Bull Braz Math Soc, New Series (2024) 55:18
https://doi.org/10.1007/s00574-024-00392-w

Some Results on Shadowing and Local Entropy Properties of Dynamical Systems

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Received: 25 September 2023 / Accepted: 5 March 2024 / Published online: 2 April 2024

Abstract
We consider some local entropy properties of dynamical systems under the assumption of shadowing. In the first part, we give necessary and sufficient conditions for shadowable points to be certain entropy points. In the second part, we give some necessary and sufficient conditions for (non) h-expansiveness under the assumption of shadowing and chain transitivity; and use the result to present a counter-example for a question raised by Artigue et al. (Proc Am Math Soc 150:3369–3378, 2022).

Keywords Shadowable points · Entropy points · Shadowing · h-expansive · s-limit shadowing

Mathematics Subject Classification 37B40 · 37B65

1 Introduction

Shadowing, introduced by Anosov (1967) and Bowen (1975), is a feature of hyperbolic dynamical systems and has played an important role in the global theory of dynamical systems [see Aoki and Hiraide (1994) or Pilyugin (1999) for background]. It generally refers to a phenomenon in which coarse orbits, or *pseudo-orbits*, are approximated by true orbits. In Morales (2016), by splitting the global shadowing into pointwise shadowings, Morales introduced the notion of *shadowable points*, which gives a tool for a local description of the shadowing phenomena. The study of shadowable points has been extended to shadowable points for flows (Aponte and Villavicencio 2018); pointwise stability and persistence (Das et al. 2020; Das and Khan 2022; Dong et al. 2019; Jung et al. 2020; Koo et al. 2018); shadowable measures (Shin 2019); average shadowable and specification points (Das et al. 2019; Das and Khan 2021); eventually

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shadowable points (Dong et al. 2020); shadowable points of set-valued dynamical systems (Luo et al. 2020); and so on.

In Kawaguchi (2017b), some sufficient conditions are given for a shadowable point to be an entropy point. Recall that the notion of *entropy points* is obtained by a concentration of positive topological entropy at a point (Ye and Zhang 2007). Also, in Arbieto and Rego (2020), the notion of shadowable points is applied to obtain pointwise sufficient conditions for positive topological entropy (see also Arbieto and Rego 2022). In the first part of this paper, we improve the result of Kawaguchi (2017b) by giving necessary and sufficient conditions for shadowable points to be certain entropy points. The *h-expansiveness* is another local entropy property of dynamical systems (Bowen 1972). In the second part of this paper, we present several necessary and sufficient conditions for (non) h-expansiveness under the assumption of shadowing and chain transitivity; and use the result to obtain a counter-example for a question in Artigue et al. (2022).

We begin with a definition. Throughout, X denotes a compact metric space endowed with a metric d.

Definition 1.1 Given a continuous map $f\colon X \to X$ and $\delta > 0$, a finite sequence $(x_i)_{i=0}^k$ of points in X, where $k > 0$ is a positive integer, is called a δ-*chain* of f if $d(f(x_i), x_{i+1}) \le \delta$ for every $0 \le i \le k-1$. A δ-chain $(x_i)_{i=0}^k$ of f with $x_0 = x_k$ is said to be a δ-*cycle* of f.

Let $f\colon X \to X$ be a continuous map. For any $x, y \in X$ and $\delta > 0$, the notation $x \to_\delta y$ means that there is a δ-chain $(x_i)_{i=0}^k$ of f with $x_0 = x$ and $x_k = y$. We write $x \to y$ if $x \to_\delta y$ for all $\delta > 0$. We say that $x \in X$ is a *chain recurrent point* for f if $x \to x$, or equivalently, for any $\delta > 0$, there is a δ-cycle $(x_i)_{i=0}^k$ of f with $x_0 = x_k = x$. Let $CR(f)$ denote the set of chain recurrent points for f. We define a relation $\leftrightarrow$ in

$$CR(f)^2 = CR(f) \times CR(f)$$

by: for any $x, y \in CR(f)$, $x \leftrightarrow y$ if and only if $x \to y$ and $y \to x$. Note that $\leftrightarrow$ is a closed equivalence relation in $CR(f)^2$ and satisfies $x \leftrightarrow f(x)$ for all $x \in CR(f)$. An equivalence class C of $\leftrightarrow$ is called a *chain component* for f. We denote by $\mathcal{C}(f)$ the set of chain components for f.

