Bull Braz Math Soc, New Series (2024) 55:41
https://doi.org/10.1007/s00574-024-00415-6

λ-Limited Sets in Banach and Dual Banach Spaces

Aleena Philip[1] · Manjul Gupta[2] · Deepika Baweja[1]

Received: 16 December 2023 / Accepted: 30 July 2024 / Published online: 12 August 2024

Abstract
In this paper, we introduce the notions of λ-limited sets and λ-L-sets in a Banach space X and its dual X^* respectively, using the vector valued sequence spaces $\lambda^{w^*}(X^*)$ and $\lambda^w(X)$. We find characterizations for these sets in terms of absolutely λ-summing operators and investigate the relationship between λ-compact sets and λ-limited sets, with a particular focus on the crucial role played by a norm iteration property. We also consider λ-limited operators and show that this class is an operator ideal containing the ideal of λ-compact operators for a suitably restricted λ. Furthermore, we define a generalized Gelfand-Philips property for Banach spaces corresponding to an abstract sequence space.

Keywords Banach sequence spaces · Absolutely λ-summing operators · λ-limited sets · λ-L-sets

Mathematics Subject Classification 46A45 · 46B45 · 47B10

1 Introduction

Limited sets were first introduced by Phillips (1940) in 1940 as a counter example to disprove the following characterization for compact sets given by Gelfand (1938): a subset of a Banach space is compact if and only if every weak* null sequence converges uniformly on that set. The existence of a non-compact set satisfying this property has provoked the interest of several researchers, and since then limited sets and its related notions have been studied extensively in the literature (see Chen et al. 2014; Delgado and Pineiro 2014; Galindo and Miranda 2022, 2024; Schlumprecht 1987). Recently,

✉ Deepika Baweja
deepika@hyderabad.bits-pilani.ac.in

Aleena Philip
aleenaphilip11@gmail.com

Manjul Gupta
manjul@iitk.ac.in

1 Department of Mathematics, BITS Pilani Hyderabad, Hyderabad 500078, India

2 Department of Mathematics and Statistics, IIT Kanpur, Kanpur 208016, India

Karn and Sinha (2014) introduced the concept of p-limited sets for $1 \leq p < \infty$ by replacing c_0 with ℓ_p in an alternate definition of limited sets. In the same direction, the so-called L-sets are defined in a similar way to limited sets in dual Banach spaces and it also has been defined corresponding to ℓ_p sequence spaces, called p-L-sets for $1 \leq p < \infty$. In this paper, we generalize the above notions for abstract sequence spaces λ.

In Sect. 3, we study the vector valued sequence space $\lambda^{w^*}(X^*)$ corresponding to a sequence space λ and the dual X^* of a Banach space X. Using the identification of $\lambda^{w^*}(X^*)$ with a class of bounded linear operators, we obtain its relationship with other vector valued sequence spaces. Section 4 is devoted to the study of the norm iteration property which plays a vital role in the study of λ-limited sets. We prove that the dual norm of a monotone Banach AK-space has this norm iteration property. Our results hold for several sequence spaces including ℓ_p spaces, Orlicz, Modular, Lorentz spaces (cf. Kamthan and Gupta 1981; Lindenstrauss and Tzafriri 1977), and the sequence spaces $\mu_{a,p}$ and $\nu_{a,p}$ given by Garling (1969). In Sect. 5, we introduce the notion of λ-limited sets in Banach spaces and λ-limited operators between Banach spaces. We investigate the relations between λ-compact sets (Gupta and Bhar 2013) and λ-limited sets, and show that the class of λ-limited operators forms an operator ideal. Furthermore, we introduce the concept of generalized Gelfand-Philips property for Banach spaces. The final section focuses on the concept of λ-L-sets in the dual of a Banach space. The results of this paper generalize some of the results proved by Karn and Sinha (2014), Delgado and Pineiro (2014) and Ghenciu (2023).

2 Preliminaries

We use the letters X and Y to denote Banach spaces over the field $\mathbb{K}$ of real or complex numbers, and B_X to denote the closed unit ball of X. For a Banach space $(X, \|\cdot\|)$, the symbol X^* denotes its topological dual equipped with the operator norm topology. We denote by $\mathcal{L}(X, Y)$, the space of continuous linear operators from X to Y.

Let ω denote the vector space of all scalar sequences defined over the field $\mathbb{K}$ with respect to the usual vector addition and scalar multiplication. The symbol e_n represents the n^{th} unit vector in ω, and ϕ the vector subspace spanned by the set $\{e_n : n \geq 1\}$. A ***sequence space*** λ is a subspace of ω such that $\phi \subseteq \lambda$. A sequence space λ is said to be **(i)** ***symmetric*** if $(\alpha_{\pi(n)})_n \in \lambda$ for all permutations π, whenever $(\alpha_n)_n \in \lambda$; **(ii)** ***normal*** if $(\beta_n)_n \in \lambda$, whenever $|\beta_n| \leq |\alpha_n|, \forall n \in \mathbb{N}$ and some $(\alpha_n)_n \in \lambda$. The ***cross dual*** or K*öthe dual* of λ is the sequence space $\lambda^{\times}$ defined as

$$\lambda^{\times} = \left\{ (\beta_n)_n \in \omega : \sum_n |\alpha_n||\beta_n| < \infty, \forall (\alpha_n)_n \in \lambda \right\}.$$

If $\lambda = \lambda^{\times\times}$, then λ is called a ***perfect sequence space***. For sequence spaces λ and μ, we write

$$\lambda.\mu = \{(\alpha_j \beta_j)_j : (\alpha_j)_j \in \lambda, (\beta_j)_j \in \mu\}.$$

A sequence space λ is said to be ***monotone*** if $m_0\lambda \subseteq \lambda$, where m_0 is the span of the set of all sequences of zeros and ones. A sequence space λ equipped with a linear topology is said to be a ***K-space*** if each of the projections $P_n : \lambda \to \mathbb{K}$, given by $P_n((\alpha_i)_i) = \alpha_n$, are continuous. A Banach K-space $(\lambda, \|\cdot\|_\lambda)$ is called a ***BK-space***. A BK-space $(\lambda, \|\cdot\|_\lambda)$ is said to be an ***AK-space*** if $\sum_{n=1}^m \alpha_n e_n$ converges to $(\alpha_n)_n$ for every $(\alpha_n)_n \in \lambda$. For a BK-space $(\lambda, \|\cdot\|_\lambda)$ with $0 < \sup_n \|e_n\|_\lambda < \infty$, the space $\lambda^\times$ is a BK-space endowed with the norm, $\|(\beta_n)_n\|_{\lambda^\times} = \sup_{(\alpha_n)_n \in B_\lambda}\{\sum_{n=1}^\infty |\alpha_n\beta_n|\}$.

For a sequence space $(\lambda, \|\cdot\|_\lambda)$, the norm $\|\cdot\|_\lambda$ is said to be **(i)** ***k-symmetric*** if $\|(\alpha_n)_n\|_\lambda = \|(\alpha_{\pi(n)})_n\|_\lambda$ for all permutations π, and **(ii)** ***monotone*** if $\|(\alpha_n)_n\|_\lambda \leq \|(\beta_n)_n\|_\lambda$ whenever $|\alpha_n| \leq |\beta_n| \,\forall n$. It is proved in (Garling 1974) that for a normal BK-space, there exists an equivalent norm which is monotone. Therefore, we henceforth assume that a normal BK-space λ is equipped with a monotone norm.

Proposition 2.1 (Gupta and Bhar 2013, Proposition 3.3) *Let* $(\lambda, \|\cdot\|_\lambda)$ *be a normal Banach* AK*-space such that* $0 < \sup_n \|e_n\|_\lambda < \infty$. *Then the topological dual* λ^* *of* λ*, is isometrically isomorphic to* $\lambda^\times$ *and we write*

$$(\lambda^*, \|\cdot\|) \cong (\lambda^\times, \|\cdot\|_{\lambda^\times}).$$

The vector valued sequence spaces $\lambda^s(X)$ and $\lambda^w(X)$ associated to a sequence space λ and a Banach space X were introduced by A. Pietsch in Pietsch (1962) as

$$\lambda^s(X) = \{(x_n)_n \subset X : (\|x_n\|)_n \in \lambda\}$$

and

$$\lambda^w(X) = \{(x_n)_n \subset X : (f(x_n))_n \in \lambda, \forall f \in X^*\}.$$

These spaces are Banach spaces equipped with the norm $\|(x_n)_n\|_\lambda^s = \|(\|x_n\|)_n\|_\lambda$ and $\|(x_n)_n\|_\lambda^w = \sup_{f \in B_{X^*}} \|(f(x_n))_n\|_\lambda$ respectively.

