method of superposition and which furthermore seems more intuitive to us. As we shall show, the resulting system of axioms can directly be seen to be equivalent to Hilbert's system.

Hilbert's axioms are arranged in five groups. The first two groups are the axioms of incidence and the axioms of betweenness. The third group, the axioms of congruence, falls into two subgroups, the axioms of congruence (III1)–(III3) for line segments, and the axioms of congruence (III4) and (III5) for angles. Here, we deal mainly with the latter. In citing Hilbert in more detail in the following, we restrict the discussion to plane geometry for simplicity.

As for notation, we write \overline{AB} for the line joining the points A and B , AB for the segment with end points A and B , \overrightarrow{AB} for the ray originating at the point A and containing the point B , and $\angle BAC$ for the angle formed by the rays \overrightarrow{AB} and \overrightarrow{AC} , having vertex A . The sign \equiv will denote congruence of segments and of angles.

Hilbert's axioms of congruence for angles may be paraphrased as follows:

(III4) Angle protraction. *Given an angle $\angle BAC$ and a ray \overrightarrow{DE} , there exists a unique ray \overrightarrow{DF} on a given side of the line \overline{DE} such that $\angle BAC \equiv \angle EDF$.*

(III5) Side-angle-side criterion. *If for two triangles ABC and $A'B'C'$ the congruences*

$$AB \equiv A'B', AC \equiv A'C', \angle BAC \equiv \angle B'A'C'$$

hold, then the angle congruence

$$\angle ABC \equiv \angle A'B'C'$$

is also satisfied.

After stating this axiom, Hilbert remarks that by exchanging the roles of B and C , one immediately obtains that also

$$\angle ACB \equiv \angle A'C'B' .$$

Furthermore, Hilbert shows in the subsequent Theorem 12 that the assumptions of axiom (III5) also entail

$$AC \equiv A'C' ,$$

so that, for the two triangles ABC and $A'B'C'$, all the corresponding sides and all the corresponding angles are congruent. We express this in short by saying that the two triangles ABC and $A'B'C'$ are congruent.

For the sake of completeness, it should be noted that in (III4) we have omitted a statement from Hilbert's original version concerning the reflexivity of the congruence relation for angles. We shall come back to this (minor) topic later.

We now propose to replace these two axioms by the following three axioms (while retaining Hilbert's axioms of incidence, of betweenness, and of congruence of segments as basis). First, we have to reinforce Hilbert's axiom (III1) about segments in the following way:

(US) Uniqueness of segments. *Given two points A, B , then on a given ray originating at a point A' there is at most one point B' such that $AB \equiv A'B'$.*

This uniqueness property is proved by Hilbert using his axiom (III5).—Conversely, we may weaken Hilbert's axiom (III4), postulating only the uniqueness, but not the existence of the ray \overrightarrow{DF} .

(UA) Uniqueness of angles. *Given an angle $\angle BAC$ and given a ray \overrightarrow{DE} , there exists at most one ray \overrightarrow{DF} on a given side of the line \overline{DE} such that $\angle BAC \equiv \angle EDF$.*

The principles inherent in the uniqueness axioms (US) and (UA), which are quite similar, are often used implicitly by Euclid.—The following axiom justifies Euclid's method of superposition.

(CT) Existence of congruent triangles. *Given a triangle ABC and a segment DE such that $DE \equiv AB$, then on a given side of the line \overline{DE} there is a point F such that the triangle ABC is congruent to the triangle DEF .*

This axiom in itself is by no means new. It has been used by various authors, but not in the context for which we propose it here, namely to render Hilbert's axioms more amenable while staying closely within their system, and to justify certain of Euclid's original proofs, in particular the much incriminated method of

superposition. At the end of the article, we shall give indications concerning other occurrences of axioms of this kind, and the relation of (CT) to other approaches via rigid motions.

We now prove, assuming Hilbert's axioms of incidence, betweenness, and congruence of segments, that our system of axioms (US), (UA), and (CT) is equivalent to Hilbert's system of axioms (III4) and (III5).