A subset S of X is said to be f-invariant if $f(S) \subset S$. For an f-invariant subset S of X, we say that $f|_S\colon S \to S$ is *chain transitive* if for any $x, y \in S$ and $\delta > 0$, there is a δ-chain $(x_i)_{i=0}^k$ of $f|_S$ with $x_0 = x$ and $x_k = y$.

Remark 1.1 The following properties hold

- $CR(f) = \bigsqcup_{C \in \mathcal{C}(f)} C$,
- Every $C \in \mathcal{C}(f)$ is a closed f-invariant subset of $CR(f)$,
- $f|_C\colon C \to C$ is chain transitive for all $C \in \mathcal{C}(f)$,
- For any f-invariant subset S of X, if $f|_S\colon S \to S$ is chain transitive, then $S \subset C$ for some $C \in \mathcal{C}(f)$.

Let $f : X \to X$ be a continuous map. For $x \in X$, we define a subset $C(x)$ of X by

$$C(x) = \{x\} \cup \{y \in X : x \to y\}.$$

By this definition, we easily see that for any $x \in X$, $C(x)$ is a closed f-invariant subset of X. We say that a closed f-invariant subset S of X is *chain stable* if for any $\epsilon > 0$, there is $\delta > 0$ for which every δ-chain $(x_i)_{i=0}^k$ of f with $x_0 \in S$ satisfies $d(x_i, S) \leq \epsilon$ for all $0 \leq i \leq k$. A proof of the following lemma is given in Sect. 3.

Lemma 1.1 *$C(x)$ is chain stable for all $x \in X$.*

Remark 1.2 For any $x \in X$, since $C(x)$ is chain stable, it satisfies the following properties

- $CR(f|_{C(x)}) = C(x) \cap CR(f)$,
- for every $C \in \mathcal{C}(f)$, $C \subset C(x)$ if and only if $C \cap C(x) \neq \emptyset$,
- $\mathcal{C}(f|_{C(x)}) = \{C \in \mathcal{C}(f) : C \subset C(x)\}$.

Let $f : X \to X$ be a continuous map and let $\xi = (x_i)_{i \geq 0}$ be a sequence of points in X. For $\delta > 0$, ξ is called a δ-*pseudo orbit* of f if $d(f(x_i), x_{i+1}) \leq \delta$ for all $i \geq 0$. For $\epsilon > 0$, ξ is said to be ϵ-*shadowed* by $x \in X$ if $d(f^i(x), x_i) \leq \epsilon$ for all $i \geq 0$.

Definition 1.2 Given a continuous map $f : X \to X$, $x \in X$ is called a *shadowable point* for f if for any $\epsilon > 0$, there is $\delta > 0$ such that every δ-pseudo orbit $(x_i)_{i \geq 0}$ of f with $x_0 = x$ is ϵ-shadowed by some $y \in X$. We denote by $Sh(f)$ the set of shadowable points for f.

For a continuous map $f : X \to X$ and a subset S of X, we say that f has the *shadowing on S* if for any $\epsilon > 0$, there is $\delta > 0$ such that every δ-pseudo orbit $(x_i)_{i \geq 0}$ of f with $x_i \in S$ for all $i \geq 0$ is ϵ-shadowed by some $y \in X$. We say that f has the *shadowing property* if f has the shadowing on X.

The next lemma is a basis for the formulation of Theorems 1.1 and 1.2.

Lemma 1.2 *For a continuous map $f : X \to X$ and $x \in X$, the following conditions are equivalent*

(1) $x \in Sh(f)$,
(2) $C(x) \subset Sh(f)$,
(3) f has the shadowing on $C(x)$.

Next, we recall the definition of entropy points from Ye and Zhang (2007). Let $f : X \to X$ be a continuous map. For $n \geq 1$, the metric d_n on X is defined by

$$d_n(x, y) = \max_{0 \leq i \leq n-1} d(f^i(x), f^i(y))$$

for all $x, y \in X$. For $n \geq 1$ and $r > 0$, a subset E of X is said to be (n, r)-*separated* if $d_n(x, y) > r$ for all $x, y \in E$ with $x \neq y$. Let K be a subset of X. For $n \geq 1$ and

$r > 0$, let $s_n(f, K, r)$ denote the largest cardinality of an (n, r)-separated subset of K. We define $h(f, K, r)$ and $h(f, K)$ by

$$h(f, K, r) = \limsup_{n\to\infty} \frac{1}{n} \log s_n(f, K, r)$$

and

$$h(f, K) = \lim_{r\to 0} h(f, K, r).$$

We also define the topological entropy $h_{\mathrm{top}}(f)$ of f by $h_{\mathrm{top}}(f) = h(f, X)$.

