Bull Braz Math Soc, New Series (2024) 55:30
https://doi.org/10.1007/s00574-024-00406-7

On Real Analytic Levi-Flat Hypersurfaces Associated with Milnor Fibrations

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Received: 29 January 2024 / Accepted: 25 May 2024 / Published online: 7 June 2024

Abstract
Let G and X be germs of holomorphic vector fields at $0 \in \mathbb{C}^n$. Consider the real analytic map $\psi_{G,X} : \mathbb{C}^n \to \mathbb{C}$ defined by $\psi_{G,X}(z) = \langle G(z), X(z) \rangle$, where $\langle \cdot, \cdot \rangle$ represents the usual Hermitian product. In this paper, we investigate the following question: under which conditions on the germs of holomorphic vector fields G and X is the real analytic hypersurface $M = \{F(z) = 2\mathrm{Re}(\psi_{G,X}(z)) = 0\}$ Levi-flat? This problem was posed by Maria A. Soares Ruas.

Keywords Levi-flat hypersurfaces · Holomorphic foliations · Milnor fibrations

Mathematics Subject Classification Primary 32V40 · 32S65

1 Introduction and Statement of Results

Let M be a real analytic hypersurface at the origin $0 \in \mathbb{C}^n$, $n \geq 2$, defined by the equation $F(z_1, \ldots, z_n) = 0$, where F is a real analytic function vanishing at 0. We say that M is *nondegenerate* if the *Levi form*

$$LF(z, \bar{z}) = \sum_{1 \leq \alpha, \beta \leq n} g_{\alpha\bar{\beta}} z^{\alpha} \bar{z}^{\beta}, \quad g_{\alpha\bar{\beta}} = \left(\frac{\partial^2 F}{\partial z^{\alpha} \partial \bar{z}^{\beta}} \right)_0 \tag{1}$$

The second author acknowledges support from CNPq Projeto Universal 408687/2023-1 “Geometria das Equações Diferenciais Algébricas” and CNPq Projeto PQ 2023 306011/2023-9.

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is nondegenerate at 0. Otherwise, we say that M is *Levi-flat*. The purpose of this paper is to study the degeneracy of the Levi form of real analytic hypersurfaces obtained from *real singularities with a Milnor fibration*. More precisely, Milnor proved in Milnor (1968, Theorem 11.2) that if $f : (\mathbb{R}^n, 0) \to (\mathbb{R}^p, 0)$, $n > p$, is a real analytic map whose derivative Df has rank p on a punctured neighborhood of $0 \in \mathbb{R}^n$, then, for every sufficiently small sphere $\mathbb{S}_\epsilon \subset \mathbb{R}^n$, the mapping

$$\psi := \frac{f}{\|f\|} : \mathbb{S}_\epsilon - N_K \to \mathbb{S}^{p-1} \tag{2}$$

is a locally trivial fibration, where $K = f^{-1}(0) \cap \mathbb{S}_\epsilon$ is the singularity link, and N_K is a tubular neighborhood of K in $\mathbb{S}_\epsilon$. The map ψ can always be extended to $\mathbb{S}_\epsilon - K$ as the projection of a fibration, but this extension is not necessarily as $\dfrac{f}{\|f\|}$. Follows Ruas et al. (2002, Definition 1.1), we will say that $f : (\mathbb{R}^n, 0) \to (\mathbb{R}^p, 0)$, $n > p$, satisfies *the Milnor condition* at 0 if Df has rank p on a punctured neighborhood of 0. When f satisfies the Milnor condition at 0, and furthermore, the map $\dfrac{f}{\|f\|} : \mathbb{S}_\epsilon - K \to \mathbb{S}^{p-1}$ is a fibration for every sufficiently small sphere $\mathbb{S}_\epsilon \subset \mathbb{R}^n$, we say that f satisfies the *strong Milnor condition* at 0, see for instance (Ruas et al. 2002, Definition 2.5). Maps of this type induce an *open book decomposition* on the sphere $\mathbb{S}_\epsilon$. Milnor pointed out in his book that is difficult to find examples satisfying the strong Milnor condition, (see Milnor 1968, p. 100). In Seade (1997) and Seade (1996), Seade presented a method for constructing families of nontrivial maps $f : \mathbb{R}^{2n} \to \mathbb{R}^2$ that satisfy the strong Milnor condition at 0. This construction is given as follows: let $\chi(\mathbb{C}^n, 0)$ denote the space of all germs of holomorphic vector fields at $0 \in \mathbb{C}^n$, and let G, X be elements in $\chi(\mathbb{C}^n, 0)$. Consider the real analytic map

$$\psi_{G,X} : \mathbb{C}^n \cong \mathbb{R}^{2n} \to \mathbb{C} \cong \mathbb{R}^2$$

defined by $\psi_{G,X}(z) = \langle G(z), X(z) \rangle$, where

$$\langle G(z), X(z) \rangle = \sum_{i=1}^{n} G_i(z) \cdot \overline{X}_i(z), \tag{3}$$

is the usual Hermitian product. Note that the argument of $i\langle G(z), X(z) \rangle$ is the angle by which we rotate the field G so that it becomes orthogonal to the field X. Thus, the real analytic variety $\psi_{G,X}^{-1}(0)$, called the *polar variety of* G *and* X, is the set of points where G and X are orthogonal. Consequently, on the polar variety, the holomorphic foliations defined by the fields G and X are transversal, and their intersection gives rise to a foliation by real curves in $\psi_{G,X}^{-1}(0)$. In the particular case where X is the gradient field of a real analytic function $f : \mathbb{R}^{2n} \to \mathbb{R}^2$, the polar variety is the set of points where the foliations defined by the field G and the level curves of f are tangent.

Furthermore, $\psi_{G,X}^{-1}(0)$ is a complete intersection defined by the equations

$$\text{Re}\,\langle G(z), X(z)\rangle = \text{Im}\,\langle G(z), X(z)\rangle = 0.$$

In Seade (1997), Seade proved that if $X = (z_1, \dots, z_n)$ is the radial field and $G = (\lambda_1 z_1^{a_1}, \dots, \lambda_n z_n^{a_n})$, then $\psi_{G,X}$ satisfies the Milnor condition for any $\lambda_k \in \mathbb{C}^*$ and integers $a_k > 1$. On the other hand, given $\sigma \in S_n$, a permutation of the set $\underline{n} := \{1, \dots, n\}$, families of vector fields of the form $G = (\lambda_1 z_{\sigma_1}^{a_1}, \dots, \lambda_n z_{\sigma_n}^{a_n})$ and $X = (\beta_1 z_1^{b_1}, \dots, \beta_n z_n^{b_n})$ that satisfy the Milnor condition or the strong Milnor condition at the origin were classified by Ruas–Seade–Verjovsky (Ruas et al. 2002, Theorem 2.7).

In this paper, we consider $M = \{F(z) = 0\}$ defined by

$$F(z) := 2\text{Re}(\psi_{G,X}(z)) = \psi_{G,X}(z) + \overline{\psi_{G,X}(z)}, \tag{4}$$

where $G, X \in \chi(\mathbb{C}^n, 0)$. A simple example in $(\mathbb{C}^2, 0)$ is when we consider $G(z_1, z_2) = (z_1, z_2)$ and $X = (z_2, -z_1)$. Then

$$M = \{F(z_1, z_2) = 2\text{Re}(\psi_{G,X}(z)) = z_1\bar{z}_2 - z_2\bar{z}_1 = 0\} \tag{5}$$

is a real analytic Levi-flat hypersurface at $0 \in \mathbb{C}^2$ whose Levi foliation admits as leaves the complex curves $z_1 = c \cdot z_2$, where $c \in \mathbb{C}$ (see Burns and Gong 2003, p. 51). Motivated by this, M.A. Soares Ruas have posed the following problem:

Problem 1 *Under which conditions on the germs of holomorphic vector fields G and X is the real analytic hypersurface $M = \{F(z) = 2\text{Re}(\psi_{G,X}(z)) = 0\}$ Levi-flat?*

In order to answer Problem 1, we consider the vector fields G and X explored in Ruas et al. (2002, pp. 203–211). More specifically, our first result is as follows:

Theorem A *Let G and X be elements in $\chi(\mathbb{C}^n, 0)$, $n \geq 2$, of the form $G(z) = (z_1^{a_1}, \dots, z_n^{a_n})$ and $X(z) = (z_1^{b_1}, \dots, z_n^{b_n})$, where $a_k > b_k \geq 1$ are positive integers, for all $k = 1, \dots, n$. Then $M = \{F(z) = 2\text{Re}(\psi_{G,X}(z)) = 0\}$, i.e.,*

$$M = \left\{\text{Re}\left(\sum_{k=1}^{n} z_k^{a_k}\bar{z}_k^{b_k}\right) = 0\right\}.$$

is nondegenerate at $0 \in \mathbb{C}^n$.

Our second and third theorems are motivated by the example given in (5):

Theorem B *Let G and X be elements in $\chi(\mathbb{C}^n, 0)$, $n \geq 2$, of the form $G(z) = (z_1^a, z_2^b)$ and $X(z) = (z_2^b, z_1^a)$ with a, b positive integers. Then $M = \{F(z) = 2\text{Re}(\psi_{G,X}(z)) = 0\}$, i.e.,*

$$M = \left\{\text{Re}\left(z_1^a\bar{z}_2^b + z_2^b\bar{z}_1^a\right) = 0\right\}$$

is a Levi-flat hypersurface at $0 \in \mathbb{C}^2$.

Theorem C *Let G and X be elements in $\chi(\mathbb{C}^n, 0)$, $n \geq 2$, of the form $G = (z_1^{a_1}, z_2^{a_2})$ and $X = (z_2^{b_2}, z_1^{b_1})$, where $a_1 \geq b_1$ and $a_2 \geq b_2$ are positive integers satisfying $a_1 b_2 = a_2 b_1$. Then $M = \{F(z) = 2\mathrm{Re}(\psi_{G,X}(z)) = 0\}$, i.e.,*

$$M = \left\{\mathrm{Re}\left(z_1^{a_1}\bar{z}_2^{b_2} + z_2^{a_2}\bar{z}_1^{b_1}\right) = 0\right\}$$

is Levi-flat if, and only if, $a_1 = b_1$ and $a_2 = b_2$.