An ***operator ideal*** $\mathcal{I}$ is a subclass of the class $\mathcal{L}$ of all continuous linear operators between Banach spaces such that, for all Banach spaces X and Y, its components $\mathcal{I}(X, Y) = \mathcal{L}(X, Y) \cap \mathcal{I}$ satisfy the following properties:

- $\mathcal{I}$ contains the class of all finite rank operators.
- The class $\mathcal{I}(X, Y)$ is a subspace of the class $\mathcal{L}(X, Y)$.
- If $U \in \mathcal{L}(X_0, X)$, $S \in \mathcal{L}(Y, Y_0)$ and $T \in \mathcal{I}(X, Y)$, then $STU \in \mathcal{I}(X_0, Y_0)$.

The component of the **dual ideal** of $\mathcal{I}$ is given by

$$\mathcal{I}^d(X, Y) = \{T : X \to Y : T^* \in \mathcal{I}(Y^*, X^*)\}.$$

The subclass $\mathcal{I}^d$ of $\mathcal{L}$ is an operator ideal.

We refer the reader to Pietsch (1980) for a detailed theory of operator ideals.

The class of absolutely λ-summing operators was introduced by Ramanujan in Ramanujan (1970a).

Definition 2.2 A continuous linear operator $T : X \to Y$ is said to be absolutely λ-summing if for each $(x_n)_n \in \lambda^w(X)$, the sequence $(Tx_n) \in \lambda^s(Y)$.

The space of all absolutely λ-summing operators from X to Y, denoted by $\Pi_\lambda(X, Y)$, is a Banach space endowed with norm

$$\pi_\lambda(T) = \sup_{(x_n)_n \in B_{\lambda^w(X)}} \|(Tx_n)_n\|_\lambda^s.$$

Moreover, $(\Pi_\lambda, \pi_\lambda)$ is a Banach operator ideal.

Relating $\Pi_\lambda(X, Y)$ with $\Pi_\mu(X, Y)$ associated with spaces λ and μ, we have the following

Theorem 2.3 (Dubinsky and Ramanujan 1971, Theorem 3.1) *Let X, Y be Banach spaces, and λ, μ be normal sequence spaces and $\nu = \left(\lambda^\times.\mu\right)^\times$. If $\left(\nu.\lambda^\times\right)^\times \subset \mu$ and $\nu.\mu \subset \lambda$, then every absolutely λ-summing map from X to Y is μ-summing.*

Using the characteristic of compact sets of being sets of elements in the closed convex hull of null sequences, Sinha and Karn (2002) introduced the concept of p-compact sets, for $1 \leq p < \infty$. Later, Gupta and Bhar (2013) generalize the above notions for an arbitrary sequence space λ:

Definition 2.4 (i) A subset K of a Banach space X is said to be λ**-compact** if there exists $(x_n)_n \in \lambda^s(X)$ such that,

$$K \subset \left\{ \sum_{n=1}^{\infty} \alpha_n x_n : (\alpha_n)_n \in B_{\lambda^\times} \right\}.$$

(ii) An operator $T \in \mathcal{L}(X, Y)$ is said to be λ**-compact** if $T(B_X)$ is a λ-compact subset of Y.

Let λ be a monotone symmetric sequence space equipped with a k-symmetric norm $\|\cdot\|_\lambda$ such that $(\lambda, \|\cdot\|_\lambda)$ is a BK-space. Then the space of all λ-compact operators from X to Y denoted by $K_\lambda(X, Y)$ is a quasi-normed space with respect to the quasi-norm

$$k_\lambda(T) = \inf \left\{ \|(y_n)_n\|_\lambda^s : (y_n)_n \in \lambda^s(X) \text{ such that } T(B_X) \subset \left\{ \sum_{n=1}^{\infty} \alpha_n y_n : (\alpha_n)_n \in B_{\lambda^\times} \right\} \right\}.$$

In addition, if $\|\cdot\|_\lambda$ satisfies $0 < \inf_n \|e_n\|_\lambda \leq \sup_n \|e_n\|_\lambda < \infty$, then (K_λ, k_λ) is a quasi normed operator ideal.

In Gelfand (1938), Gelfand gave the following characterization for compact sets: a subset B of a Banach space is compact if and only every weak* null sequence converges uniformly on B. Phillips (1940) disproved this characterization by providing an example of a non-compact set satisfying this property. This gives rise to the notion of limited subsets of a Banach space on which every weak* null sequence converges uniformly. An equivalent definition for limited sets is given as follows:

Definition 2.5 A set $A \subset X$ is said to be **limited** if for every $(f_n)_n \in c_0^{w^*}(X^*)$, there exists a sequence $(\alpha_n)_n \in c_0$ such that $| \langle x, f_n \rangle | \leq \alpha_n$ for every $n \in \mathbb{N}$ and $x \in A$.

Based on this characterization, Karn and Sinha (2014) introduced the notion of p-limited sets and p-limited operators. Indeed, for $1 \leq p < \infty$, a subset A of a Banach space X is p**-limited** if for every $(f_n)_n \in \ell_p^{w^*}(X^*)$, there exists a sequence $(\alpha_n)_n \in \ell_p$ such that $| \langle x, f_n \rangle | \leq \alpha_n$ for every $n \in \mathbb{N}$ and $x \in A$; and an operator $T \in \mathcal{L}(X, Y)$ is p**-limited** if $T(B_X)$ is a p-limited subset of Y. Note that the vector valued sequence spaces $c_0^{w^*}(X^*)$ and $\ell_p^{w^*}(X^*)$ are specific cases of the generalized sequence space $\lambda^{w^*}(X^*)$ associated with the dual X^* of a Banach space X, and correspond to $\lambda = c_0$ and $\lambda = \ell_p$ (for $1 \leq p < \infty$), respectively, as discussed in the beginning of Sect. 3.

It is clear that every compact set is limited. But the converse need not be true (see Phillips 1940). Therefore, it is natural to ask which Banach spaces have the property that every limited (resp. p-limited) set is compact (resp. p-compact). Such Banach spaces are said to have **Gelfand-Philips property** (Diestel and Uhl 1983) (resp. **Gelfand-Philips property of order** p (Karn and Sinha 2014)).

The following notions of L-sets and p-L sets defined in the dual X^* of a Banach space X are given in Ghenciu (2023).

Definition 2.6 A subset $F \subset X^*$ is said to be an L**-set** if each weakly null sequence $(x_n)_n \subset X$ converges to 0 uniformly on F, and a p-L**-set** if for each $(x_n)_n \in \ell_p^w(X)$, there exists a sequence $(\alpha_n)_n \in \ell_p$, such that $| \langle x_n, f \rangle | \leq \alpha_n$ for every $n \in \mathbb{N}$ and $f \in F$, where $p \in [1, \infty)$.

3 Relationship between the spaces $\lambda^{w^*}(X^*)$ and $\lambda^w(X^*)$

The main result of this section is the equality $\lambda^{w^*}(X^*) = \lambda^w(X^*)$ for a reflexive AK-BK sequence space λ. We use this equality to obtain a characterization of λ-limited sets in terms of absolutely λ-summing operators.

Given a sequence space λ and the dual X^* of a Banach space X, Fourie and Swart (1979) introduced the vector valued sequence space

$$\lambda^{w^*}(X^*) = \{(f_n)_n \subset X^* : (f_n(x))_n \in \lambda, \forall x \in X\}$$

which is a Banach space endowed with the norm

$$\|(f_n)_n\|_\lambda^{w^*} = \sup_{x \in B_X} \|(f_n(x))_n\|_\lambda.$$

Now we cite the result (Fourie and Swart 1979, Proposition 2.2(a)) along with the proof for the reader's convenience.