Definition 1.3 Let $f: X \to X$ be a continuous map. For $x \in X$, we denote by $\mathcal{K}(x)$ the set of closed neighborhoods of x.

(1) $Ent(f)$ is the set of $x \in X$ such that $h(f, K) > 0$ for all $K \in \mathcal{K}(x)$,
(2) For $r > 0$, $Ent_r(f)$ is the set of $x \in X$ such that $h(f, K, r) > 0$ for all $K \in \mathcal{K}(x)$,
(3) For $r > 0$ and $b > 0$, $Ent_{r,b}(f)$ is the set of $x \in X$ such that $h(f, K, r) \geq b$ for all $K \in \mathcal{K}(x)$.

Remark 1.3 The following properties hold

- $Ent(f)$, $Ent_r(f)$, $r > 0$, and $Ent_{r,b}(f)$, $r, b > 0$, are closed f-invariant subsets of X,
-
$$Ent(f) \subset Ent_r(f) \subset Ent_{r,b}(f)$$
for all $r, b > 0$,
- for any closed subset K of X and $r > 0$, if $h(f, K, r) > 0$, then $K \cap Ent_r(f) \neq \emptyset$,
- for any closed subset K of X and $r, b > 0$, if $h(f, K, r) \geq b$, then $K \cap Ent_{r,b}(f) \neq \emptyset$.

For a continuous map $f: X \to X$, we define $\mathcal{C}_o(f)$ as the set of $C \in \mathcal{C}(f)$ such that C is a periodic orbit or an odometer. We also define $\mathcal{C}_{no}(f)$ to be

$$\mathcal{C}_{no}(f) = \mathcal{C}(f) \setminus \mathcal{C}_o(f).$$

We refer to Sect. 2.1 for various matters related to this definition.

The first theorem characterizes the shadowable points that are entropy points of a certain type.

Theorem 1.1 *Given a continuous map $f: X \to X$ and $x \in Sh(f)$,*

$$x \in \bigcup_{r>0} Ent_r(f)$$

if and only if $\mathcal{C}_{no}(f|_{C(x)}) \neq \emptyset$.

The following theorem characterizes the shadowable points that are entropy points of other type.

Theorem 1.2 *Given a continuous map* $f: X \to X$ *and* $x \in Sh(f)$*,*

$$x \in \bigcup_{r,b>0} Ent_{r,b}(f)$$

if and only if $h_{\text{top}}(f|_{C(x)}) = h(f, C(x)) > 0$.

Remark 1.4 In Ye and Zhang (2007), a point of

$$\bigcup_{r,b>0} Ent_{r,b}(f)$$

is called a *uniform entropy point* for f.

Remark 1.5 In Sect. 3, we give an example of a continuous map $f: X \to X$ such that

- f has the shadowing property and so satisfies $X = Sh(f)$,
- $$Ent(f) \setminus \bigcup_{r>0} Ent_r(f)$$

  is a non-empty set.

Before we state the next theorem, we introduce some definitions.

Definition 1.4 For a continuous map $f: X \to X$ and $(x, y) \in X^2$,

- (x, y) is called a *distal pair* for f if

$$\liminf_{i\to\infty} d(f^i(x), f^i(y)) > 0,$$

- (x, y) is called a *proximal pair* for f if

$$\liminf_{i\to\infty} d(f^i(x), f^i(y)) = 0,$$

- (x, y) is called an *asymptotic pair* for f if

$$\limsup_{i\to\infty} d(f^i(x), f^i(y)) = 0.$$

- (x, y) is called a *scrambled pair* for f if

$$\limsup_{i\to\infty} d(f^i(x), f^i(y)) > 0 \text{ and } \liminf_{i\to\infty} d(f^i(x), f^i(y)) = 0.$$

For a continuous map $f: X \to X$ and $x \in X$, the *ω-limit set* $\omega(x, f)$ of x for f is defined to be the set of $y \in X$ such that $\lim_{j\to\infty} f^{i_j}(x) = y$ for some sequence $0 \le i_1 < i_2 < \cdots$. Note that $\omega(x, f)$ is a closed f-invariant subset of X and satisfies $y \to z$ for all $y, z \in \omega(x, f)$. For every $x \in X$, we have $\omega(x, f) \subset C$ for some $C \in \mathcal{C}(f)$ and such C satisfies $C \subset C(x)$.