First, we assume Hilbert's axioms (III4) and (III5), and we show that they imply (US), (UA) and (CT). As already remarked, (US) is proved by Hilbert using (III5), and (UA) is just the uniqueness assertion of (III4). As for (CT), let ABC be a triangle and DE a segment congruent to AB . Then by angle protraction according to (III4), there is a ray \overrightarrow{DF} on the given side of \overline{DE} such that the angle $\angle BAC$ is congruent to the angle $\angle EDF$. We may assume that F is chosen in such a way that the segment DF is congruent to AC . Hilbert's axiom (III5) then says that $\angle ABC$ is congruent to the angle $\angle DEF$, and $\angle ACB$ is congruent to the angle $\angle DFE$. As already mentioned above, Hilbert shows in Theorem 12 as a direct consequence of axioms (III4) and (III5) that then EF is congruent to BC , so that the triangle ABC is congruent to the triangle DEF , as required in (CT).

Conversely, we assume (US), (UA), and (CT). In order to deduce (III4), in the situation given there, let E' be a point on the ray \overrightarrow{DE} such that $DE' \equiv AB$, and let F be a point on the given side of the line $\overline{DE'} = \overline{DE}$ such that ABC is congruent to $DE'F$; such a point exists by (CT). Then, the given angle $\angle BAC$ is congruent to the angle $\angle E'DF = \angle EDF$. Uniqueness of the ray \overrightarrow{DF} is guaranteed by (UA).

Next, we deduce (III5) from (US), (UA) and (CT). This will also explain how (CT) justifies the superposition method of Euclid. Let ABC and $A'B'C'$ be the triangles as in (III5). By (CT) there is a point \tilde{C} on the same side of $\overline{A'B'}$ as C' such that the triangles ABC and $A'B'\tilde{C}$ are congruent. In particular, the angle $\angle BAC$ is congruent to $\angle B'A'\tilde{C}$, but also to $\angle B'A'C'$ by the assumption in (III5). Thus, by (UA), the point \tilde{C} lies on the ray $\overrightarrow{A'C'}$. Since $A'C' \equiv AC \equiv A'\tilde{C}$ by the assumptions of (III5) and the choice of \tilde{C} , it follows from (US) that $\tilde{C} = C'$. Hence, the triangle ABC is congruent to $A'B'\tilde{C} = A'B'C'$, which proves the side-angle-side criterion (SAS, Euclid I.4) for the congruence of triangles, in a manner which is quite near to Euclid's argument by superposition. In particular, $\angle ACB \equiv \angle A'C'B'$, as asserted in Hilbert's axiom (III5).

We have seen just now how axiom (CT) may be used to justify Euclid's superposition method in his proof of the side-angle-side criterion (SAS) for the congruence of triangles. We now examine similarly Euclid's proof of the side-side-side criterion (SSS) in I.8. Let ABC and DEF be two triangles such that corresponding sides are congruent, $AB \equiv DE$, $AC \equiv DF$, $BC \equiv EF$. By (CT), there is a point D' on the same side of \overline{EF} as D such that the triangle ABC is congruent to $D'EF$. (We have permuted the rôles of the points involved in order to agree closely with Euclid.) One would like to have that $D' = D$, so that the triangles ABC and DEF are congruent. If not, then, with the expression used by Euclid, one would have brought together the segments $ED \equiv ED'$ and $FD \equiv FD'$ on the same side of \overline{EF} in two different points D and D' , which is impossible by Euclid I.7.

From this point, it could be established that the congruence relation for angles is an equivalence relation on the basis of our axiom system in exactly the same way

as is done by Hilbert in Theorem 19. For simplicity's sake, one could also assume this as an extra axiom. Euclid would certainly not have objected to this; the first of his common notions can be interpreted as a formulation of transitivity.

It is remarkable that, except for the proofs of SAS and SSS, Euclid seems to be reluctant to use the superposition method. This shall be illustrated by two examples.