Proposition 3.1 *Let X be a Banach space and $(\lambda, \| \cdot \|_\lambda)$ be a normal Banach AK-space such that $0 < \sup_n \|e_n\|_\lambda < \infty$. Then the linear map $H : \lambda^{w^*}(X^*) \to \mathcal{L}(X, \lambda)$ defined as $H(\bar{f}) = H_{\bar{f}}$, where $H_{\bar{f}}(x) = (f_n(x))_n$, for $\bar{f} = (f_n)_n \in \lambda^{w^*}(X^*)$, $x \in X$, is an isometric isomorphism.*

Proof Let $\bar{f} = (f_n)_n \in \lambda^{w^*}(X^*)$. Then,

$$\|H(\bar{f})\| = \|H_{\bar{f}}\| = \sup_{x \in B_X} \|(f_n(x))_n\|_\lambda = \|\bar{f}\|_\lambda^{w^*}$$

proves that H is an isometry. For proving H is onto, consider $A \in \mathcal{L}(X, \lambda)$. Then, due to Proposition 2.1, $A^* \in \mathcal{L}(\lambda^\times, X^*)$. Since $\{e_n\}_n$ is a Schauder basis for λ,

$$\big(\langle x, A^* e_n \rangle\big)_n = (\langle Ax, e_n\rangle)_n = Ax \in \lambda, \quad \forall x \in X,$$

it follows that $(A^* e_n)_n \in \lambda^{w^*}(X^*)$ and if we set $\bar{f} = (A^* e_n)_n$,

$$H_{\bar{f}}(x) = \big(\langle x, A^* e_n \rangle\big)_n = (\langle Ax, e_n\rangle)_n = Ax, \forall x \in X.$$

□

Restricting λ further to be a reflexive sequence space, we prove the following

Proposition 3.2 *Let X be a Banach space and $(\lambda, \|\cdot\|_\lambda)$ be a reflexive Banach AK-space such that $0 < \sup_n \|e_n\|_\lambda < \infty$. Then the linear map $L : \lambda^w(X) \to \mathcal{L}(\lambda^\times, X)$ defined as $L(\bar{x}) = L_{\bar{x}}$ is an isometric isomorphism, where $\bar{x} = (x_n)_n \in \lambda^w(X)$ and $L_{\bar{x}}(\bar{\alpha}) = \sum_n^\infty \alpha_n x_n, \forall\ \bar{\alpha} = (\alpha_n)_n \in \lambda^\times$.*

Proof By (Gupta and Bhar 2013, Proposition 3.5), the map L is an isometry. Consider $A \in \mathcal{L}(\lambda^\times, X)$. Then $A^* \in \mathcal{L}(X^*, (\lambda^\times)^*) = \mathcal{L}(X^*, \lambda)$, since λ is reflexive (and therefore perfect) and $\{e_n\}_n$ becomes a Schauder basis. Hence

$$(\langle Ae_n, f\rangle)_n = \big(\langle e_n, A^* f\rangle\big)_n = A^* f \in \lambda, \quad \forall f \in X^*,$$

and thus $(Ae_n)_n \in \lambda^w(X)$.

Moreover, for $\bar{x} = (Ae_n)_n$ and $\bar{\alpha} \in \lambda^\times$,

$$\begin{aligned} L_{\bar{x}}(\bar{\alpha}) &= \sum_{n=1}^\infty \alpha_n A e_n \\ &= A\left(\sum_{n=1}^\infty \alpha_n e_n\right) \\ &= A(\bar{\alpha}) \end{aligned}$$

which implies that L is onto. □

Remark 3.3 If λ is not reflexive, Proposition 3.2 need not be true. For instance, if $\lambda = (\ell_1, \|\cdot\|_1)$, then for any Banach space X, we have $\ell_1^w(X) \cong \mathcal{L}(c_0, X)$ (Diestel et al. 1995, p.36). If we consider $X = c_0$, then the identity operator on c_0 cannot be extended to ℓ_∞ since c_0 is not complemented in ℓ_∞ (Megginson 2012, p.301). Therefore, $\mathcal{L}(\ell_\infty, X) \subsetneq \mathcal{L}(c_0, X)$.

Note that there is a resemblance between (Botelho and Santiago 2024, Proposition 3.10) and Proposition 3.2, but the objectives (and thus the approaches) of the papers are different. In Botelho and Santiago (2024) the authors investigate sequence classes that can be represented by operator ideals, whereas we establish a relation between the sequence space $\lambda^w(X)$ and the space of all linear operators from $\lambda^\times$ to X. However, under the restriction that λ is a reflexive AK-BK space with $\|e_n\|_\lambda = 1$, both results are equivalent. Remark 3.3 emphasizes the importance of reflexivity in Proposition 3.2.

Finally we prove that for a suitably restricted λ, the vector valued sequence spaces $\lambda^w(X^*)$ and $\lambda^{w^*}(X^*)$ coincide for every Banach space X.

Theorem 3.4 *Let $(\lambda, \|\cdot\|_\lambda)$ be a reflexive Banach AK-space such that $0 < \sup_n \|e_n\|_\lambda < \infty$. Then, for any Banach space X, it follows that $\lambda^w(X^*) = \lambda^{w^*}(X^*)$.*

Proof It is easy to see that $\lambda^w(X^*) \subseteq \lambda^{w^*}(X^*)$. To prove the reverse inclusion, consider a sequence $\bar{f} = (f_n)_n \in \lambda^w(X^*)$. By Proposition 3.1, there exists a linear map $T_{\bar{f}} \in \mathcal{L}(X, \lambda)$ such that $T_{\bar{f}}(x) = (f_n(x))_n$. For $\bar{\alpha} \in \lambda^\times$ and $x \in X$,

$$\begin{aligned}\left\langle T^*_{\bar{f}}(\bar{\alpha}), x\right\rangle &= \left\langle \bar{\alpha}, T_{\bar{f}}x\right\rangle \\ &= \sum_{n=1}^{\infty} \alpha_n f_n(x) \quad \forall x \in X,\end{aligned}$$

where $g = \sum_{n=1}^{\infty} \alpha_n f_n \in X^*$ by Banach-Steinhaus theorem. Hence $T^*_{\bar{f}}(\bar{\alpha}) = g$.

Since $T^*_{\bar{f}} \in \mathcal{L}(\lambda^\times, X^*)$, by Proposition 3.2 there exists a sequence $\bar{h} = (h_n) \in \lambda^w(X^*)$ such that $T^*_{\bar{f}} = L_{\bar{h}}$. Therefore, for every $\bar{\alpha} = (\alpha_n)_n \in \lambda^\times$,

$$\begin{aligned}T^*_{\bar{f}}(\bar{\alpha}) &= L_{\bar{h}}(\bar{\alpha}) \\ \sum_{n=1}^{\infty} \alpha_n f_n &= \sum_{n=1}^{\infty} \alpha_n h_n.\end{aligned}$$

In particular, taking $\bar{\alpha} = e_n$, we get $f_n = h_n$ for each $n \in \mathbb{N}$ and hence $\bar{f} = \bar{h} \in \lambda^w(X^*)$. □

4 Norm iteration property of $\|\cdot\|_\lambda$

In this section, we study the norm iteration property of the dual norm of a sequence space λ and establish the conditions on λ such that the norm $\|\cdot\|_{\lambda^\times}$ has the norm iteration property.

The norm iteration property is defined in Ramanujan (1970b) by the following

Definition 4.1 A sequence space $(\lambda, \|\cdot\|_\lambda)$ is said to have the ***norm iteration property*** if for each sequence $(\bar{\alpha}_n)_n = ((\alpha_j^n)_j)_n$ in λ, the sequence $\bar{\alpha}_j = (\alpha_j^n)_n \in \lambda, \forall j \in \mathbb{N}$ and $\left\|(\|(\bar{\alpha}_n)\|_\lambda)_n\right\|_\lambda = \left\|\left(\|(\bar{\alpha}_j)\|_\lambda\right)_j\right\|_\lambda$.