Finally, we consider a family of vector fields studied in Ruas et al. (2002, Theorem 2.1).

Theorem D *Let G and X be elements in $\chi(\mathbb{C}^n, 0)$, $n \geq 2$, of the form $G = (z_1^{a_1}, \ldots, z_n^{a_n})$ and $X = (z_{\sigma_1}^{b_{\sigma_1}}, \ldots, z_{\sigma_n}^{b_{\sigma_n}})$, where $a_k \geq b_k$ are positive integers. Let us assume that for some $\ell \in \underline{n}$, the integers $a_\ell, b_\ell, a_{\sigma_\ell}, b_{\sigma_\ell}$ satisfy the following conditions: $a_\ell > b_\ell$ and $a_\ell b_{\sigma_\ell} = b_\ell a_{\sigma_\ell}$. Then, $M = \{F(z) = 2\mathrm{Re}(\psi_{G,X}(z)) = 0\}$, i.e.,*

$$M = \left\{\mathrm{Re}\left(\sum_{k=1}^{n} z_k^{a_k}\bar{z}_{\sigma_k}^{b_{\sigma_k}}\right) = 0\right\}.$$

is nondegenerate at $0 \in \mathbb{C}^n$.

Following our results it seems that the property of M being Levi-flat is related to the property that the function $\psi_{G,X}$ does not satisfy the Milnor condition, see Ruas et al. (2002, Theorem 2.1).

The paper is organized as follows: In Sect. 2, we introduce the concept of real analytic Levi-flat hypersurfaces at $(\mathbb{C}^n, 0)$, shedding light on essential properties that will play a pivotal role throughout the paper. Section 3 is dedicated to proving Theorem A. In Sect. 4, we establish the validity of Theorems B and C, while Sect. 5 focuses on the proof of Theorem D. Finally, in Sect. 6, we provide examples for further illustration.

2 Levi-Flat Hypersurfaces

In this section, we will discuss real analytic Levi-flat hypersurfaces at $(\mathbb{C}^n, 0)$. These are real analytic hypersurfaces whose regular part is foliated by immersed complex submanifolds of codimension one. Levi-flat hypersurfaces naturally arise in the theory of foliations as invariant subsets. In general, germs of codimension one holomorphic foliations that leave invariant hypersurfaces of this type admit a meromorphic first integral (see Cerveau and Lins-Neto 2011, Theorem 1). On the other hand, there are examples of holomorphic webs that leave invariant Levi-flat hypersurfaces (see Da Silva and Fernández-Pérez 2023; Fernández-Pérez 2013; Shafikov and Sukhov 2015). Levi-flat hypersurfaces are a central focus of the development in this paper.

Let $M = \{F(z) = 0\}$ be a germ of real analytic hypersurface at $0 \in \mathbb{C}^n$, where $F : (\mathbb{C}^n, 0) \to (\mathbb{R}, 0)$ is a real analytic function at $0 \in \mathbb{C}^n$. The *singular set* of M is denoted by $\mathsf{Sing}(M)$ and defined by

$$\mathsf{Sing}(M) := \{F(z) = 0\} \cap \{dF(z) = 0\}.$$

We define the *regular part* of M as $M^* := \{F(z) = 0\} \backslash \{dF(z) = 0\}$. In M^*, the *Levi distribution* is given by $L_p := Ker(\partial F(p)) \subset T_p M^*$, where $p \in M^*$. Note that L_p is the unique complex hyperplane contained in $T_p M^*$.

Definition 2.1 We say that M is *Levi-flat* if the Levi distribution on M^* is integrable. In this case, the Levi distribution induces a foliation on M^* called the *Levi foliation*, denoted by $\mathcal{L}$.

The Levi distribution can also be given by the 1-form $\eta = i(\partial F - \bar{\partial} F)$ called the *Levi 1-form*. Thus, the integrability of the Levi distribution is equivalent to the integrability of the form η in the sense of Frobenius, that is, η is integrable if and only if $\eta \wedge d\eta|_{M^*} \equiv 0$.

The simplest example of a Levi-flat hypersurface is given below.

Example 2.1 In $\mathbb{C}^n$ with coordinates $(z_1, \ldots, z_n)$, consider $M = \{Im(z_n) = 0\}$. Then M is a smooth Levi-flat real analytic hypersurface, meaning $\mathsf{Sing}(M) = \emptyset$. The Levi distribution on M is given by $L_p = \mathrm{Ker}(dz_n(p))$, $p \in M^*$. The leaves of the Levi foliation on M are given by $\{z_n = c\}$ where $c \in \mathbb{R}$.

Let's consider a slightly more elaborate example given by Brunella (2007, Example 1.2).

Example 2.2 With coordinates (z, w) in $\mathbb{C}^2$ such that $z = x + iy$ and $w = s + it$, the real analytic hypersurface M given by

$$M = \{(z, w) \in \mathbb{C}^2 : t^2 = 4(y^2 + s)y^2\}$$

is Levi-flat, with singular set $\mathsf{Sing}(M) = \{t = y = 0\}$. The leaves of the Levi foliation on M^* are given by $L_c = \{w = (z + c)^2 : Im(z) \neq 0\}$ with $c \in \mathbb{R}$.

The next result provides the local form of a smooth Levi-flat hypersurface. Essentially, it tells us that, at regular points, every Levi-flat hypersurface is locally similar to the Example 2.1.

Theorem 2.1 [Cartan's theorem (Cartan 1933)] *Let $M \subset \mathbb{C}^n$ be a real analytic Levi-flat hypersurface. In a neighborhood of each point $p \in M^*$, there exists a holomorphic coordinate system $z = (z_1, \ldots, z_n)$ such that $M = \{Im(z_n) = 0\}$.*

A criterion for the integrability of the Levi form is given in the next proposition.

Proposition 2.2 *Let $M = \{F(z) = 0\}$ be a germ of a real analytic hypersurface at $0 \in \mathbb{C}^n$. Then, M is Levi-flat if and only if $\partial F(p) \wedge \bar{\partial} F(p) \wedge \partial\bar{\partial} F(p) = 0$ for all $p \in M$.*

Proof Let $\eta = i(\partial F - \bar{\partial} F)$ denote the Levi 1-form of M. Assuming M is Levi-flat, this implies $\eta \wedge d\eta|_{M^*} = 0$, which is equivalent to $(\partial F - \bar{\partial} F) \wedge \partial\bar{\partial} F|_{M^*} = 0$. Consequently, we have $\bar{\partial} F \wedge \partial\bar{\partial} F|_{M^*} = \partial F \wedge \partial\bar{\partial} F|_{M^*}$. In particular, $\partial F(p) \wedge \bar{\partial} F(p) \wedge \partial\bar{\partial} F(p) = 0$ for all $p \in M$.

Conversely, the condition $\partial F(p) \wedge \bar{\partial} F(p) \wedge \partial\bar{\partial} F(p) = 0$ for all $p \in M$ is equivalent to $(\partial F - \bar{\partial} F) \wedge \partial\bar{\partial} F \wedge dF|_{M^*} = 0$. Hence

$$(\partial F - \bar{\partial} F) \wedge \partial\bar{\partial} F \wedge dF = F\theta, \tag{6}$$

where θ is a 4-form in some open subset of $\mathbb{C}^n$. Since $\theta \wedge dF = 0$, we can express θ as $\theta = \beta \wedge dF$, where β is a 3-form in some open subset of $\mathbb{C}^n$. Substituting this into Eq. (6), we obtain

$$\left[(\partial F - \bar{\partial} F) \wedge \partial\bar{\partial} F - F\beta\right] \wedge dF = 0.$$

Thus, there exists a 2-form κ such that

$$(\partial F - \bar{\partial} F) \wedge \partial\bar{\partial} F = F\beta + \kappa \wedge dF,$$

this expression implies that $(\partial F - \bar{\partial} F) \wedge \partial\bar{\partial} F|_{M^*} = 0$, leading to the integrability of η. □

Now, let's verify that the regular part of M is mapped to the regular part of M' for M and M' being biholomorphic (not necessarily Levi-flat).

Lemma 2.3 *Let $u \in M'$. Then $z = z(u) \in \mathrm{Sing}(M)$ if and only if $u \in \mathrm{Sing}(M')$.*

Proof Denote $M' = \{G(u) = 0\}$ where $G : (\mathbb{C}^n, 0) \to (\mathbb{R}), 0$ is a real analytic function at $0 \in \mathbb{C}^n$.

We have $dG(u) = \partial G(u) + \bar{\partial} G(u)$, where

$$\partial G = \sum_{j=1}^{n} \frac{\partial G}{\partial u_j} du_j, \quad \bar{\partial} G = \sum_{j=1}^{n} \frac{\partial G}{\partial \bar{u}_j} d\bar{u}_j.$$

A point $u \in M'$ belongs to the singular set $\mathrm{Sing}(M')$ if and only if all partial derivatives $\dfrac{\partial G}{\partial u_j}(u)$, $\dfrac{\partial G}{\partial \bar{u}_j}(u)$ are identically zero. By the chain rule, we obtain

$$\frac{\partial G}{\partial u_j}(u) = \sum_{\alpha=1}^{n} \frac{\partial G}{\partial z_\alpha}(z(u)) \frac{\partial z_\alpha}{\partial u_j}(u),$$

thus, it follows that

$$\begin{pmatrix} \frac{\partial G}{\partial u_1}(u) & \cdots & \frac{\partial G}{\partial u_n}(u) \end{pmatrix} = \begin{pmatrix} \frac{\partial G}{\partial z_1}(z(u)) & \cdots & \frac{\partial G}{\partial z_n}(z(u)) \end{pmatrix} \begin{pmatrix} \frac{\partial z_1}{\partial u_1}(u) & \cdots & \frac{\partial z_1}{\partial u_n}(u) \\ \vdots & \ddots & \vdots \\ \frac{\partial z_n}{\partial u_1}(u) & \cdots & \frac{\partial z_n}{\partial u_n}(u) \end{pmatrix}.$$

Since $z = z(u)$ is a biholomorphism, the change of coordinates matrix is invertible, and therefore,

$$\begin{pmatrix} \frac{\partial G}{\partial u_1}(u) & \cdots & \frac{\partial G}{\partial u_n}(u) \end{pmatrix} = 0 \iff \begin{pmatrix} \frac{\partial G}{\partial z_1}(z(u)) & \cdots & \frac{\partial G}{\partial z_n}(z(u)) \end{pmatrix} = 0.$$

□

In the next proposition, we will see that the Levi-flat property is invariant under a change of coordinates. More specifically, we establish the following result.