Remark 1.6 Let $f: X \to X$ be a continuous map.

- For a closed f-invariant subset S of X and $e > 0$, we say that $x \in S$ is an *e-sensitive point* for $f|_S: S \to S$ if for any $\epsilon > 0$, there is $y \in S$ such that $d(x, y) \le \epsilon$ and $d(f^i(x), f^i(y)) > e$ for some $i \ge 0$. We define $Sen_e(f|_S)$ to be the set of e-sensitive points for $f|_S$ and

$$Sen(f|_S) = \bigcup_{e>0} Sen_e(f|_S).$$

- A closed f-invariant subset M of X is said to be a *minimal set* for f if closed f-invariant subsets of M are only $\emptyset$ and M. This is equivalent to $M = \omega(x, f)$ for all $x \in M$.

In Kawaguchi (2017b), the author gave three sufficient conditions for a shadowable point to be an entropy point. The next theorem refines Corollary 1.1 of Kawaguchi (2017b).

Theorem 1.3 *Let $f: X \to X$ be a continuous map. For any $x \in X$ and $C \in \mathcal{C}(f)$ with $\omega(x, f) \subset C$, if one of the following conditions is satisfied, then $C \in \mathcal{C}_{no}(f|_{C(x)})$.*

(1) $\omega(x, f) \cap Sen(f|_{CR(f)}) \neq \emptyset$,
(2) there is $y \in X$ such that $(x, y) \in X^2$ is a scrambled pair for f,
(3) $\omega(x, f)$ is not a minimal set for f.

Remark 1.7 For a continuous map $f: X \to X$, $y \in X$ is called a *minimal point* for f if $y \in \omega(y, f)$ and $\omega(y, f)$ is a minimal set for f. Due to Theorem 8.7 of Furstenberg (1981), we know that for any $x \in X$, there is a minimal point $y \in X$ for f such that (x, y) is a proximal pair for f. If $\omega(x, f)$ is not a minimal set for f, then it follows that (x, y) is a scrambled pair for f, thus (3) always implies (2).

We consider another local property of dynamical systems so-called h-expansiveness Bowen (1972). Let $f: X \to X$ be a continuous map. For $x \in X$ and $\epsilon > 0$, let

$$\Phi_\epsilon(x) = \{y \in X: d(f^i(x), f^i(y)) \le \epsilon \text{ for all } i \ge 0\}$$

and

$$h^*_f(\epsilon) = \sup_{x\in X} h(f, \Phi_\epsilon(x)).$$

We say that f is *h-expansive* if $h^*_f(\epsilon) = 0$ for some $\epsilon > 0$. The following theorem gives several conditions equivalent to (non) h-expansiveness under the assumption of shadowing and chain transitivity.

Theorem 1.4 *Let $f: X \to X$ be a continuous map. If f is chain transitive and has the shadowing property, then the following conditions are equivalent*

(1) f is not h-expansive,
(2) for any $\epsilon > 0$, there is $r > 0$ such that for every $\delta > 0$, there is a pair

$$((x_i)_{i=0}^k, (y_i)_{i=0}^k)$$

of δ-chains of f with $(x_0, x_k) = (y_0, y_k)$ and

$$r \le \max_{0 \le i \le k} d(x_i, y_i) \le \epsilon,$$

(3) for any $\epsilon > 0$, there are $m \ge 1$ and a closed f^m-invariant subset Y of X such that

$$\sup_{i \ge 0} d(f^i(x), f^i(y)) \le \epsilon$$

for all $x, y \in Y$ and there is a factor map

$$\pi : (Y, f^m) \to (\{0, 1\}^{\mathbb{N}}, \sigma),$$

where $\sigma : \{0, 1\}^{\mathbb{N}} \to \{0, 1\}^{\mathbb{N}}$ is the shift map.
(4) for any $\epsilon > 0$, there is a scrambled pair $(x, y) \in X^2$ for f such that

$$\sup_{i \ge 0} d(f^i(x), f^i(y)) \le \epsilon.$$

In Sect. 4, we use this theorem to obtain a counter-example for a question in Artigue et al. (2022). We shall make some definitions to precisely state the properties that are satisfied by the example.