To prove the main result of this section we need the following lemma, cf. (Nogueira 2016, Lema 2.4.10):

Lemma 4.2 *Let A and B be non empty sets and $f : A \times B \to \mathbb{R}$ be a function. Then*

$$\sup_{a\in A}\sup_{b\in B} f(a,b) = \sup_{b\in B}\sup_{a\in A} f(a,b).$$

Theorem 4.3 *Let $(\lambda, \|\cdot\|_\lambda)$ be a monotone Banach AK-space such that $0 < \inf_n \|e_n\|_\lambda \le \sup_n \|e_n\|_\lambda < \infty$. Then the norm $\|\cdot\|_{\lambda^\times}$ has the norm iteration property.*

Proof Since $(\lambda^\times, \|\cdot\|_{\lambda^\times})$ is topologically isomorphic to $(\lambda^*, \|\cdot\|)$, we have

$$\|\bar{\alpha}\|_{\lambda^\times} = \sup_{(\beta_n)_n\in B_\lambda} \sum_{n=1}^\infty |\alpha_n\beta_n|$$

for any $\bar{\alpha} = (\alpha_n)_n \in \lambda^\times$. Clearly $\|\cdot\|_{\lambda^\times}$ is monotone. To prove that $\|\cdot\|_{\lambda^\times}$ has norm iteration property, consider the sequence $(\bar{\alpha_n})_n \in (\lambda^\times)^s(\lambda^\times)$ where $\bar{\alpha_n} = (\alpha_n^j)_j \in \lambda^\times$, for all $n \in \mathbb{N}$. Then for each $j \in \mathbb{N}$, $\alpha_n^j e_j \in \lambda^\times$ and

$$|\alpha_n^j|\|e_j\|_{\lambda^\times} = \|\alpha_n^j e_j\|_{\lambda^\times} \le \|\bar{\alpha_n}\|_{\lambda^\times}.$$

Thus, we have

$$\sum_{n=1}^\infty |\alpha_n^j\gamma_n| \le \frac{1}{\|e_j\|_{\lambda^\times}} \sum_{n=1}^\infty \|\bar{\alpha_n}\|_{\lambda^\times}|\gamma_n| \le \frac{1}{c}\sum_{n=1}^\infty \|\bar{\alpha_n}\|_{\lambda^\times}|\gamma_n| < \infty \tag{1}$$

where $(\gamma_n)_n \in \lambda$ and $c = \inf_j \|e_j\|_{\lambda^\times}$. Hence $\widehat{\alpha^j} = (\alpha_n^j)_n \in \lambda^\times$ for each $j \in \mathbb{N}$.

In order to prove that $(\widehat{\alpha^j})_j \in (\lambda^\times)^s(\lambda^\times)$ consider $\bar{\beta} = (\beta_j)_j \in \lambda$. Then for $m \in \mathbb{N}$,

$$\begin{aligned}
\sum_{j=1}^m \|\widehat{\alpha^j}\|_{\lambda^\times}|\beta_j| &= \sum_{j=1}^m \left(\sup_{(\gamma_n)_n\in B_\lambda}\sum_{n=1}^\infty |\alpha_n^j\gamma_n|\right)|\beta_j| = \sup_{(\gamma_n)_n\in B_\lambda}\sum_{j=1}^m\sum_{n=1}^\infty |\alpha_n^j\beta_j\gamma_n| \\
&= \sup_{(\gamma_n)_n\in B_\lambda}\lim_k \sum_{j=1}^m\sum_{n=1}^k |\alpha_n^j\beta_j\gamma_n| = \sup_{(\gamma_n)_n\in B_\lambda}\lim_k \sum_{n=1}^k\left(\sum_{j=1}^m |\alpha_n^j\beta_j|\right)|\gamma_n| \\
&\le \sup_{(\gamma_n)_n\in B_\lambda}\sum_{n=1}^\infty \left(\|\bar{\alpha_n}\|_{\lambda^\times}\|(\beta_j)_j\|_\lambda\right)|\gamma_n| \le \|(\beta_j)_j\|_\lambda\|(\bar{\alpha_n})_n\|_{\lambda^\times}^s.
\end{aligned}$$

Hence $(\widehat{\alpha^j})_j \in (\lambda^\times)^s(\lambda^\times)$. Observing that

$$\begin{aligned}\|(\bar{\alpha_n})_n\|_{\lambda^\times} &= \sup_{(\gamma_n)_n\in B_\lambda} \sum_{n=1}^{\infty} \left(\sup_{(\beta_j)_j\in B_\lambda} \sum_{j=1}^{\infty} |\alpha_n^j \beta_j| \right) |\gamma_n| \\ &= \sup_{(\gamma_n)_n\in B_\lambda} \sup_{(\beta_j)_j\in B_\lambda} \sum_{n=1}^{\infty}\sum_{j=1}^{\infty} |\alpha_n^j \beta_j \gamma_n|\end{aligned}$$

and

$$\|(\widehat{\alpha^j})_j\|_{\lambda^\times} = \sup_{(\beta_j)_j\in B_\lambda} \sup_{(\gamma_n)_n\in B_\lambda} \sum_{j=1}^{\infty}\sum_{n=1}^{\infty} |\alpha_n^j \beta_j \gamma_n|$$

it follows that the sequence $((\alpha_n^j \beta_j \gamma_n)_j)_n \in \ell_1^s(\ell_1)$ and hence

$$\sum_{n=1}^{\infty}\sum_{j=1}^{\infty} |\alpha_n^j \beta_j \gamma_n| = \sum_{j=1}^{\infty}\sum_{n=1}^{\infty} |\alpha_n^j \beta_j \gamma_n|,$$

using the norm iteration property of the norm of ℓ_1.

Define $f : \lambda \times \lambda \to \mathbb{K}$ by $f((\beta_j)_j, (\gamma_n)_n) = \sum_{j=1}^{\infty}\sum_{n=1}^{\infty} |\alpha_n^j \beta_j \gamma_n|$. Then applying Lemma 4.2 to the function f, we get $\|(\bar{\alpha_n})_n\|_{\lambda^\times} = \|(\widehat{\alpha^j})_j\|_{\lambda^\times}$ and thus the norm iteration property of $\|\cdot\|_{\lambda^\times}$ follows. □

From the above theorem, we have the following examples of sequence spaces with its norm having norm iteration property:

- For $1 \le p < \infty$, $\ell_p = \{(\alpha_j)_j \in \omega : \sum\limits_{j=1}^{\infty} |\alpha_j|^p < \infty\}$ with norm $\|(\alpha_j)_j\|_p = \left(\sum\limits_{j=1}^{\infty} |\alpha_j|^p\right)^{1/p}$.
- **Orlicz sequence space** ℓ_M (Kamthan and Gupta 1981, p.297), defined as

$$\ell_M = \{(\alpha_j)_j \in \omega : \sum_{j=1}^{\infty} M\left(\frac{|\alpha_j|}{k}\right) < \infty \text{ for some } k > 0\}$$

  with respect to the norm

$$\|(\alpha_j)_j\|_{(M)} = \inf\left\{k > 0 : \sum_{j=1}^{\infty} M\left(\frac{|\alpha_j|}{k}\right) \le 1\right\},$$

  where M is an Orlicz function satisfying Δ_2 condition (for instance, $M(x) = e^x - 1$).

- **Modular sequence spaces** $\ell_{\{M_j\}}$ (Kamthan and Gupta 1981, p.319) defined as

$$\ell_{\{M_j\}} = \{(\alpha_j)_j \in \omega : \sum_{j=1}^{\infty} M_j \left(\frac{|\alpha_j|}{k}\right) < \infty \text{ for some } k > 0\}$$

with respect to the norm,

$$\|(\alpha_j)_j\|_{(M_j)} = \inf \left\{ k > 0 : \sum_{j=1}^{\infty} M_j \left(\frac{|\alpha_j|}{k}\right) \leq 1 \right\},$$

where $(M_j)_j$ is a sequence of Orlicz functions.