Proposition 2.4 *Let $G : V \subset \mathbb{C}^n \to \mathbb{R}$ be a real-analytic function in coordinates $z = (z_1, \ldots, z_n)$ and let $z = z(u)$ be a change of coordinates, that is, a biholomorphism from the open set $V \subset \mathbb{C}^n$ to an open set $U \subset \mathbb{C}^n$. Then, the hypersurface $M' = \{u \in U : G(u) = G(z(u)) = 0\}$ is Levi-flat if and only if $M = \{z \in V : G(z) = 0\}$ is Levi-flat.*

Proof The partial derivatives of G in the coordinates $z = (z_1, \ldots, z_n)$ are given by

$$\partial G = \sum_{\alpha=1}^{n} \frac{\partial G}{\partial z_\alpha} dz_\alpha, \quad \overline{\partial} G = \sum_{\beta=1}^{n} \frac{\partial G}{\partial \overline{z}_\beta} d\overline{z}_\beta, \quad \partial\overline{\partial} G = \sum_{\delta,\beta=1}^{n} \frac{\partial^2 G}{\partial z_\delta \partial \overline{z}_\beta} dz_\delta \wedge d\overline{z}_\beta.$$

Therefore, in coordinates $z = (z_1, \ldots, z_n)$, we have

$$\partial G \wedge \overline{\partial} G \wedge \partial\overline{\partial} G = \sum_{\alpha,\beta,\gamma,\delta} \left(\frac{\partial G}{\partial z_\alpha} \frac{\partial G}{\partial \overline{z}_\beta} \frac{\partial^2 G}{\partial z_\delta \partial \overline{z}_\gamma} \right) dz_\alpha \wedge d\overline{z}_\beta \wedge dz_\delta \wedge d\overline{z}_\gamma. \quad (7)$$

Now, by making the change of coordinates $z = z(u)$, we obtain

$$dz_\alpha = \sum_{j=1}^{n} \frac{\partial z_\alpha}{\partial u_j} du_j, \qquad d\overline{z}_\beta = \sum_{k=1}^{n} \frac{\partial \overline{z}_\beta}{\partial \overline{u}_k} d\overline{u}_k,$$

from which it follows that

$$dz_\alpha \wedge d\overline{z}_\beta \wedge dz_\delta \wedge d\overline{z}_\gamma = \sum_{j,k,\ell,m} \frac{\partial z_\alpha}{\partial u_j} \frac{\partial \overline{z}_\beta}{\partial \overline{u}_k} \frac{\partial z_\delta}{\partial u_\ell} \frac{\partial \overline{z}_\gamma}{\partial \overline{u}_m} du_j \wedge d\overline{u}_k \wedge du_\ell \wedge d\overline{u}_m.$$

Hence, in coordinates $u = (u_1, \ldots, u_n)$,

$$\partial G \wedge \overline{\partial} G \wedge \partial\overline{\partial} G = \sum_{\alpha,\beta,\gamma,\delta} \left[\left(\frac{\partial G}{\partial z_\alpha} \frac{\partial G}{\partial \overline{z}_\beta} \frac{\partial^2 G}{\partial z_\delta \partial \overline{z}_\gamma} \right) \left(\sum_{j,k,\ell,m} \left(\frac{\partial z_\alpha}{\partial u_j} \frac{\partial \overline{z}_\beta}{\partial \overline{u}_k} \frac{\partial z_\delta}{\partial u_\ell} \frac{\partial \overline{z}_\gamma}{\partial \overline{u}_m} \right) du_j \wedge d\overline{u}_k \wedge du_\ell \wedge d\overline{u}_m \right) \right].$$

From the expression above, combined with Eq. (7), it follows that M' is Levi-flat if and only if M is Levi-flat. This concludes the proof of the Proposition 2.4. □

2.1 Complexification

Let $F : (\mathbb{C}^n, 0) \to (\mathbb{R}, 0)$ be a real analytic function at $0 \in \mathbb{C}^n$. The *complexification of* F is defined by

$$F_{\mathbb{C}}(z, w) = \sum_{\mu,\nu} F_{\mu,\nu} z^{\mu} w^{\nu},$$

where $F(z) = \sum_{\mu,\nu} F_{\mu,\nu} z^{\mu} \overline{z}^{\nu}$ is a power series of F convergent in a neighborhood of the origin. We observe that $F_{\mathbb{C}}$ is holomorphic at $(\mathbb{C}^n \times \mathbb{C}^n, 0)$. The complexification of M is defined as $M_{\mathbb{C}} = \{F_{\mathbb{C}} = 0\}$, and the complexification of the Levi 1-form is given by

$$\eta_{\mathbb{C}} = i \sum_{k=1}^{n} \left(\frac{\partial F_{\mathbb{C}}}{\partial z_k} dz_k - \frac{\partial F_{\mathbb{C}}}{\partial w_k} dw_k \right).$$

Given that η is integrable on M^*, it follows that $\eta_{\mathbb{C}}$ is also integrable on $M^*_{\mathbb{C}}$. It's worth noting that we can express $dF_{\mathbb{C}} = \alpha + \beta$ and $\eta_{\mathbb{C}} = i(\alpha - \beta)$, where

$$\alpha = \sum_{k=1}^{n} \frac{\partial F_{\mathbb{C}}}{\partial z_k} dz_k \text{ and } \beta = \sum_{k=1}^{n} \frac{\partial F_{\mathbb{C}}}{\partial w_k} dw_k. \tag{8}$$

Furthermore, we observe that α and β define the same foliation as $\eta_{\mathbb{C}}$ on $M^*_{\mathbb{C}}$. Thus, the integrability of $\eta_{\mathbb{C}}$ is equivalent to

$$\alpha(z, w) \wedge d\alpha(z, w) \wedge \beta(z, w) = 0 \quad \text{for all} \quad (z, w) \in M^*_{\mathbb{C}}. \tag{9}$$

This condition will be used later to verify that the real part of a certain family of polar varieties is not Levi-flat.

3 Proof of Theorem A

Before proving Theorem A, we will make some considerations for the case where the vector fields G and X are given by $G = (z_1^{a_1}, \ldots, z_n^{a_n})$ and $X = (z_1^{b_1}, \ldots, z_n^{b_n})$, with a_k, b_k positive integers such that $a_k \geq b_k$, for $k \in \underline{n}$. In this case, the Hermitian product of G and X is given by

$$\langle G(z), X(z) \rangle = \sum_{k=1}^{n} z_k^{a_k} \overline{z}_k^{b_k}$$

and

$$F(z) = 2\text{Re}\langle G(z), X(z) \rangle = \sum_{k=1}^{n} z_k^{a_k} \overline{z}_k^{b_k} + z_k^{b_k} \overline{z}_k^{a_k}. \tag{10}$$

Note that, if $a_k = b_k$ for every $k \in \underline{n}$, then $M = \{F = 0\}$ is a point. Indeed, if $a_k = b_k$ for every $k \in \underline{n}$, we have

$$F(z) = \sum_{k=1}^{n} 2|z_k|^{a_k}$$

and consequently, $M = \{0\}$. Therefore, in the statement of Theorem A, we do not consider $a_k = b_k$ for every $k \in \underline{n}$.

Regarding the singular set of M, we have the following proposition:

Proposition 3.1 *Let $G = (z_1^{a_1}, \ldots, z_n^{a_n})$ and $X = (z_1^{b_1}, \ldots, z_n^{b_n})$, with $a_k \geq b_k \geq 1$, and $M = \{F = 0\}$, where $F(z) = 2\mathrm{Re}\langle G(z), X(z)\rangle$. Then $\mathrm{Sing}(M) = \{0\}$.*

Proof Taking the partial derivatives of F, we have

$$\partial F = \sum_{k=1}^{n} \left(a_k z_k^{(a_k-1)} \overline{z}_k^{b_k} + b_k z_k^{(b_k-1)} \overline{z}_k^{a_k} \right) dz_k$$

and

$$\overline{\partial} F = \sum_{k=1}^{n} \left(b_k z_k^{a_k} \overline{z}_k^{(b_k-1)} + a_k z_k^{b_k} \overline{z}_k^{(a_k-1)} \right) dz_k.$$

Then $dF(z) = \partial F(z) + \overline{\partial} F(z) = 0$ if and only if

$$\begin{cases} a_1 z_1^{(a_1-1)} \overline{z}_1^{b_1} + b_1 z_1^{(b_1-1)} \overline{z}_1^{a_1} & = 0 \\ b_1 z_1^{a_1} \overline{z}_1^{(b_1-1)} + a_1 z_1^{b_1} \overline{z}_1^{(a_1-1)} & = 0 \\ \vdots & \\ a_n z_n^{(a_n-1)} \overline{z}_n^{b_n} + b_n z_n^{(b_n-1)} \overline{z}_n^{a_n} & = 0 \\ b_n z_n^{a_n} \overline{z}_n^{(b_n-1)} + a_n z_n^{b_n} \overline{z}_n^{(a_n-1)} & = 0 \end{cases}$$

Clearly, $0 \in \mathrm{Sing}(M)$. Now let's denote $I := \{k \in \underline{n}; a_k = b_k\}$ and $J := \{k \in \underline{n}; a_k > b_k\}$. We have $\underline{n} = I \cup J$ and $I \cap J = \emptyset$. Consider the following equations from the system above:

$$a_k z_k^{(a_k-1)} \overline{z}_k^{b_k} + b_k z_k^{(b_k-1)} \overline{z}_k^{a_k} = 0 \tag{11}$$

$$b_k z_k^{a_k} \overline{z}_k^{(b_k-1)} + a_k z_k^{b_k} \overline{z}_k^{(a_k-1)} = 0 \tag{12}$$

If $k \in I$, the equations above are rewritten as:

$$2a_k z_k^{(a_k-1)} \overline{z}_k^{a_k} = 0$$

$$2a_k z_k^{a_k} \overline{z}_k^{(a_k-1)} = 0$$

and it follows that $z = (z_1, \ldots, z_n) \in \mathsf{Sing}(M)$ implies $z_k = 0$, for all $k \in I$. Now, let's assume that $z = (z_1, \ldots, z_n) \in \mathsf{Sing}(M)$ and $z_k \neq 0$, for some $k \in J$. From (11), we obtain

$$a_k z_k^{(a_k-1)} \overline{z}_k^{b_k} = -b_k z_k^{(b_k-1)} \overline{z}_k^{a_k},$$

that yields,

$$\frac{a_k}{b_k} = -\left(\frac{\overline{z}_k}{z_k}\right)^{c_k}.$$

From (12) we get

$$b_k z_k^{a_k} \overline{z}_k^{(b_k-1)} = -a_k z_k^{b_k} \overline{z}_k^{(a_k-1)},$$

that is,

$$\frac{b_k}{a_k} = -\left(\frac{\overline{z}_k}{z_k}\right)^{c_k}.$$

Thus, from (11) and (12), we deduce $\frac{a_k}{b_k} = \frac{b_k}{a_k}$, but this is a contradiction, because $a_k > b_k$. Hence, $\mathsf{Sing}(M) = \{0\}$ and $M^* = M - \{0\}$. □

Now, we prove a technical lemma that will be used in the proof of Theorem A.