Definition 1.5 Let $f: X \to X$ be a continuous map and let $\xi = (x_i)_{i \ge 0}$ be a sequence of points in X. For $\delta > 0$, ξ is called a δ-*limit-pseudo orbit* of f if $d(f(x_i), x_{i+1}) \le \delta$ for all $i \ge 0$, and

$$\lim_{i \to \infty} d(f(x_i), x_{i+1}) = 0.$$

For $\epsilon > 0$, ξ is said to be ϵ-*limit shadowed* by $x \in X$ if $d(f^i(x), x_i) \le \epsilon$ for all $i \ge 0$, and

$$\lim_{i \to \infty} d(f^i(x), x_i) = 0.$$

We say that f has the *s-limit shadowing property* if for any $\epsilon > 0$, there is $\delta > 0$ such that every δ-limit-pseudo orbit of f is ϵ-limit shadowed by some point of X.

Remark 1.8 If f has the s-limit shadowing property, then f satisfies the shadowing property.

Definition 1.6 Let $f: X \to X$ be a homeomorphism. For $x \in X$ and $\epsilon > 0$, let

$$\Gamma_\epsilon(x) = \{y \in X: d(f^i(x), f^i(y)) \le \epsilon \text{ for all } i \in \mathbb{Z}\}.$$

We say that f is

- *expansive* if there is $e > 0$ such that $\Gamma_e(x) = \{x\}$ for all $x \in X$,
- *countably-expansive* if there is $e > 0$ such that $\Gamma_e(x)$ is a countable set for all $x \in X$,
- *cw-expansive* if there is $e > 0$ such that $\Gamma_e(x)$ is totally disconnected for all $x \in X$.

A continuous map $f: X \to X$ is said to be *transitive* (resp. *mixing*) if for any non-empty open subsets U, V of X, it holds that $f^j(U) \cap V \neq \emptyset$ for some $j > 0$ (resp. for all $j \geq i$ for some $i > 0$).

Remark 1.9 If f is transitive, then f is chain transitive, and the converse holds when f has the shadowing property.

In Sect. 4, by using Theorem 1.4, we give an example of a homeomorphism $f: X \to X$ (Example 4.1) such that

(1) X is totally disconnected and so f is cw-expansive,
(2) f is mixing,
(3) f is h-expansive,
(4) f has the s-limit shadowing property,
(5) f is not countably-expansive,
(6) f satisfies $X_e = \emptyset$, where

$$X_e = \{x \in X: \Gamma_\epsilon(x) = \{x\} \text{ for some } \epsilon > 0\}.$$

In Artigue et al. (2020), it is proved that if a homeomorphism $f: X \to X$ has the *L-shadowing property*, that is, a kind of two-sided s-limit shadowing property, then

$$f|_{CR(f)}: CR(f) \to CR(f)$$

is expansive if and only if $f|_{CR(f)}$ is countably-expansive if and only if $f|_{CR(f)}$ is h-expansive (see Corollary C of Artigue et al. (2020)). Example 4.1 shows that even if a homeomorphism $f: X \to X$ satisfies the s-limit shadowing property, this equivalence does not hold. The example also gives a negative answer to the following question in Artigue et al. (2022) (see Question 3 of Artigue et al. (2022)):

Question *Is X_e non-empty for every transitive h-expansive and cw-expansive homeomorphism $f: X \to X$ satisfying the shadowing property?*

This paper consists of four sections. In Sect. 2, we collect some definitions, notations, and facts that are used in this paper. In Sect. 3, we prove Lemmas 1.1 and 1.2; prove Theorems 1.1, 1.2, and 1.3; and give an example mentioned in Remark 1.5. In Sect. 4, we prove Theorem 1.4 and give an example mentioned above (Example 4.1) after proving some auxiliary lemmas.

2 Preliminaries

In this section, we briefly collect some definitions, notations, and facts that are used in this paper.

2.1 Odometers, Equicontinuity, and Chain Continuity

An *odometer* (also called an *adding machine*) is defined as follows. Given an increasing sequence $m = (m_k)_{k\geq 1}$ of positive integers such that $m_1 \geq 1$ and m_k divides m_{k+1} for each $k = 1, 2, \dots$, we define

- $X(k) = \{0, 1, \dots, m_k - 1\}$ (with the discrete topology),
- $$X_m = \left\{ (x_k)_{k\geq 1} \in \prod_{k\geq 1} X(k) \colon x_k \equiv x_{k+1} \pmod{m_k} \text{ for all } k \geq 1 \right\},$$
- $g_m(x)_k = x_k + 1 \pmod{m_k}$ for all $x = (x_k)_{k\geq 1} \in X_m$ and $k \geq 1$.