- For $1 \leq p < \infty$, **Lorentz sequence spaces of order** p (Kamthan and Gupta 1981, p.323) defined as

$$d(x, p) = \left\{ (\alpha_j)_j \in c_0 : \sup \left\{ \sum_{j=1}^{\infty} x_j |\alpha_{\sigma(j)}|^p : \sigma \in \Pi \right\} < \infty \right\},$$

where $x = (x_j)_j \in c_0, x \notin \ell_1$ such that $x_j > 0 \;\; \forall j$ and $1 = x_1 \geq x_2 \geq \cdots$ endowed with the norm

$$\|(\alpha_j)_j; p\| = \sup \left\{ \sum_{j=1}^{\infty} x_j |\alpha_{\sigma(j)}|^p : \sigma \in \Pi \right\}.$$

- **The sequence spaces** $m(\bar{\phi})$ and $n(\bar{\phi})$ introduced by Sargent in Sargent (1960). For a sequence $\bar{\alpha} = (\alpha_j)_j$, define $\Delta\alpha_j = (\alpha_j - \alpha_{j-1})$, $\alpha_0 = 0$; $S(\bar{\alpha})$ denotes the collection of all sequences which are permutations of $\bar{\alpha}$. $\mathfrak{C}$ is the set of all finite sequences of positive integers. For $\sigma \in \mathfrak{C}$ define $c(\sigma) = (c_j(\sigma))$, where $c_j(\sigma) = 1$ if $j \in \sigma$ and 0 otherwise. Let $\mathfrak{C}_s = \{\sigma \in \mathfrak{C} : \sum_{j=1}^{\infty} c_j(\sigma) \leq s\}$. Let $\bar{\phi} = (\phi_j)$ be a given sequence such that for each j, $0 < \phi_1 \leq \phi_j \leq \phi_{j+1}$ and $(j+1)\phi_j > j\phi_{j+1}$. Then the sequence spaces

$$m(\bar{\phi}) = \left\{ \bar{\alpha} : \|\bar{\alpha}\| = \sup_{s \geq 1} \sup_{\sigma \in \mathfrak{C}_s} \left(\frac{1}{\phi_s} \sum_{j \in \sigma} |\alpha_j| \right) < \infty \right\}$$

and

$$n(\bar{\phi}) = \left\{ \bar{\alpha} : \|\bar{\alpha}\| = \sup_{u \in S(\bar{\alpha})} \sum_j |u_j| \Delta\phi_j < \infty \right\}$$

are perfect BK-spaces which are Kothe duals of each other. Additionally, the sequence space $n(\bar{\phi})$ is an AK-space and, by Theorem 4.3, the norm of $m(\bar{\phi})$ has the norm iteration property.

Remark 4.4 Note that there are sequence spaces with the norm iteration property without being cross duals of other sequence spaces. For instance, the AK-BK space c_0, of all sequences converging to zero with respect to supremum norm, has the norm iteration property.

Remark 4.5 Additionally, it should be noted that $\ell_\infty^\times = \ell_1$ possesses the norm iteration property, despite ℓ_∞ not being an AK space.

Remarks 4.4 and 4.5 indicate that the conditions stated in Theorem 4.3 are only necessary, not sufficient.

5 λ-limited sets

In this section, we introduce the concepts of λ-limited sets and λ-limited operators associated to a sequence space λ and give characterizations in terms of absolutely λ-summing operators.

Definition 5.1 Let λ be a sequence space. Then a subset A of a Banach space X is said to be λ-limited, if for each $(f_n)_n \in \lambda^{w^*}(X^*)$, there exists a sequence $(\alpha_n)_n \in \lambda$ such that

$$|f_n(x)| \leq \alpha_n, \qquad \forall\, n \in \mathbb{N}, x \in A.$$

Thus p-limited (limited) sets are precisely λ-limited sets for $\lambda = \ell_p (\lambda = c_0)$. Some elementary results about λ-limited sets are given in the following

Proposition 5.2 *Let λ be a Banach sequence space and A, B be subsets of a Banach space X.*

(i) *Every λ-limited set is bounded.*
(ii) *If A is λ-limited, then $\bar{A}$ is λ-limited.*
(iii) *If A, B are λ-limited, then $A + B$, $A \cup B$, $A \cap B$ are λ-limited.*
(iv) *If $A \subseteq B$ and B is λ-limited, then A is λ-limited.*
(v) *If A is λ-limited, then $T(A)$ is λ-limited for each $T \in \mathcal{L}(X, Y)$.*
(vi) *If λ is normal and A is λ-limited, then αA is λ-limited for each $\alpha \in \mathbb{K}$.*

Proof (i) If A is λ-limited, then for each $f \in X^*$, $(f, 0, 0, \cdots) \in \lambda^{w^*}(X^*)$ and there exists a sequence $\bar{\alpha_f} = (\alpha_{fn})_n \in \lambda$ such that

$$|f(x)| \leq \alpha_{f1}, \ \ \forall\, x \in A.$$

Therefore A is weakly bounded and hence bounded.

(ii) If $x \in \bar{A}$, then there exists a sequence $(x_k)_k \subseteq A$ such that x_k converges to x. For each $(f_n)_n \in \lambda^{w^*}(X^*)$ there exists $(\alpha_n)_n \in \lambda$ such that $|f_n(x_k)| \leq \alpha_n, \forall k, n \in \mathbb{N}$. Letting $k \to \infty$, we get

$$|f_n(x)| \leq \alpha_n, \forall\, n \in \mathbb{N}, x \in \bar{A}$$

and so $\bar{A}$ is λ-limited. (iii), (iv) and (vi) have straightforward proofs.

(v) For every $(g_n)_n \in \lambda^{w^*}(Y^*)$, $(T^* g_n)_n \in \lambda^{w^*}(X^*)$. As A is λ-limited, there exists $(\alpha_n)_n \in \lambda$ such that

$$|g_n(Tx)| = |\langle x, T^* g_n \rangle| \leq \alpha_n \quad \forall\, x \in A, n \in \mathbb{N}.$$

Thus $T(A)$ is λ-limited. □

A necessary and sufficient condition for the closed unit ball in X to be λ-limited is given by the following

Theorem 5.3 *Let λ be a normal sequence space and X be a Banach space. Then B_X is λ-limited if and only if $\lambda^{w^*}(X^*) = \lambda^s(X^*)$.*

Proof The result follows directly from the definition of λ-limited sets. □

As a consequence of the above result, we derive

Corollary 5.4 1. *The closed unit ball B_X of an infinite dimensional Banach space X is not limited in X.*
2. *For $1 \leq p < \infty$, the closed unit ball B_X of an infinite dimensional Banach space X is not p-limited in X.*

Proof 1. By the Josefson-Neissenzweig theorem (Diestel 2012, p.219), there exists a weak* null sequence $(f_n)_n$ with $\|f_n\| = 1$ in X^*. Since $\|f_n\| = \sup_{x \in B_X} |f_n(x)| = 1$, B_X is not a limited subset of X.
2. Since $\ell_p^s(X^*) \subsetneq \ell_p^w(X^*)$, by Dvoretzky-Rogers theorem (Diestel et al. 1995, p.50), the result follows. □

Let us recall from Delgado and Pineiro (2014); Gupta and Bhar (2013), the bounded linear operator $U_A : \ell_1(A) \to X$ associated to a bounded subset A of a Banach space X, defined by $U_A((\eta_x)_x) = \sum_{x \in A} \eta_x x$. We obtain the following characterization for λ-limited sets using this operator.

Proposition 5.5 *Let X be a Banach space and $(\lambda, \|\cdot\|_\lambda)$ be a reflexive Banach AK-space such that $0 < \sup_n \|e_n\|_\lambda < \infty$. Then a subset A of X is λ-limited if and only if U_A^* is λ-summing.*

Proof Let $A \subseteq X$ be λ-limited. Since A is bounded, by Proposition 5.2, the operator U_A is bounded. To prove that U_A^* is absolutely λ-summing, consider $(f_n)_n \in \lambda^w(X^*) \subseteq \lambda^{w^*}(X^*)$. Since A is λ-limited, there exists a sequence $(\alpha_n)_n \in \lambda$ such that

$$\|U_A^*(f_n)\|_\infty = \sup_{x \in A} |f_n(x)| \leq \alpha_n \quad \forall\, x \in A, n \in \mathbb{N},$$

which implies that $\left(\|U_A^*(f_n)\|_\infty\right)_n \in \lambda$ and hence U_A^* is absolutely λ-summing.