Lemma 3.2 *Let $M_{\mathbb{C}}$ be the complexification of* $M = \left\{F(z) = 2\mathrm{Re}\left(\sum_{k=1}^{n} z_k^{a_k} \overline{z}_k^{b_k}\right) = 0\right\}$, $n \geq 3$. *Consider the functions*

$$\begin{aligned} g_j(z, w) &= a_j w_j^{(a_j-1)} z_j^{b_j} + b_j w_j^{(b_j-1)} z_j^{a_j} \\ f_k(z, w) &= a_k z_k^{(a_k-1)} w_k^{b_k} + b_k z_k^{(b_k-1)} w_k^{a_k} \\ h_\ell(z, w) &= a_\ell b_\ell \left(z_\ell^{(a_\ell-1)} w_\ell^{(b_\ell-1)} + z_\ell^{(b_\ell-1)} w_\ell^{(a_\ell-1)}\right), \end{aligned}$$

where $a_m > b_m \geq 1$ are integers for each $m = 1, \ldots, n$, and $n \geq 3$. For each triple (j, k, ℓ) of indices $j, k, \ell = 1, \ldots, n$; there exists $(z_0, w_0) \in M_{\mathbb{C}}$ such that

$$f_k(z_0, w_0) g_j(z_0, w_0) h_\ell(z_0, w_0) \neq 0.$$

Proof In this case, the complexification of F is given by

$$F_{\mathbb{C}}(z, w) = \sum_{k=1}^{n} z_k^{a_k} w_k^{b_k} + w_k^{a_k} z_k^{b_k}. \tag{13}$$

For each triple (j, k, ℓ), let's choose $(z_0, w_0) = (z_1, \ldots, z_n, w_1, \ldots, w_n)$ satisfying:

1. $z_j = z_k = z_\ell = w_j = w_k = 1$,
2. w_ℓ such that $w_\ell^{b_\ell} + w_\ell^{a_\ell} = -4$,
3. $z_m = w_m = 0$ for the remaining indices.

We observe that $(z_0, w_0) \in M_{\mathbb{C}}$. In fact, we have

$$F_{\mathbb{C}}(z_0, w_0) = z_j^{a_j} w_j^{b_j} + w_j^{a_j} z_j^{b_j} + z_k^{a_k} w_k^{b_k} + w_k^{a_k} z_k^{b_k} + z_\ell^{a_\ell} w_\ell^{b_\ell} + w_\ell^{a_\ell} z_\ell^{b_\ell}$$
$$= 4 + w_\ell^{b_\ell} + w_\ell^{a_\ell}$$

then $F_{\mathbb{C}}(z_0, w_0) = 0$ on $M_{\mathbb{C}}^*$. From item (3) above, we obtain

$$w_\ell^{b_\ell - 1} + w_\ell^{a_\ell - 1} = -\frac{4}{w_\ell}.$$

Furthermore, we have

$$g_j(z_0, w_0) = a_j + b_j,$$

$$f_k(z_0, w_0) = a_k + b_k,$$

$$h_\ell(z_0, w_0) = a_\ell b_\ell \left(w_\ell^{(b_\ell - 1)} + w_\ell^{(a_\ell - 1)} \right)$$
$$= -\frac{4 a_\ell b_\ell}{w_\ell}.$$

Hence

$$f_k(z_0, w_0) g_j(z_0, w_0) h_\ell(z_0, w_0) = -\frac{4 a_\ell b_\ell}{w_\ell}(a_j + b_j)(a_k + b_k)$$
$$\neq 0.$$

□

Now, let's restate Theorem A for completeness.

Theorem A *Let G and X be elements in $\chi(\mathbb{C}^n, 0)$, $n \geq 2$, of the form $G(z) = (z_1^{a_1}, \ldots, z_n^{a_n})$ and $X(z) = (z_1^{b_1}, \ldots, z_n^{b_n})$, where $a_k > b_k \geq 1$ are positive integers, for all $k = 1, \ldots, n$. Then $M = \{F(z) = 2\mathrm{Re}(\psi_{G,X}(z)) = 0\}$, i.e.,*

$$M = \left\{ \mathrm{Re}\left(\sum_{k=1}^{n} z_k^{a_k} \bar{z}_k^{b_k} \right) = 0 \right\}.$$

is nondegenerate at $0 \in \mathbb{C}^n$.

Proof We divide the proof into two cases $n = 2$ and $n \geq 3$. For $n = 2$, we examine the Eq. (9) associated to the complexification of $M = \{F = 0\}$, where

$$F(z) = z_1^{a_1}\overline{z}_1^{b_1} + z_2^{a_2}\overline{z}_2^{b_2} + \overline{z}_1^{a_1}z_1^{b_1} + \overline{z}_2^{a_2}z_2^{b_2}. \tag{14}$$

Note that, the complexification of F is given by

$$F_{\mathbb{C}}(z, w) = z_1^{a_1}w_1^{b_1} + z_2^{a_2}w_2^{b_2} + w_1^{a_1}z_1^{b_1} + w_2^{a_2}z_2^{b_2} \tag{15}$$

then $dF_{\mathbb{C}} = \alpha + \beta$, where

$$\alpha = (a_1z_1^{(a_1-1)}w_1^{b_1} + b_1z_1^{(b_1-1)}w_1^{a_1})dz_1 + (a_2z_2^{(a_2-1)}w_2^{b_2} + b_2z_2^{(b_2-1)}w_2^{a_2})dz_2,$$
$$\beta = (a_1w_1^{(a_1-1)}z_1^{b_1} + b_1w_1^{(b_1-1)}z_1^{a_1})dw_1 + (a_2w_2^{(a_2-1)}z_2^{b_2} + b_2w_2^{(b_2-1)}z_2^{a_2})dw_2.$$

Now we have

$$\begin{aligned} d\alpha = a_1b_1 \Big(z_1^{(a_1-1)}w_1^{(b_1-1)} + z_1^{(b_1-1)}w_1^{(a_1-1)})dw_1 \wedge dz_1 \\ +a_2b_2(z_2^{(a_2-1)}w_2^{(b_2-1)} + z_2^{(b_2-1)}w_2^{(a_2-1)}\Big) dw_2 \wedge dz_2. \end{aligned}$$

We use the following notations

$$\begin{aligned} \alpha_1 &= \left(a_1z_1^{(a_1-1)}w_1^{b_1} + b_1z_1^{(b_1-1)}w_1^{a_1}\right), \\ \alpha_2 &= \left(a_2z_2^{(a_2-1)}w_2^{b_2} + b_2z_2^{(b_2-1)}w_2^{a_2}\right), \\ \beta_1 &= \left(a_1w_1^{(a_1-1)}z_1^{b_1} + b_1w_1^{(b_1-1)}z_1^{a_1}\right), \\ \beta_2 &= \left(a_2w_2^{(a_2-1)}z_2^{b_2} + b_2w_2^{(b_2-1)}z_2^{a_2}\right), \\ g_1 &= a_1b_1\left(z_1^{(a_1-1)}w_1^{(b_1-1)} + z_1^{(b_1-1)}w_1^{(a_1-1)}\right), \\ g_2 &= a_2b_2\left(z_2^{(a_2-1)}w_2^{(b_2-1)} + z_2^{(b_2-1)}w_2^{(a_2-1)}\right). \end{aligned}$$

So, we have $\alpha \wedge d\alpha \wedge \beta = (\alpha_1\beta_1g_2 + \alpha_2\beta_2g_1)dz_1 \wedge dz_2 \wedge dw_1 \wedge dw_2$. Therefore,

$$\begin{aligned} \alpha_1\beta_1g_2 &= a_2b_2\left(a_1^2 + b_1^2\right)\left(z_2^{(a_2-1)}w_2^{(b_2-1)} + z_2^{(b_2-1)}w_2^{(a_2-1)}\right)z_1^{(a_1+b_1-1)}w_1^{(a_1+b_1-1)} \\ &\quad + a_1b_1a_2b_2\left(z_1^{(2a_1-1)}w_1^{(2b_1-1)} + z_1^{(2b_1-1)}w_1^{(2a_1-1)}\right)\left(z_2^{(a_2-1)}w_2^{(b_2-1)} + z_2^{(b_2-1)}w_2^{(a_2-1)}\right), \\ \alpha_2\beta_2g_1 &= a_1b_1\left(a_2^2 + b_2^2\right)\left(z_1^{(a_1-1)}w_1^{(b_1-1)} + z_1^{(b_1-1)}w_1^{(a_1-1)}\right)z_2^{(a_2+b_2-1)}w_2^{(a_2+b_2-1)} \\ &\quad + a_1b_1a_2b_2\left(z_2^{(2a_2-1)}w_2^{(2b_2-1)} + z_2^{(2b_2-1)}w_2^{(2a_2-1)}\right)\left(z_1^{(a_1-1)}w_1^{(b_1-1)} + z_1^{(b_1-1)}w_1^{(a_1-1)}\right). \end{aligned}$$