We regard X_m as a subspace of the product space $\prod_{k\geq 1} X(k)$. The homeomorphism

$$g_m \colon X_m \to X_m$$

(or (X_m, g_m)) is called an odometer with the periodic structure m.

Let $f \colon X \to X$ be a continuous map and let S be a closed f-invariant subset of X. We say that $f|_S \colon S \to S$ is

- *equicontinuous* if for every $\epsilon > 0$, there is $\delta > 0$ such that any $x, y \in S$ with $d(x, y) \leq \delta$ satisfies
$$\sup_{i\geq 0} d(f^i(x), f^i(y)) \leq \epsilon,$$
- *chain continuous* if for every $\epsilon > 0$, there is $\delta > 0$ such that any δ-pseudo orbits $(x_i)_{i\geq 0}$ and $(y_i)_{i\geq 0}$ of $f|_S$ with $x_0 = y_0$ satisfies
$$\sup_{i\geq 0} d(x_i, y_i) \leq \epsilon.$$

Recall that for a continuous map $f \colon X \to X$, $\mathcal{C}_o(f)$ is defined as the set of $C \in \mathcal{C}(f)$ such that C is a periodic orbit or an odometer, that is, $(C, f|_C)$ is topologically conjugate to an odometer.

Lemma 2.1 *For a continuous map $f \colon X \to X$, the following conditions are equivalent*

(1) $\mathcal{C}(f) = \mathcal{C}_o(f)$,
(2) $f|_{CR(f)} \colon CR(f) \to CR(f)$ is an equicontinuous homeomorphism and $CR(f)$ is totally disconnected,

(3) $f|_{CR(f)}\colon CR(f) \to CR(f)$ *is chain continuous.*

Proof We prove the implication (1) $\implies$ (2). Since $\mathcal{C}(f) = \mathcal{C}_o(f)$,

(A) every $C \in \mathcal{C}(f)$ is totally disconnected,
(B) $f|_{CR(f)}$ is a distal homeomorphism, that is, every $(x, y) \in CR(f)^2$ is a distal pair for $f|_{CR(f)}$.

Since the quotient space

$$\mathcal{C}(f) = CR(f)/\leftrightarrow$$

is totally disconnected, by (A), we obtain that $CR(f)$ is totally disconnected. By (B) and Corollary 1.9 of Auslander et al. (2007), we conclude that $f|_{CR(f)}$ is an equicontinuous homeomorphism. For a proof of (2) $\implies$ (3) (resp. (3) $\implies$ (1)), we refer to Lemma 3.3 (resp. Section 6) of Kawaguchi (2021a). □

By applying Lemma 2.1 to $f|_C\colon C \to C$, $C \in \mathcal{C}(f)$, we obtain the following corollary.

Corollary 2.1 *For a continuous map $f\colon X \to X$ and $C \in \mathcal{C}(f)$, the following conditions are equivalent*

(1) $C \in \mathcal{C}_o(f)$*,*
(2) $f|_C\colon C \to C$ *is an equicontinuous homeomorphism and* C *is totally disconnected,*
(3) $f|_C\colon C \to C$ *is chain continuous.*

2.2 Factor Maps and Inverse Limit

For two continuous maps $f\colon X \to X$, $g\colon Y \to Y$, where X, Y are compact metric spaces, a continuous map $\pi\colon X \to Y$ is said to be a *factor map* if π is surjective and satisfies $\pi \circ f = g \circ \pi$. A factor map $\pi\colon X \to Y$ is also denoted as

$$\pi\colon (X, f) \to (Y, g).$$

Given an inverse sequence of factor maps

$$\pi = (\pi_n\colon (X_{n+1}, f_{n+1}) \to (X_n, f_n))_{n\geq 1},$$

let

$$X = \{x = (x_n)_{n\geq 1} \in \prod_{n\geq 1} X_n\colon \pi_n(x_{n+1}) = x_n \text{ for all } n \geq 1\},$$

which is a compact metric space. Then, a continuous map $f\colon X \to X$ is well-defined by $f(x) = (f_n(x_n))_{n\geq 1}$ for all $x = (x_n)_{n\geq 1} \in X$. We call

$$(X, f) = \lim_{\pi}(X_n, f_n)$$

the *inverse limit system*. It is easy to see that f is transitive (resp. mixing) if and only if $f_n : X_n \to X_n$ is transitive (resp. mixing) for all $n \geq 1$. It is also easy to see that f has the shadowing property if $f_n : X_n \to X_n$ has the shadowing property for all $n \geq 1$.