Conversely, let U_A^* be λ-summing. Then using Theorem 3.4, for $(f_n)_n \in \lambda^{w^*}(X^*) = \lambda^w(X^*)$ and $x \in A$,

$$|f_n(x)| \leq \sup_{x \in A} |f_n(x)| = \|U_A^*(f_n)\|_\infty$$

proves that A is λ-limited. □

The next result follows from Proposition 5.5 and Theorem 2.3.

Theorem 5.6 *Let X be a Banach space and $(\lambda, \|\cdot\|_\lambda)$, $(\mu, \|\cdot\|_\mu)$ be normal Banach AK-spaces such that $0 < \sup_n \|e_n\|_\lambda, \sup_n \|e_n\|_\mu < \infty$. Also assume that μ is reflexive and $\nu = \left(\lambda^\times.\mu\right)^\times$. If $\left(\nu.\lambda^\times\right)^\times \subset \mu$ and $\nu.\mu \subset \lambda$, then every λ-limited subset of X is μ-limited.*

Corollary 5.7 *(*Delgado and Pineiro 2014, Proposition 2.1(3)*) If $1 \leq p \leq q < \infty$, then every p-limited set is q-limited.*

Proof Let $\frac{1}{r} = \frac{1}{p} - \frac{1}{q}$. Then for $(\alpha_n)_n \in \ell_q$ and $(\beta_n)_n \in \ell_r$, the sequence $(\alpha_n\beta_n)_n \in \ell_p$ by generalized Holder's inequality

$$\left(\sum_n |\alpha_n\beta_n|^p\right)^{1/p} \leq \left(\sum_n |\alpha_n|^q\right)^{1/q} \left(\sum_n |\beta_n|^r\right)^{1/r}. \tag{2}$$

Consider $\lambda = \ell_p$ and $\mu = \ell_q$, we get $\nu = (\ell_{p'} \cdot \ell_q)^\times = \ell_r$, and $(\nu.\lambda^\times)^\times = \ell_q = \mu$ and $\nu \cdot \mu = \ell_p = \lambda$ by (2). Thus, by Theorem 5.6, every p-limited set is q-limited. □

Corollary 5.8 *For a reflexive sequence space μ, every 1-limited set is μ-limited.*

Proof Let $\lambda = \ell_1$ and μ be any reflexive sequence space. Then μ being reflexive, is perfect and hence normal. Note that $\nu = (\ell_\infty \cdot \mu)^\times = \mu^\times$, $(\nu \cdot \lambda^\times)^\times = (\mu)^{\times\times} = \mu$ and $\nu \cdot \mu = \mu^\times \cdot \mu \subseteq \ell_1 = \lambda$. Hence, by Propositon 5.6, every 1-limited set is μ-limited. □

Remark 5.9 For $\lambda = \mu = c_0$, $\nu = \ell_\infty$ and $\nu \cdot \mu = c_0$, $(\nu.\lambda^\times)^\times = \ell_\infty \nsubseteq \mu$. Thus each λ-limited set is μ-limited set, but the condition $(\nu.\lambda^\times)^\times \subseteq \mu$ is being violated.

The next result establishes the relation between λ-compact subsets and λ-limited subsets of a Banach space.

Proposition 5.10 *Let λ be a normal sequence space such that $\|\cdot\|_\lambda$ has the norm iteration property. Then every λ-compact subset of a Banach space X is λ-limited.*

Proof Let $K \subseteq X$ be a λ-compact subset. Then there exists a sequence $(x_n)_n \in \lambda^s(X)$, such that

$$K \subseteq \left\{\sum_n \alpha_n x_n : (\alpha_n)_n \in B_{\lambda^\times}\right\}.$$

Let $(f_n)_n \in \lambda^{w^*}(X^*)$. Then

$$\|(f_n(x_k))_n\|_\lambda = \|x_k\| \left\| \left(f_n \left(\frac{x_k}{\|x_k\|} \right) \right)_n \right\|_\lambda \\ \leq \|x_k\| \|(f_n)_n\|_\lambda^{w^*} \quad \forall\, k \in \mathbb{N}.$$

Since λ is normal and $(x_k)_k \in \lambda^s(X)$, we get $(\|(f_n(x_k))_n\|_\lambda)_k \in \lambda$. Therefore, by the monotonicity and norm iteration property of $\|\cdot\|_\lambda$,

$$\left\| (\|(f_n(x_k))_k\|_\lambda)_n \right\|_\lambda = \left\| (\|(f_n(x_k))_n\|_\lambda)_k \right\|_\lambda \\ \leq \left\| \left(\|x_k\| \|(f_n)_n\|_\lambda^{w^*} \right)_k \right\|_\lambda < \infty.$$

Set $\beta_n = \|(f_n(x_k))_k\|_\lambda$ for each $n \in \mathbb{N}$. It is clear that $(\beta_n)_n \in \lambda$. Consider $x \in K$. Then $x = \sum_{k=1}^\infty \alpha_k x_k$ for some $(\alpha_k)_k \in B_{\lambda^\times}$. For $(f_n)_n \in \lambda^{w^*}(X^*)$,

$$|f_n(x)| = \left| \sum_k \alpha_k f_n(x_k) \right| \\ \leq \|(\alpha_k)_k\|_{\lambda^\times} \|(f_n(x_k))_k\|_\lambda \\ \leq \beta_n \quad \forall\, n \in \mathbb{N}.$$

Hence K is a λ-limited set. □

Example 5.11 For $\lambda = (\ell_p, \|\cdot\|_p)$, $1 \leq p < \infty$, the unit vector basis of c_0 is λ-limited but not λ-compact. Indeed $A = \{e_n : n \in \mathbb{N}\} \subset c_0$. Clearly A is not compact, and hence not p-compact. Since $U_A^* : c_0^* \to \ell_\infty(A)$ is the inclusion operator from ℓ_1 to ℓ_∞, by Grothendieck's theorem (Diestel et al. 1995, p.15), U_A^* is p-summing for $1 \leq p < \infty$. Therefore, by Proposition 5.5, A is p-limited.

Example 5.11 shows that the converse of Proposition 5.10 is not necessarily true. This leads to the following

Definition 5.12 A Banach space X is said to have the generalized Gelfand-Philips property if every λ-limited subset of X is λ-compact.

For proving a characterization for spaces having the generalized Gelfand-Philips property, we introduce the concept of λ-limited operators as follows:

Definition 5.13 A linear operator $T : X \to Y$ is said to be λ-limited if $T(B_X)$ is a λ-limited subset of Y.

Let us denote the class of λ-limited operators from X to Y by $\Pi_{\lambda,L}(X, Y)$. Then, if λ is normal, $\Pi_{\lambda,L}(X, Y)$ is a subspace of $\mathcal{L}(X, Y) = \Pi_\phi(X, Y)$, cf. (Dubinsky and Ramanujan 1971), where the former assertion follows from Proposition 5.2 and the second equality follows from the fact that $\phi^s(X) = \phi^w(X)$. Therefore we have, the following

Proposition 5.14 *$\Pi_{\lambda,L}$ is an operator ideal if λ is normal.*

Proof It is easy to see that every finite rank operator from X to Y is λ-limited and $\Pi_{\lambda,L}(X,Y)$ is a subspace of $\mathcal{L}(X,Y)$, for every Banach spaces X and Y. To prove the ideal property, let $T \in \Pi_{\lambda,L}(X,Y)$, $U \in \mathcal{L}(X_0,X)$ and $S \in \mathcal{L}(Y,Y_0)$. Then, for every $(f_j)_j \in \lambda^{w^*}(Y_0^*)$ and $x \in B_{X_0}$, there exists $(\alpha_j)_j \in \lambda$ such that,

$$\left|f_j(STU(x))\right| \leq \alpha_j \|U(x)\| \leq \alpha_j \|U\|.$$

This proves that $STU \in \Pi_{\lambda,L}(X_0,Y_0)$ and hence $\Pi_{\lambda,L}$ is an operator ideal. □

Moreover, by Proposition 5.10 and (Gupta and Bhar 2013, Theorem 3.10), a monotone symmetric sequence space λ equipped with a k-symmetric norm $\|\cdot\|_\lambda$ such that $(\lambda,\|\cdot\|_\lambda)$ is a BK-space with the norm iteration property, and $0 < \inf_n \|e_n\|_\lambda < \sup_n \|e_n\|_\lambda < \infty$, $K_\lambda(X,Y) \subseteq \Pi_{\lambda,L}(X,Y)$.