Now, using the expression (15), we obtain the following relationship on $M_{\mathbb{C}}^*$:

$$\left(z_2^{(a_2-1)}w_2^{(b_2-1)}+z_2^{(b_2-1)}w_2^{(a_2-1)}\right)=-\frac{z_1w_1}{z_2w_2}\left(z_1^{(a_1-1)}w_1^{(b_1-1)}+z_1^{(b_1-1)}w_1^{(a_1-1)}\right).$$

With this, we can rewrite

$$\begin{aligned}\alpha_1\beta_1g_2 &= \left(\frac{z_1^{(a_1-1)}w_1^{(b_1-1)}+z_1^{(b_1-1)}w_1^{(a_1-1)}}{z_2w_2}\right)\Big[-a_2b_2\left(a_1^2+b_1^2\right)z_1^{(a_1+b_1)}w_1^{(a_1+b_1)}\\ &\quad - a_1b_1a_2b_2\left(z_1^{2a_1}w_1^{2b_1}+z_1^{2b_1}w_1^{2a_1}\right)\Big],\\ \alpha_2\beta_2g_1 &= \left(\frac{z_1^{(a_1-1)}w_1^{(b_1-1)}+z_1^{(b_1-1)}w_1^{(a_1-1)}}{z_2w_2}\right)\Big[a_1b_1\left(a_2^2+b_2^2\right)z_2^{(a_2+b_2)}w_2^{(a_2+b_2)}\\ &\quad + a_1b_1a_2b_2\left(z_2^{2a_2}w_2^{2b_2}+z_2^{2b_2}w_2^{2a_2}\right)\Big].\end{aligned}$$

In this way, we get

$$\begin{aligned}\alpha_2\beta_2g_1+\alpha_1\beta_1g_2 &= \left(\frac{z_1^{(a_1-1)}w_1^{(b_1-1)}+z_1^{(b_1-1)}w_1^{(a_1-1)}}{z_2w_2}\right)\Big[a_1b_1\left(a_2^2+b_2^2\right)z_2^{(a_2+b_2)}w_2^{(a_2+b_2)}\\ &\quad + a_1b_1a_2b_2\left(z_2^{2a_2}w_2^{2b_2}+z_2^{2b_2}w_2^{2a_2}-z_1^{2a_1}w_1^{2b_1}+z_1^{2b_1}w_1^{2a_1}\right)\\ &\quad - a_2b_2\left(a_1^2+b_1^2\right)z_1^{(a_1+b_1)}w_1^{(a_1+b_1)}\Big].\end{aligned}$$

Again using (15), we obtain

$$\left(z_2^{2a_2}w_2^{2b_2}+z_2^{2b_2}w_2^{2a_2}-z_1^{2a_1}w_1^{2b_1}-z_1^{2b_1}w_1^{2a_1}\right)=2\left(z_1^{(a_1+b_1)}w_1^{(a_1+b_1)}-z_2^{(a_2+b_2)}w_2^{(a_2+b_2)}\right).$$

Hence

$$\begin{aligned}\alpha_2\beta_2g_1+\alpha_1\beta_1g_2 &= \left(\frac{z_1^{(a_1-1)}w_1^{(b_1-1)}+z_1^{(b_1-1)}w_1^{(a_1-1)}}{z_2w_2}\right)\Big[a_1b_1\left(a_2^2+b_2^2\right)z_2^{(a_2+b_2)}w_2^{(a_2+b_2)}\\ &\quad + 2a_1b_1a_2b_2\left(z_1^{(a_1+b_1)}w_1^{(a_1+b_1)}-z_2^{(a_2+b_2)}w_2^{(a_2+b_2)}\right)\\ &\quad - a_2b_2\left(a_1^2+b_1^2\right)z_1^{(a_1+b_1)}w_1^{(a_1+b_1)}\Big],\end{aligned}$$

that is,

$$\begin{aligned}\alpha_2\beta_2g_1+\alpha_1\beta_1g_2 &= \left(\frac{z_1^{(a_1-1)}w_1^{(b_1-1)}+z_1^{(b_1-1)}w_1^{(a_1-1)}}{z_2w_2}\right)\Big[a_1b_1(a_2-b_2)^2\left(z_2^{(a_2+b_2)}w_2^{(a_2+b_2)}\right)\\ &\quad - a_2b_2(a_1-b_1)^2\left(z_1^{(a_1+b_1)}w_1^{(a_1+b_1)}\right)\Big].\end{aligned}$$

Therefore, by (9), we conclude that $\eta|_{M^*}$ is integrable if and only if $a_1b_1(a_2-b_2)^2=0$ and $a_2b_2(a_1-b_1)^2=0$, that is, $a_2=b_2$ and $a_1=b_1$. This completes the proof for $n=2$.

Now, we consider $n\geq 3$. With the notation introduced in Lemma 3.2, we can rewrite

$$\alpha=\sum_{k=1}^{n} f_k dz_k, \qquad \beta=\sum_{k=1}^{n} g_k dw_k.$$

Then $d\alpha=\sum_{\ell=1}^{n} h_\ell dw_\ell\wedge dz_\ell$. Indeed, we have

$$d\alpha=\sum_{k,\ell=1}^{n}\frac{\partial f_k}{\partial z_\ell}dz_\ell\wedge dz_k+\sum_{k,\ell=1}^{n}\frac{\partial f_k}{\partial w_\ell}dw_\ell\wedge dz_k.$$

Furthermore, we have the following relationships:

$$\begin{aligned}
\frac{\partial f_k}{\partial z_\ell}&=0,\ \text{ for } k\neq\ell,\\
\frac{\partial f_k}{\partial w_\ell}&=0,\ \text{ for } k\neq\ell,\\
\frac{\partial f_\ell}{\partial w_\ell}&=a_\ell b_\ell\left(z_\ell^{(a_\ell-1)}w_\ell^{(b_\ell-1)}+z_\ell^{(b_\ell-1)}w_\ell^{(a_\ell-1)}\right).
\end{aligned}$$

Hence, it follows that

$$d\alpha=\sum_{\ell=1}^{n}a_\ell b_\ell\left(z_\ell^{(a_\ell-1)}w_\ell^{(b_\ell-1)}+z_\ell^{(b_\ell-1)}w_\ell^{(a_\ell-1)}\right)dw_\ell\wedge dz_\ell=\sum_{\ell=1}^{n}h_\ell\, dw_\ell\wedge dz_\ell.$$

Thus, we have

$$\alpha\wedge d\alpha\wedge\beta=\sum_{\ell\neq j,k} f_k g_j h_\ell\, dz_k\wedge dw_\ell\wedge dz_\ell\wedge dw_j. \tag{16}$$

We observe that for $j\neq k$, the coefficient of the term $dz_k\wedge dw_\ell\wedge dz_\ell\wedge dw_j$ in 4-form $\alpha\wedge d\alpha\wedge\beta$ is exactly $f_kg_jh_\ell$. In other words, if $j\neq k$,

$$f_kg_jh_\ell\, dz_k\wedge dw_\ell\wedge dz_\ell\wedge dw_j$$

is the only term in Eq. (16) that is of the form $\rho\, dz_k\wedge dw_\ell\wedge dz_\ell\wedge dw_j$. Thus, given a triple (j,k,ℓ) of pairwise distinct indices, by Lemma (3.2) there exists $(z_0,w_0)\in M_{\mathbb{C}}$ such that

$$\alpha(z_0,w_0)\wedge d\alpha(z_0,w_0)\wedge\beta(z_0,w_0)\neq 0.$$

Therefore, M is not Levi-flat, and we conclude the proof of Theorem A. □

4 The Case of Vector Fields $G = (z_1^{a_1}, z_2^{a_2})$ and $X = (z_{\sigma_1}^{b_{\sigma_1}}, z_{\sigma_2}^{b_{\sigma_2}})$

In this section, we will explore some results regarding the singular set of the hypersurface. $M = \{F(z) = 0\}$, where $F(z) = 2\mathrm{Re}\langle G(z), X(z)\rangle$, $G = (z_1^{a_1}, z_2^{a_2})$, and $X = (z_{\sigma_1}^{b_{\sigma_1}}, z_{\sigma_2}^{b_{\sigma_2}})$, where $\sigma \in \mathcal{S}_n$ is a permutation of the set $\{1, \ldots, n\}$. Let's start by considering the particular case where $G = (z_1^a, z_2^b)$ and $X = (z_2^b, z_1^a)$ with a, b being positive integers. In this case, the Hermitian product of G with X is given by

$$\langle G, X\rangle = z_1^a \overline{z}_2^b + \overline{z}_1^a z_2^b.$$

We observe that $\langle G, X\rangle \in \mathbb{R}$. Therefore, let's consider

$$M = \{F(z_1, z_2) = z_1^a \overline{z}_2^b + \overline{z}_1^a z_2^b = 0\}. \tag{17}$$

We observe that for these fields, the map $\psi_{G,X}$ does not satisfy the Milnor condition, as stated in theorem (Ruas et al. 2002, Theorem 2.7). In this case, the hypersurface M will be Levi-flat, as we will show in Theorem B. First, we will verify the following result regarding the singular set of M.

Proposition 4.1 *Let M be the hypersurface described in Eq. (17). Then we have the following options for* $\mathrm{Sing}(M)$*:*

1. $\mathrm{Sing}(M) = \{0\}$, *if* $b = a = 1$,
2. $\mathrm{Sing}(M) = \{z_1 = 0\}$, *if* $b = 1$ *and* $a > 1$,
3. $\mathrm{Sing}(M) = \{z_2 = 0\}$, *if* $a = 1$ *and* $b > 1$,
4. $\mathrm{Sing}(M) = \{z_1 = 0\} \cup \{z_2 = 0\}$, *if* $a, b > 1$.