In I.24, Euclid considers two triangles ABC and DEF such that $AB \equiv DE$, $AC \equiv DF$ and $\angle BAC > \angle EDF$ and shows that then $BC > EF$. In order to do so, he constructs a point G by angle protraction such that $\angle BAC \equiv \angle EDG$ and $DG \equiv DF$, so that $DG \equiv AC$; then he uses the side-angle-side criterion to conclude that the triangles ABC and DEG are congruent. We shall not pursue the proof further; we just remark that such a point G could have been obtained directly by the superposition method, or, to be more precise, by axiom (CT), instead of using superposition indirectly via the side-angle-side criterion, whose proof by Euclid depends on the superposition method.

In I.26 Euclid shows the congruence criterion for two triangles ABC and DEF involving two pairs of congruent angles $\angle ABC \equiv \angle DEF$, $\angle BCA \equiv \angle EFD$ and one side. First, he considers the case that the two angles are adjacent to the side in question, i.e. the case that $BC \equiv EF$. Assuming $AB > DE$, he finds a point G on AB such that $BG \equiv DE$. He then concludes using the side-angle-side criterion twice.

Instead, by (CT), one could find a point G on the same side of \overline{BC} as A such that the triangles GBC and DEF are congruent. Then $\angle GBC \equiv \angle DEF \equiv \angle ABC$ and $\angle GCB \equiv \angle ACB$. By the uniqueness axiom (UA) for angles it follows that $\overrightarrow{BG} = \overrightarrow{BA}$ and $\overrightarrow{CG} = \overrightarrow{CA}$ and hence $G = A$, so that triangle ABC is congruent to DEF .

In the case that $AB \equiv DE$, and assuming $BC > EF$, Euclid considers a point H on the segment BC such that $BH \equiv EF$. He then concludes using twice the side-angle-side criterion and the theorem asserting that an exterior angle of a triangle is larger than each of the opposite interior angles (Euclid I.16).

Instead, by (CT), one could find a point H on the same side of \overline{AB} as C such that the triangles ABH and DEF are congruent. Then $\angle ABH \equiv \angle DEF \equiv \angle ABC$, so that by the uniqueness axiom (UA) for angles $\overrightarrow{BH} = \overrightarrow{BC}$. Now we can conclude that $H = C$, which ends the proof. Indeed, if H and C were different points, then because of $\angle BHA \equiv \angle EFD \equiv \angle BCA$ the triangle AHC would have an exterior angle equal to an opposite interior angle, which contradicts the above-mentioned result (Euclid I.16).

Thus, whenever a triangle congruent to a given triangle, but in a different position is required in a proof, Euclid seems to prefer to construct such a triangle via the side-angle-side criterion (SAS) once it is established rather than using the superposition method, although he had used this method for the proof of SAS. In other words, he prefers to use the superposition method indirectly instead of using it directly. This casts some doubt on whether our axiom (CT) is really close to what Euclid had in mind when using superposition. It would be interesting to know what historians of mathematics think about these matters.

We now comment on the relation of axiom (CT) to axiom systems employing rigid motions. A (rigid) motion is a bijection of the point set onto itself mapping every

pair of points onto a congruent pair of points. It can be shown that (CT) is closely related to the existence of sufficiently many motions (in the sense that the group of motions has certain transitivity properties). There are approaches to euclidean geometry in which the existence of sufficiently many motions is the central axiom, see e.g. [8]. In the same sense, Hartshorne investigates in §17 of [3] the relation of the side-angle-side criterion SAS to the existence of sufficiently many motions. Euclid however never considers mappings, only figures, so we felt that an axiom like (CT) is more in his spirit.

Finally, we indicate other occurrences of axioms similar to (CT) in the literature. Versions of it appear in Karzel, Sörensen and Windelberg [7] (Axiom V6 in §16 p. 97), and in Rigby [10] (Axiom (C7) p. 22). These versions are stronger than (CT) in that they do not only postulate the existence of the point F , but also uniqueness. The use which the cited authors make of this axiom is different from our purpose here. Rigby remarks in [10, p. 23] that his axioms (C1)–(C5) together with the mentioned axiom (C7) are equivalent to Hilbert's axioms without further comment.

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Conflict of interest The authors declare that they have no conflict of interest.

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