Proposition 5.15 *Let $(\lambda,\|\cdot\|_\lambda)$ be a reflexive Banach AK-space such that $0 < \sup_n \|e_n\|_\lambda < \infty$. Then $T \in \Pi_{\lambda,L}(X,Y)$ if and only if T^* is absolutely λ-summing.*

Proof Let $T \in \mathcal{L}(X,Y)$ be a λ-limited operator. Then for each $(g_n)_n \in \lambda^{w^*}(Y^*) = \lambda^w(Y^*)$, there exists a sequence $(\alpha_n)_n \in \lambda$ such that $|g(Tx)| \leq \alpha_n$, for all $x \in B_X$ and $n \in \mathbb{N}$. To prove that T^* is λ-summing, consider $(g_n)_n \in \lambda^w(Y^*) = \lambda^{w^*}(Y^*)$. Thus, we have

$$\|T^* g_n\| = \sup_{x \in B_X} \left|T^* g_n(x)\right| \leq \alpha_n, \ \forall\, n \in \mathbb{N}.$$

Since λ is normal, $(T^* g_n)_n \in \lambda^s(X^*)$ and therefore T^* is λ-summing. Tracing back the above proof, converse can be easily proved. □

Proposition 5.16 *Let X and Y be Banach spaces and $(\lambda,\|\cdot\|_\lambda)$ be a reflexive, symmetric Banach AK-space such that $0 < \sup_n \|e_n\|_\lambda < \infty$. Then the following are equivalent:*

(i) *Y has the generalized Gelfand-Philips property.*
(ii) *$K_\lambda(X,Y) = \Pi_\lambda^d(X,Y)$ for each Banach space X.*

Proof Suppose that Y has the generalized Gelfand-Philips property. By (Gupta and Bhar 2013, Proposition 4.3), we have $K_\lambda(X,Y) \subset \Pi_\lambda^d(X,Y)$ for each Banach space X. To prove the equality, consider $T \in \Pi_\lambda^d(X,Y)$. Then, by Proposition 5.15, $T(B_X)$ is λ-limited in Y and hence T is a λ-compact operator.

Conversly, assume that $K_\lambda(X,Y) = \Pi_\lambda^d(X,Y)$. Let $A \subset Y$ be a λ-limited set. Then, using Proposition 5.5, U_A^* is λ-summing and hence $U_A(B_{\ell_1(A)})$ is λ-compact. It is easy to see that $A \subset U_A(B_{\ell_1(A)})$. Therefore A is a λ-compact set. □

6 *λ-L*-sets

Analogous to the definition of a λ-limited set, we introduce the notion of a λ-L-set in the dual of a Banach space X as follows:

Definition 6.1 Let λ be a sequence space and X be a Banach space. Then a subset F of the dual X^* is said to be a λ-L-set if for each $(x_n)_n \in \lambda^w(X)$, there exists a sequence $(\alpha_n)_n \in \lambda$ such that,

$$|f(x_n)| \leq \alpha_n, \quad \forall\, n \in \mathbb{N},\ f \in F.$$

Some elementary facts about λ-L-sets are presented in Proposition 6.2.

Proposition 6.2 *Let λ be a normal sequence space and X be a Banach space.*

(i) *Every λ-L-subset F of X^* is bounded.*
(ii) *If F is a λ-L-set, then the closures $\bar{F}^{w^*}$, $\bar{F}^{w}$ and $\bar{F}$ of F, with respect to the weak*, weak and norm topology respectively, are λ-L-sets.*
(iii) *If F, G are λ-L-sets, then $F + G$, $F \cup G$, $F \cap G$ are λ-L-sets.*
(iv) *If $F \subseteq G$ and G is a λ-L-set, then F is λ-L-set.*
(v) *If F is λ-L-set in Y^*, then $T^*(F)$ is a λ-L-set in X^* $\forall\, T \in \mathcal{L}(X, Y)$.*

Proof (i) Let F be a λ-L-set. Since $(x, 0, 0, \dots) \in \lambda^w(X)$ for each $x \in X$, there exists a sequence $\bar{\alpha_x} = (\alpha_{xn})_n \in \lambda$ such that

$$|f(x)| \leq \alpha_{x1}, \quad \forall\, f \in F$$

which proves that F is weak* bounded and hence bounded by Banach Steinhaus theorem.

(ii) Let $f \in \bar{F}^{w^*}$. Then there exists a net $(f_\delta)_\delta \subseteq F$ such that f_δ converges to f. For each $(x_n)_n \in \lambda^w(X)$ there exists $(\alpha_n)_n \in \lambda$ such that

$$|f_\delta(x_n)| \leq \alpha_n \quad \forall n \in \mathbb{N}.$$

Thus $|f(x_n)| \leq \alpha_n$ $\forall n \in \mathbb{N}, \forall f \in \bar{F}$, and hence $\bar{F}^{w^*}$ is a λ-L-set.
Similar arguments can be used to prove that $\bar{F}^{w}$ and $\bar{F}$ are λ-L-sets. (iii) and (iv) have straighforward proofs.

(v) Note that for every $(x_n)_n \in \lambda^w(X)$, $(Tx_n)_n \in \lambda^w(Y)$. Since F is a λ-L-set, there exists $(\alpha_n)_n \in \lambda$ such that

$$|\left\langle x_n, T^* g\right\rangle| = |\langle Tx_n, g\rangle| \leq \alpha_n \quad \forall g \in F, n \in \mathbb{N}.$$

Hence $T^*(F)$ is a λ-L-set in X^*. □

One can easily establish the next result.

Proposition 6.3 *For a Banach space X, the following sentences are equivalent:*

1. *X has the Schur's property.*
2. *B_{X^*} is an L-set.*
3. *B_{X^*} is a c-L-set.*

The following result gives the relation between λ-limited sets and λ-L-sets in a dual Banach space.

Proposition 6.4 *For a normal sequence space λ and a Banach space X, every λ-limited set in X^* is a λ-L-set.*

Proof Let F be a λ-limited set in X^*, $(x_n)_n \in \lambda^w(X)$ and $J : X \to X^{**}$ be the canonical inclusion. Then for every $f \in X^*$, the sequence $(\langle f, Jx_n\rangle)_n = (f(x_n))_n \in \lambda$. Therefore, $(Jx_n)_n \in \lambda^{w^*}(X^{**})$. Since F is λ-limited, there exists $(\alpha_n)_n \in \lambda$ such that

$$|\langle x_n, f\rangle| \le \alpha_n \quad \forall n \in \mathbb{N},\ f \in F.$$

□

The converse of the above proposition is not necessarily true as exhibited in the following result.

Proposition 6.5 *Let X be an infinite dimensional Banach space having the Schur's property. Then B_{X^*} is a L-set which is not limited.*

Proof By Proposition 6.3, B_{X^*} is a L-set, but it is not limited by Corollary 5.4.1. □

Let F be a bounded subset of X^*. Define a continuous linear operator $E_F : X \to \ell_\infty(F)$ as

$$E_F(x) = (f(x))_{f\in F},\ x \in X.$$

Next, we obtain a characterization for λ-L sets.