Proof The partial derivatives of F are given by

$$\partial F = a z_1^{(a-1)} \overline{z}_2^b dz_1 + b \overline{z}_1^a z_2^{(b-1)} dz_2,$$
$$\overline{\partial} F = a \overline{z}_1^{(a-1)} z_2^b d\overline{z}_1 + b z_1^a \overline{z}_2^{(b-1)} d\overline{z}_2.$$

Thus, $(z_1, z_2) \in \mathrm{Sing}(M)$ if and only if the following equations are satisfied:

$$a z_1^{(a-1)} \overline{z}_2^b = 0$$
$$b \overline{z}_1^a z_2^{(b-1)} = 0$$
$$a \overline{z}_1^{(a-1)} z_2^b = 0$$
$$b z_1^a \overline{z}_2^{(b-1)} = 0$$

If $a, b > 1$, it follows that $\mathrm{Sing}(M) = \{z_1 = 0\} \cup \{z_2 = 0\}$. If $a = 1$ or $b = 1$, we will have $\mathrm{Sing}(M) = \{z_2 = 0\}$ or $\mathrm{Sing}(M) = \{z_1 = 0\}$, respectively. Finally, if $a = b = 1$, we obtain $\mathrm{Sing}(M) = \{0\}$. □

Let's restate Theorem B for completeness.

Theorem B *Let G and X be elements in $\chi(\mathbb{C}^n, 0)$, $n \geq 2$, of the form $G(z) = (z_1^a, z_2^b)$ and $X(z) = (z_2^b, z_1^a)$ with a, b positive integers. Then $M = \{F(z) = 2\mathrm{Re}(\psi_{G,X}(z)) = 0\}$, i.e.,*

$$M = \left\{ z_1^a \bar{z}_2^b + z_2^b \bar{z}_1^a = 0 \right\}$$

is a Levi-flat hypersurface at $0 \in \mathbb{C}^2$.

Proof In this case, F is given by $F(z) = z_1^a \bar{z}_2^b + \bar{z}_1^a z_2^b$ and its complexification will be given by

$$F_{\mathbb{C}}(z, w) = z_1^a w_2^b + w_1^a z_2^b. \tag{18}$$

Therefore, we have on $M_{\mathbb{C}}^*$

$$\begin{aligned} \alpha &= a z_1^{(a-1)} w_2^b dz_1 + b w_1^a z_2^{(b-1)} dz_2, \\ \beta &= a w_1^{(a-1)} z_2^b dw_1 + b z_1^a w_2^{(b-1)} dw_2, \\ d\alpha &= ab z_1^{(a-1)} w_2^{(b-1)} dw_2 \wedge dz_1 + ab w_1^{(a-1)} z_2^{(b-1)} dw_1 \wedge dz_2. \end{aligned}$$

Hence

$$\begin{aligned} \alpha \wedge d\alpha \wedge \beta = \Big(& a^2 b^2 z_1^{(2a-1)} w_2^{(2b-1)} w_1^{(a-1)} z_2^{(b-1)} \\ & + a^2 b^2 w_1^{(2a-1)} z_2^{(2b-1)} z_1^{(a-1)} w_2^{(b-1)} \Big) dz_1 \wedge dw_1 \wedge dz_2 \wedge dw_2. \end{aligned}$$

From (18), we get

$$z_1^{(a-1)} w_2^{(b-1)} = -\left(\frac{w_1 z_2}{z_1 w_2} \right) w_1^{(a-1)} z_2^{(b-1)},$$

which implies that $\alpha \wedge d\alpha \wedge \beta$ is equal to

$$\begin{aligned} \Big[& a^2 b^2 z_1^{(2a-1)} w_2^{(2b-1)} w_1^{(a-1)} z_2^{(b-1)} \\ & - a^2 b^2 \left(\frac{w_1 z_2}{z_1 w_2} \right) w_1^{(2a-1)} z_2^{(2b-1)} w_1^{(a-1)} z_2^{(b-1)}) dz_1 \Big] dz_1 \wedge dw_1 \wedge dz_2 \wedge dw_2, \end{aligned}$$

that is,

$$\alpha \wedge d\alpha \wedge \beta = \left[a^2 b^2 \frac{w_1^{(a-1)} z_2^{(b-1)}}{z_1 w_2} \left(z_1^{2a} w_2^{2b} - w_1^{2a} z_2^{2b} \right) \right] dz_1 \wedge dw_1 \wedge dz_2 \wedge dw_2.$$

Again, from (18), we have $z_1^a w_2^b = -w_1^a z_2^b$, which yields $z_1^{2a} w_2^{2b} = w_1^{2a} z_2^{2b}$ in $M_{\mathbb{C}}$. Thus, M is Levi-flat. □

Now, we consider

$$M = \{F(z) = \mathrm{Re}(z_1^{a_1}\bar{z}_2^{b_2} + z_2^{a_2}\bar{z}_1^{b_1}) = 0\} \tag{19}$$

In Proposition 4.1, we saw that $\mathrm{Sing}(M) = \{z_1 = 0\} \cup \{z_2 = 0\}$ in the case where $a_1 = b_1 > 1$ and $a_2 = b_2 > 1$. Now, let's verify that this also occurs in the case where $a_1 > b_1 > 1$ and $a_2 > b_2 > 1$ (even without the assumption $a_1 b_2 = a_2 b_1$).

Proposition 4.2 *Let M be the hypersurface described in Eq. (19). Then, we have the following options for* $\mathrm{Sing}(M)$*:*

1. $\mathrm{Sing}(M) = \{0\}$, *if* $b_1 = b_2 = 1$*;*
2. $\mathrm{Sing}(M) = \{z_1 = 0\}$, *if* $b_2 = 1$ *and* $b_1 > 1$*;*
3. $\mathrm{Sing}(M) = \{z_2 = 0\}$, *if* $b_1 = 1$ *and* $b_2 > 1$*;*
4. $\mathrm{Sing}(M) = \{z_1 = 0\} \cup \{z_2 = 0\}$ *if* $b_1, b_2 > 1$.

Proof We observe that F is given by

$$F(z) = z_1^{a_1}\bar{z}_2^{b_2} + \bar{z}_1^{a_1} z_2^{b_2} + z_2^{a_2}\bar{z}_1^{b_1} + \bar{z}_2^{a_2} z_1^{b_1},$$

and its partial derivatives are given by

$$\partial F = \frac{1}{2}\left[\left(a_1 z_1^{(a_1-1)}\bar{z}_2^{b_2} + b_1 z_1^{(b_1-1)}\bar{z}_2^{a_2}\right) dz_1 + \left(a_2 z_2^{(a_2-1)}\bar{z}_1^{b_1} + b_2 z_2^{(b_2-1)}\bar{z}_1^{a_1}\right) dz_2\right],$$
$$\bar{\partial} F = \frac{1}{2}\left[\left(a_1 \bar{z}_1^{(a_1-1)} z_2^{b_2} + b_1 \bar{z}_1^{(b_1-1)} z_2^{a_2}\right) d\bar{z}_1 + \left(a_2 \bar{z}_2^{(a_2-1)} z_1^{b_1} + b_2 \bar{z}_2^{(b_2-1)} z_1^{a_1}\right) d\bar{z}_2\right].$$

Therefore, $(z_1, z_2) \in M$ belongs to the singular set if it satisfies the equations

$$a_1 z_1^{(a_1-1)}\bar{z}_2^{b_2} + b_1 z_1^{(b_1-1)}\bar{z}_2^{a_2} = 0 \tag{20}$$
$$a_1 \bar{z}_1^{(a_1-1)} z_2^{b_2} + b_1 \bar{z}_1^{(b_1-1)} z_2^{a_2} = 0 \tag{21}$$
$$a_2 z_2^{(a_2-1)}\bar{z}_1^{b_1} + b_2 z_2^{(b_2-1)}\bar{z}_1^{a_1} = 0 \tag{22}$$
$$a_2 \bar{z}_2^{(a_2-1)} z_1^{b_1} + b_2 \bar{z}_2^{(b_2-1)} z_1^{a_1} = 0 \tag{23}$$

Let's assume that $b_1, b_2 > 1$. Then we see that $\{z_1 = 0\} \cup \{z_2 = 0\} \subset \mathrm{Sing}(M)$. Now, suppose by contradiction that $(z_1, z_2) \in \mathrm{Sing}(M)$ with $z_1 \neq 0$ and $z_2 \neq 0$. From Eq. (20), we obtain

$$\frac{a_1}{b_1} = -\frac{z_1^{(b_1-1)}\bar{z}_2^{a_2}}{z_1^{(a_1-1)}\bar{z}_2^{b_2}} = -\frac{\bar{z}_2^{(a_2-b_2)}}{z_1^{(a_1-b_1)}},$$

and from Eq. (23), we obtain

$$\frac{a_2}{b_2} = -\frac{\bar{z}_2^{(b_2-1)} z_1^{a_1}}{\bar{z}_2^{(a_2-1)} z_1^{b_1}} = -\frac{z_1^{(a_1-b_1)}}{\bar{z}_2^{(a_2-b_2)}},$$

which implies

$$\frac{a_1}{b_1} = \frac{b_2}{a_2},$$

In other words, $a_1 a_2 = b_1 b_2$. This is absurd, since $a_1 > b_1$ and $a_2 > b_2$ by assumption. We conclude that $\mathrm{Sing}(M) = \{z_1 = 0\} \cup \{z_2 = 0\}$, if $b_1, b_2 > 1$. Now let's assume $b_1 = 1$ and $b_2 > 1$. Then, the equations for the singular set are given by

$$a_1 z_1^{(a_1-1)} \overline{z}_2^{b_2} + \overline{z}_2^{a_2} = 0 \tag{24}$$
$$a_1 \overline{z}_1^{(a_1-1)} z_2^{b_2} + z_2^{a_2} = 0 \tag{25}$$
$$a_2 z_2^{(a_2-1)} \overline{z}_1 + b_2 z_2^{(b_2-1)} \overline{z}_1^{a_1} = 0 \tag{26}$$
$$a_2 \overline{z}_2^{(a_2-1)} z_1 + b_2 \overline{z}_2^{(b_2-1)} z_1^{a_1} = 0 \tag{27}$$

Clearly, $\{z_2 = 0\} \subset \mathrm{Sing}(M)$. Now suppose $z_2 \neq 0$, then we necessarily have $z_1 \neq 0$ by Eq. (24). Thus, it follows from Eqs. (24) and (27):

$$a_1 = -\frac{\overline{z}_2^{(a_2-b_2)}}{z_1^{(a_1-1)}},$$
$$\frac{a_2}{b_2} = -\frac{z_1^{(a_1-1)}}{\overline{z}_2^{(a_2-b_2)}},$$

which implies $a_1 = \frac{b_2}{a_2}$, that is, $a_1 a_2 = b_2$. However, this contradicts $a_2 > b_2$ and $a_1 > 1$. Therefore, $\mathrm{Sing}(M) = \{z_2 = 0\}$ if $b_1 = 1$ and $b_2 > 1$. In a similar way we obtain $\mathrm{Sing}(M) = \{z_1 = 0\}$ if $b_2 = 1$ and $b_1 > 1$, and we also obtain $\mathrm{Sing}(M) = \{0\}$ if $b_1 = b_2 = 1$. □

To prove Theorem C, we will use the following lemma

Lemma 4.3 *Let $M_{\mathbb{C}}$ be the complexification of*

$$M = \{F(z) = z_1^{a_1} \overline{z}_2^{b_2} + \overline{z}_1^{a_1} z_2^{b_2} + z_2^{a_2} \overline{z}_1^{b_1} + \overline{z}_2^{a_2} z_1^{b_1} = 0\},$$

we have $\alpha \wedge d\alpha \wedge \beta = h\, dz_1 \wedge dw_1 \wedge dz_2 \wedge dw_2$, where

$$h = c(a_1 a_2 + b_1 b_1 - 2c)\left(z_1^{a_1+b_1} w_2^{a_2+b_2} - z_2^{(a_2+b_2)} w_1^{(a_1+b_1)}\right)\left(\frac{z_2^{(a_2-1)} w_1^{(b_1-1)} + z_2^{(b_2-1)} w_1^{(a_1-1)}}{z_1 w_2}\right)$$

and $c = a_1 b_2 = a_2 b_1$.

Proof The complexification of F is given by

$$F_{\mathbb{C}} = z_1^{a_1} w_2^{b_2} + w_1^{a_1} z_2^{b_2} + z_2^{a_2} w_1^{b_1} + w_2^{a_2} z_1^{b_1}.$$

Therefore, we have

$$\begin{aligned}
\alpha &= \left(a_1 z_1^{(a_1-1)} w_2^{b_2} + b_1 z_1^{(b_1-1)} w_2^{a_2}\right) dz_1 + \left(a_2 z_2^{(a_2-1)} w_1^{b_1} + b_2 z_2^{(b_2-1)} w_1^{a_1}\right) dz_2,\\
\beta &= \left(a_1 w_1^{(a_1-1)} z_2^{b_2} + b_1 w_1^{(b_1-1)} z_2^{a_2}\right) dw_1 + \left(a_2 w_2^{(a_2-1)} z_1^{b_1} + b_2 w_2^{(b_2-1)} z_1^{a_1}\right) dw_2,\\
d\alpha &= \left(a_1 b_2 z_1^{(a_1-1)} w_2^{(b_2-1)} + a_2 b_1 z_1^{(b_1-1)} w_2^{(a_2-1)}\right) dw_2 \wedge dz_1\\
&\quad + \left(a_2 b_1 z_2^{(a_2-1)} w_1^{(b_1-1)} + a_1 b_2 z_2^{(b_2-1)} w_1^{(a_1-1)}\right) dw_1 \wedge dz_2.
\end{aligned}$$

We will use the notations

$$\begin{aligned}
\alpha_1 &= a_1 z_1^{(a_1-1)} w_2^{b_2} + b_1 z_1^{(b_1-1)} w_2^{a_2};\\
\alpha_2 &= a_2 w_1^{b_1} z_2^{(a_2-1)} + b_2 w_1^{a_1} z_2^{(b_2-1)};\\
\beta_1 &= a_1 w_1^{(a_1-1)} z_2^{b_2} + b_1 w_1^{(b_1-1)} z_2^{a_2};\\
\beta_2 &= a_2 z_1^{b_1} w_2^{(a_2-1)} + b_2 z_1^{a_1} w_2^{(b_2-1)};\\
g_1 &= a_1 b_2 z_1^{(a_1-1)} w_2^{(b_2-1)} + a_2 b_1 z_1^{(b_1-1)} w_2^{(a_2-1)};\\
g_2 &= a_2 b_1 w_1^{(b_1-1)} z_2^{(a_2-1)} + a_1 b_2 w_1^{(a_1-1)} z_2^{(b_2-1)}.
\end{aligned}$$

so that we can write:

$$\begin{aligned}
\alpha &= \alpha_1 dz_1 + \alpha_2 dz_2;\\
\beta &= \beta_1 dw_1 + \beta_2 dw_2;\\
d\alpha &= g_1 dw_2 \wedge dz_1 + g_2 dw_1 \wedge dz_2.
\end{aligned}$$

With these notations, we get

$$\alpha \wedge d\alpha \wedge \beta = (\alpha_1 \beta_2 g_2 + \alpha_2 \beta_1 g_1) dz_1 \wedge dw_1 \wedge dz_2 \wedge dw_2.$$

In $M_{\mathbb{C}}$, we have

$$z_1^{a_1} w_2^{b_2} + z_1^{b_1} w_2^{a_2} = -(w_1^{a_1} z_2^{b_2} + w_1^{b_1} z_2^{a_2}), \tag{28}$$

which implies on $M_{\mathbb{C}}^*$

$$z_1^{(a_1-1)} w_2^{(b_2-1)} + z_1^{(b_1-1)} w_2^{(a_2-1)} = -\left(\frac{w_1 z_2}{z_1 w_2}\right)(w_1^{(a_1-1)} z_2^{(b_2-1)} + w_1^{(b_1-1)} z_2^{(a_2-1)}).$$

Taking $c = a_1 b_2 = b_1 a_2$, we have

$$g_1 = c(z_1^{(a_1-1)} w_2^{(b_2-1)} + z_1^{(b_1-1)} w_2^{(a_2-1)}) = -c\left(\frac{w_1 z_2}{z_1 w_2}\right)(w_1^{(a_1-1)} z_2^{(b_2-1)} + w_1^{(b_1-1)} z_2^{(a_2-1)}),$$

and it follows that

$$g_1 = -\left(\frac{w_1 z_2}{z_1 w_2}\right) g_2.$$

Therefore, on $M^*_{\mathbb{C}}$, we obtain

$$\alpha_1\beta_2 g_2 + \alpha_2\beta_1 g_1 = \alpha_1\beta_2 g_2 - \alpha_2\beta_1 \left(\frac{w_1 z_2}{z_1 w_2}\right) g_2 = \frac{g_2}{z_1 w_2}(z_1 w_2 \alpha_1 \beta_2 - w_1 z_2 \alpha_2 \beta_1),$$

thus

$$\alpha \wedge d\alpha \wedge \beta = \left[\frac{g_2}{z_1 w_2}(z_1 w_2 \alpha_1 \beta_2 - w_1 z_2 \alpha_2 \beta_1)\right] dz_1 \wedge dw_1 \wedge dz_2 \wedge dw_2. \quad (29)$$

Now, we observe

$$\begin{aligned}\alpha_1\beta_2 &= \left(a_1 z_1^{(a_1-1)} w_2^{b_2} + b_1 z_1^{(b_1-1)} w_2^{a_2}\right)\left(a_2 z_1^{b_1} w_2^{(a_2-1)} + b_2 z_1^{a_1} w_2^{(b_2-1)}\right)\\ &= (a_1a_2 + b_1b_2)\, z_1^{(a_1+b_1-1)} w_2^{(a_2+b_2-1)} + c\left(z_1^{(2a_1-1)} w_2^{(2b_2-1)} + z_1^{(2b_1-1)} w_2^{(2a_2-1)}\right),\end{aligned}$$

which implies

$$z_1 w_2 \alpha_1 \beta_2 = (a_1a_2 + b_1b_2)\, z_1^{(a_1+b_1)} w_2^{(a_2+b_2)} + c\left(z_1^{2a_1} w_2^{2b_2} + z_1^{2b_1} w_2^{2a_2}\right).$$

Similarly, we obtain

$$w_1 z_2 \alpha_2 \beta_1 = (a_1a_2 + b_1b_2) w_1^{(a_1+b_1)} z_2^{(a_2+b_2)} + c(w_1^{2b_1} z_2^{2a_2} + w_1^{2a_1} z_2^{2b_2}),$$

so that $z_1 w_2 \alpha_1 \beta_2 - w_1 z_2 \alpha_2 \beta_1$ is equal to

$$\begin{aligned}&(a_1a_2 + b_1b_2)\left(z_1^{(a_1+b_1)} w_2^{(a_2+b_2)} - w_1^{(a_1+b_1)} z_2^{(a_2+b_2)}\right)\\ &\quad + c\left(z_1^{2a_1} w_2^{2b_2} + z_1^{2b_1} w_2^{2a_2} - w_1^{2b_1} z_2^{2a_2} - w_1^{2a_1} z_2^{2b_2}\right).\end{aligned}$$

Again, from Eq. (28), on $M^*_{\mathbb{C}}$ we have:

$$(z_1^{a_1} w_2^{b_2} + z_1^{b_1} w_2^{a_2})^2 = (w_1^{a_1} z_2^{b_2} + w_1^{b_1} z_2^{a_2})^2,$$

that is,

$$2(w_1^{(a_1+b_1)} z_2^{(a_2+b_2)} - z_1^{(a_1+b_1)} w_2^{(a_2+b_2)}) = z_1^{2a_1} w_2^{2b_2} + z_1^{2b_1} w_2^{2a_2} - w_1^{2b_1} z_2^{2a_2} - w_1^{2a_1} z_2^{2b_2},$$

which implies that $z_1 w_2 \alpha_1 \beta_2 - w_1 z_2 \alpha_2 \beta_1$ is equal to

$$\begin{aligned}&(a_1a_2 + b_1b_2)\left(z_1^{(a_1+b_1)} w_2^{(a_2+b_2)} - w_1^{(a_1+b_1)} z_2^{(a_2+b_2)}\right)\\ &\quad +2c\left(w_1^{(a_1+b_1)} z_2^{(a_2+b_2)} - z_1^{(a_1+b_1)} w_2^{(a_2+b_2)}\right).\end{aligned}$$

So, $z_1w_2\alpha_1\beta_2 - w_1z_2\alpha_2\beta_1$ is equal to

$$(a_1a_2 + b_1b_2 - 2c)\left(z_1^{(a_1+b_1)}w_2^{(a_2+b_2)} - w_1^{(a_1+b_1)}z_2^{(a_2+b_2)}\right).$$

Finally, we conclude

$$\frac{g_2}{z_1w_2}(z_1w_2\alpha_1\beta_2 - w_1z_2\alpha_2\beta_1) = \frac{g_2}{z_1w_2}(a_1a_2 + b_1b_2 - 2c) \left(z_1^{(a_1+b_1)}w_2^{(a_2+b_2)} - w_1^{(a_1+b_1)}z_2^{(a_2+b_2)}\right)$$

and the lemma follows by substituting the above expression and the expression for g_2 into Eq. (29). □

Now, we prove Theorem C.