Proposition 6.6 *Let λ be a normal sequence space and X be a Banach space. A subset F of X^* is a λ-L-set if and only if the operator E_F is absolutely λ-summing.*

Proof Let F be a λ-L-set. Then E_F is well defined and continuous. For $(x_n)_n \in \lambda^w(X)$, we have

$$\|E_F(x_n)\|_\infty = \sup_{f\in F} |f(x_n)| \le \alpha_n \ \ \forall n \in \mathbb{N}$$

for some $(\alpha_n)_n \in \lambda$. Since λ is normal, the sequence $(E_F(x_n))_n \in \lambda^s(X)$. Tracing back the proof, the converse can be easily proved. □

The above proposition along with Theorem 2.3 leads to the following

Proposition 6.7 *Let λ and μ be normal sequence spaces, and $\nu = \left(\lambda^\times \cdot \mu\right)^\times$. If $\left(\nu.\lambda^\times\right)^\times \subset \mu$ and $\nu.\mu \subset \lambda$, then every λ-L-subset of X^* is a μ-L-set.*

Similar to Corollaries 5.7 and 5.8, the above proposition yields the next result.

Corollary 6.8 *For $1 \le p \le q < \infty$, every p-L-set is a q-L-set; and every 1-L-set is a μ-L-set for a perfect sequence space μ.*

Proposition 6.9 *Every L-set is a ϕ-L-set.*

Proof Let $F \subset X^*$ be an L-set. Then $E_F : X \to \ell_\infty(F)$ defined as $E_F(x) = (f(x))_{f\in F}$ is continuous and hence absolutely ϕ-summing. By Proposition 6.6, it is clear that F is a ϕ-L-set. □

Proposition 6.10 *Let X, Y be Banach spaces, $T \in \mathcal{L}(X, Y)$ and λ be a normal sequence space. Then T is absolutely λ-summing if and only if $T^*(B_{Y^*})$ is a λ-L-set in X^*.*

Proof If $T \in \mathcal{L}(X, Y)$ is absolutely λ-summing, then for $(x_n)_n \in \lambda^w(X)$, $(\|Tx_n\|)_n \in \lambda$. For $g \in B_{Y^*}$,

$$\left|\left\langle x_n, T^*g\right\rangle\right| \leq \sup_{h\in B_{Y^*}} |\langle Tx_n, h\rangle| = \|Tx_n\|.$$

Hence $T^*(B_{Y^*})$ is a λ-L-set. The converse follows easily since λ is normal. □

As a consequence of the above proposition, we derive the following

Corollary 6.11 *For Banach spaces X and Y, where X has the Schur's property, $\mathcal{L}(X, Y) = \Pi_{c_0}(X, Y) = \Pi_c(X, Y)$.*

Proof Assume that X has the Schur's property and $T \in \mathcal{L}(X, Y)$. Then, by Proposition 6.3, $T^*(B_{Y^*})$ is a L-set in X. Hence, by Proposition 6.10, $T \in \Pi_{c_0}(X, Y)$. As $\Pi_c(X, Y) = \Pi_{c_0}(X, Y)$, the result follows. □

Remark 6.12 Since ℓ_1 has the Schur's property, we get the solution of (Megginson 2012, Ex.3.51(a)) using the above Corollary.

Proposition 6.13 *Let X, Y be Banach spaces and λ be a normal sequence space. If $\Pi_\lambda(X, Y) = K_\lambda^d(X, Y)$, then X^* has the generalized Gelfand-Philips property.*

Proof Let A be a λ-limited set in X^*. By Propositions 6.4 and 6.6, A is a λ-L-set and the operator $E_A : X \to \ell_\infty(A)$ is absolutely λ-summing. Since $\Pi_\lambda(X, Y) = K_\lambda^d(X, Y)$, $E_A^*(B_{(\ell_\infty(A))^*})$ is λ-compact. For $f \in A$, δ_f denotes the point mass at f, that is, $E_A^*(\delta_f) = f$. Then $A = E_A^*\left(\{\delta_f : f \in A\}\right) \subset E_A^*(B_{(\ell_\infty(A))^*})$ and then A is a λ-compact set. □

Acknowledgements The authors express their sincere gratitude to the anonymous referee for their careful reading of the manuscript and the valuable suggestions that significantly improved the paper. The first and third authors acknowledge financial support from the Department of Science and Technology, India (Grant Nos. DST/INSPIRE Fellowship/2019/IF190043 and SPG/2021/004500, respectively).

References

Botelho, G., Santiago, A.S.: Representation of sequence classes by operator ideals. arXiv preprint arXiv:2402.04496 (2024)

Chen, J.X., Chen, Z.L., Ji, G.X.: Almost limited sets in Banach lattices. J. Math. Anal. Appl. **412**(1), 547–553 (2014)

Delgado, J., M., Pineiro, C,: A note on p-limited sets. J. Math. Anal. Appl. **410**(2), 713–718 (2014)
Diestel, J., Uhl Jr., J.J.: Progress in Vector Measures 1977-83, in: Measure theory and its applications. Proc. Con. Sherbrooke/Canada 1982, Lecture Notes 1033, Springer Verlag, 144-192 (1983)
Diestel, J., Jarchow, H., Tonge, A.: Absolutely summing operators. Cambridge University Press, Cambridge (1995)
Diestel, J.: Sequences and series in Banach spaces, vol. 92. Springer Science, Business Media (2012)
Dubinsky, E., Ramanujan, M.S.: Inclusion theorems for absolutely λ-summing maps. Math. Ann. **192**, 177–190 (1971)
Fourie, J.H., Swart, J.: Operators factoring compactly through a normal BK-space with AK. Council for Scientific and Industrial Research [CSIR], National Research Institute for Mathematical Sciences [NRIMS] (1979)
Garling, D.J.H.: A class of reflexive symmetric BK-spaces. Can. J. Math. **21**, 602–608 (1969)
Garling, D.J.H.: Diagonal mappings between sequence spaces. Stud. Math. **51**, 129–138 (1974)
Galindo, P., Miranda, V.C.C.: A class of sets in a Banach space coarser than limited sets. Bull. Braz. Math. Soc. New Series **53**, 941–955 (2022)
Galindo, P., Miranda, V.C.C.: Some properties of p-limited sets. Proc. Am. Math. Soc. **152**(02), 749–763 (2024)
Gelfand, I.M.: Abstrakte Funktionen und lineare operatoren. Rev. Roumaine Math. Pures Appl. **5**, 742–752 (1938)
Ghenciu, I.: A note on p-limited sets in dual Banach spaces. Monatsh. Math. **200**(2), 255–270 (2023)
Gupta, M., Bhar, A.: On λ-compact operators. Indian J. Pure Appl. Math. **44**, 355–374 (2013)
Kamthan, P.K., Gupta, M.: Sequence spaces and series (Vol. 65). Marcel Dekker Incorporated (1981)
Karn, A.K., Sinha, D.P.: An operator summability of sequences in Banach spaces. Glasgow Math. J. **56**(2), 427–437 (2014)
Lindenstrauss, J., Tzafriri, L.: Classical Banach Spaces I. Sequence Spaces. Springer-Verlag, Berlin (1977)
Megginson, R.E.: An introduction to Banach space theory (Vol. 183). Springer Science Business Media (2012)
Nogueira, D.F.: Espacos de Sequências Vetoriais e Ideais de Operadores. Master's Thesis, Federal University of Uberlândia, (2016)
Phillips, R.S.: On linear transformations. Trans. Am. Math. Soc. **48**(3), 516–541 (1940)
Pietsch, A.: Verallgemeinerte Vollkommene Folgenraum. Akademi-Verlag, East Berlin (1962)
Pietsch, A.: Operator ideals. Elsevier, Amsterdam (1980)
Ramanujan, M.S.: Absolutely λ-summing operators, λ a symmetric sequence space. Math. Z. **114**, 187–193 (1970)
Ramanujan, M.S.: Generalized nuclear maps in normed linear spaces. J. Reine Angew. Math. **244**, 190–197 (1970)
Sargent, W.L.C.: Some sequence spaces related to the ℓ_p spaces. J. London Math. Soc. **1**(2), 161–171 (1960)
Schlumprecht, T.: Limited sets in Banach spaces. Ph. D. Dissertation, Munich (1987)
Sinha, D.P., Karn, A.K.: Compact operators whose adjoints factor through subspaces of ℓ_p. Stud. Math. **1**(150), 17–33 (2002)