Theorem C *Let G and X be elements in $\chi(\mathbb{C}^n, 0)$, $n \geq 2$, of the form $G = (z_1^{a_1}, z_2^{a_2})$ and $X = (z_2^{b_2}, z_1^{b_1})$, where $a_1 \geq b_1$ and $a_2 \geq b_2$ are positive integers satisfying $a_1b_2 = a_2b_1$. Then $M = \{F(z) = 2\mathrm{Re}(\psi_{G,X}(z)) = 0\}$, i.e.,*

$$M = \left\{\mathrm{Re}\left(z_1^{a_1}\bar{z}_2^{b_2} + z_2^{a_2}\bar{z}_1^{b_1}\right) = 0\right\}$$

is Levi-flat if, and only if, $a_1 = b_1$ and $a_2 = b_2$.

Proof From Lemma 4.3, we have $\alpha(z, w) \wedge \beta(z, w) \wedge d\alpha(z, w) = 0$ for all $(z, w) \in M_{\mathbb{C}}$ if, and only if, $a_1a_2 + b_1b_2 - 2c = 0$, i.e., $2c = a_1a_2 + b_1b_2$. Using the fact $c = a_1b_2 = b_1a_2$, it follows that

$$2c = a_1a_2 + b_1b_2 \iff a_1b_2 + b_1a_2 = a_1a_2 + b_1b_2 \iff b_1(a_2 - b_2) = a_1(a_2 - b_2).$$

Therefore, $M_{\mathbb{C}}$ is Levi-flat if, and only if, $a_1 = b_1$ and $a_2 = b_2$. This concludes the proof of Theorem C. □

Remark 4.1 From the above result, we conclude that for $n = 2$, the hypersurface M will not Levi-flat precisely when the map $\psi_{G,X}$ satisfies the Milnor condition at the origin.

5 Proof of Theorem D

Now, we study the case with permutations in higher dimensions. Consider $\sigma \in S_n$ a permutation on the set $\underline{n} := \{1, \ldots, n\}$, and let's use the notation $\sigma_k := \sigma(k)$. We employ a transversality argument, along with the dimension 2 case (see Theorem C), to obtain the following result:

Theorem D *Let G and X be elements in $\chi(\mathbb{C}^n, 0)$, $n \geq 2$, of the form $G = (z_1^{a_1}, \ldots, z_n^{a_n})$ and $X = (z_{\sigma_1}^{b_{\sigma_1}}, \ldots, z_{\sigma_n}^{b_{\sigma_n}})$, where $a_k \geq b_k$ are positive integers. Let* *us assume that for some* $\ell \in \underline{n}$*, the integers* $a_\ell, b_\ell, a_{\sigma_\ell}, b_{\sigma_\ell}$ *satisfy the following conditions:* $a_\ell > b_\ell$ *and* $a_\ell b_{\sigma_\ell} = b_\ell a_{\sigma_\ell}$*. Then,* $M = \{F(z) = 2\mathrm{Re}(\psi_{G,X}(z)) = 0\}$*, i.e.,*

$$M = \left\{\mathrm{Re}\left(\sum_{k=1}^{n} z_k^{a_k} \bar{z}_{\sigma_k}^{b_{\sigma_k}}\right) = 0\right\}.$$

is nondegenerate at $0 \in \mathbb{C}^n$.

Proof Let's assume by contradiction that M is Levi-flat. Therefore, the regular part M^* is foliated by complex submanifolds of (complex) dimension $n-1$. Thus, we can choose $i : \mathbb{C}^2 \hookrightarrow \mathbb{C}^n$ to be a transversal embedding to M (see Cerveau and Lins-Neto 2011, Corollary 3.3), so that the regular part of $i^{-1}(M)$ is also foliated by complex submanifolds of dimension 1 (Riemann surfaces). Without loss of generality, we can assume that $\sigma(1) = 2$ and that a_1, b_1, a_2, b_2 are the integers satisfying the conditions in the statement. Making a change of coordinates and using the fact that the Levi-flat property is invariant under biholomorphisms (Proposition 2.4), we can assume that the embedding $i : \mathbb{C}^2 \to \mathbb{C}^n$ is given by $i : (z_1, z_2) \mapsto (z_1, z_2, 0, \ldots, 0)$. We observe

$$i^{-1}(M) = \{(z_1, z_2) \in \mathbb{C}^2 : z_1^{a_1}\bar{z}_2^{b_2} + \bar{z}_1^{a_1} z_2^{b_2} + z_2^{a_2}\bar{z}_1^{b_1} + \bar{z}_2^{a_2} z_1^{b_1} = 0\}.$$

But by Theorem C, $i^{-1}(M)$ does not have the regular part foliated by complex submanifolds. Therefore, M is not Levi-flat. □

6 Examples

In this section, we will explore examples where the fields G and X do not satisfy the Milnor condition. In these examples, our hypersurfaces are all Levi-flat.

Example 6.1 Given the Pham–Brieskorn polynomial $f : \mathbb{C}^2 \to \mathbb{C}$ defined by $f(z) = z_1^p + z_2^q$, where $p, q > 2$. Consider the holomorphic vector field

$$G(z) = \left(\frac{\partial f(z)}{\partial z_2}, -\frac{\partial f(z)}{\partial z_1}\right) = \left(q z_2^{(q-1)}, -p z_1^{(p-1)}\right)$$

whose solutions represent the fibers of f. Also, take the vector field $X = (a_1, a_2)$. Thus, the Hermitian product of G and X is given by

$$\Psi_{G,X}(z) = \langle G(z), X(z)\rangle = \overline{a}_1 q z_2^{(q-1)} - \overline{a_2} p z_1^{(p-1)}.$$

Clearly, $\Psi_{G,X}$ does not satisfy the Milnor fibration condition at the origin. Now we take

$$M = \{F(z) = 2\mathrm{Re}\Psi_{G,X}(z) = 0\}.$$

Since $\Psi_{G,X}$ is a holomorphic function, then M is Levi-flat.

Example 6.2 Let $G = (z_1, iz_2, (-1-i)z_3)$ and $X = (z_1, z_2, z_3)$. According to Seade (1997, Theorem 1), $\psi_{G,X}$ does not satisfy the Milnor fibration condition at the origin. The Hermitian product of G and X is given by

$$\langle G, X\rangle = z_1\overline{z}_1 + iz_2\overline{z}_2 + (-1-i)z_3\overline{z}_3.$$

Consider $M = \{F(z) = 2\mathrm{Re}\,\langle G(z), X(z)\rangle = z_1\overline{z}_1 - z_3\overline{z}_3 = 0\}$. Let's us verify that M is Levi-flat. First, we note that $F(z) = z_1\overline{z}_1 - z_3\overline{z}_3$, and the partial derivatives of F are given by

$$\partial F = \overline{z}_1 dz_1 - \overline{z}_3 dz_3,$$
$$\overline{\partial} F = z_1 d\overline{z}_1 - z_3 d\overline{z}_3.$$

Then, $\mathsf{Sing}(M) = \{(0, z_2, 0) \in \mathbb{C}^3 : z_2 \in \mathbb{C}\}$. Clearly, $\dim_{\mathbb{R}} \mathsf{Sing}(M) = 2$. Moreover,

$$M^* = \{(z_1, z_2, z_3) \in \mathbb{C}^3 : z_1\overline{z}_1 - z_3\overline{z}_3 = 0, z_1 \neq 0 \text{ and } z_3 \neq 0\}.$$

We have $\partial\overline{\partial}F = dz_1 \wedge d\overline{z}_1 - dz_3 \wedge d\overline{z}_3$, and

$$\partial F \wedge \overline{\partial} F = z_1\overline{z}_1 dz_1 \wedge d\overline{z}_1 - \overline{z}_1 z_3 dz_1 \wedge d\overline{z}_3 - \overline{z}_3 z_1 dz_3 \wedge d\overline{z}_1 + z_3\overline{z}_3 dz_3 \wedge d\overline{z}_3,$$

that yields $\partial F \wedge \overline{\partial} F \wedge \partial\overline{\partial} F = (z_3\overline{z}_3 - z_1\overline{z}_1) dz_1 \wedge d\overline{z}_1 \wedge dz_3 \wedge d\overline{z}_3$. Thus, $\partial F(p) \wedge \overline{\partial} F(p) \wedge \partial\overline{\partial} F(p) = 0$ for all $p \in M$ and therefore, M is Levi-flat.

Example 6.3 Let $G(z) = (z_1, z_2)$ and $X(z) = (-iz_2, iz_1)$. Then

$$M = \{F(z) = 2\mathrm{Re}\,\langle G(z), X(z)\rangle = i(z_2\overline{z}_1 - z_1\overline{z}_2) = 0\}$$

is clearly Levi-flat whose singular set is $\mathsf{Sing}(M) = \{0\}$.

In general, when $G(z) = (z_1, z_2)$ is the radial vector field, $X = (\lambda_1 z_1, \lambda_2 z_2)$, and if $\mathrm{Re}(\lambda_k) \neq 0$ for $k = 1, 2$, then the singular set of $M = \{2\mathrm{Re}\,\langle G(z), X(z)\rangle = 0\}$ is just the origin $0 \in \mathbb{C}^2$, and M is Levi-flat. However, in this case, the map $\psi_{G,X}$ also does not satisfy the Milnor fibration condition at the origin, see Seade (1996, Example 3.4).

Acknowledgements Ayane A. Da Silva acknowledges support by a Ph.D grant from Capes-Brazil, and the results of this paper are part of her Ph.D thesis. Fernández-Pérez acknowledges support from CNPq Projeto Universal 408687/2023-1 "Geometria das Equações Diferenciais Algébricas" and CNPq Projeto PQ 2023 306011/2023-9. Finally, we also would like to thank the referee for suggestions and pointing out corrections.

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