3 Proofs of Theorems 1.1, 1.2, and 1.3

In this section, we prove Lemmas 1.1 and 1.2; prove Theorems 1.1, 1.2, and 1.3; and give an example mentioned in Remark 1.5.

First, we prove Lemma 1.1.

Proof of Lemma 1.1 If $C(x)$ is not chain stable, then there is $r > 0$ such that for any $\delta > 0$, there is a δ-chain $x^{(\delta)} = (x_i^{(\delta)})_{i=0}^{k_\delta}$ of f with $x_0^{(\delta)} \in C(x)$ and

$$d(x_{k_\delta}^{(\delta)}, C(x)) \geq r.$$

Then, there are a sequence $0 < \delta_1 > \delta_2 > \cdots$ and $y, z \in X$ such that the following conditions are satisfied

- $\lim_{j\to\infty} \delta_j = 0$,
- $\lim_{j\to\infty} x_0^{(\delta_j)} = y$ and $\lim_{j\to\infty} x_{k_{\delta_j}}^{(\delta_j)} = z$.

It follows that $y \in C(x)$, $d(z, C(x)) \geq r > 0$ and so $z \notin C(x)$; and $y \to z$. However, if $y = x$, we obtain $x \to z$ implying $z \in C(x)$, a contradiction. If $y \neq x$, by $x \to y$ and $y \to z$, we obtain $x \to z$ implying $z \in C(x)$, a contradiction. Thus, the lemma has been proved. □

Next, we prove Lemma 1.2.

Proof of Lemma 1.2 We prove the implication (1) $\implies$ (2). Let $x \in Sh(f)$ and $y \in C(x) \setminus \{x\}$. For any $\epsilon > 0$, since $x \in Sh(f)$, there is $\delta > 0$ such that every δ-pseudo orbit $(x_i)_{i\geq 0}$ of f with $x_0 = x$ is ϵ-shadowed by some $z \in X$. Since $y \in C(x) \setminus \{x\}$ and so $x \to y$, we have a δ-chain $\alpha = (y_i)_{i=0}^k$ of f with $y_0 = x$ and $y_k = y$. For any δ-pseudo orbit $\beta = (z_i)_{i\geq 0}$ of f with $z_0 = y$, we consider a δ-pseudo orbit

$$\xi = \alpha\beta = (x_i)_{i\geq 0} = (y_0, y_1, \ldots, y_{k-1}, z_0, z_1, z_2, \ldots)$$

of f. Then, since $x_0 = y_0 = x$, ξ is ϵ-shadowed by some $z \in X$ and so β is ϵ-shadowed by $f^k(z)$. Since $\epsilon > 0$ is arbitrary, we obtain $y \in Sh(f)$, thus (1) $\implies$ (2) has been proved.

Next, we prove the implication (2) $\implies$ (3). For a closed subset K of X, if $K \subset Sh(f)$, then by Lemma 2.4 of Kawaguchi (2017a), for any $\epsilon > 0$, there is $\delta > 0$ such that every δ-pseudo orbit $(x_i)_{i\geq 0}$ of f with $x_0 \in K$ is ϵ-shadowed by some $y \in X$. Since $C(x)$ is a closed subset of X, this clearly implies that if $C(x) \subset Sh(f)$, then f has the shadowing on $C(x)$.

Finally, we prove the implication (3) $\implies$ (1). If f has the shadowing on $C(x)$, then for any $\epsilon > 0$, there is $\delta > 0$ such that every δ-pseudo orbit $(x_i)_{i\geq 0}$ of f with $x_i \in C(x)$ for all $i \geq 0$ is ϵ-shadowed by some $y \in X$. Since $x \in C(x)$ and $C(x)$ is chain stable, if $\gamma > 0$ is sufficiently small, then for every γ-pseudo orbit $\xi = (y_i)_{i\geq 0}$ of f with $y_0 = x$, by taking $x_i \in C(x)$, $i > 0$, with $d(y_i, C(x)) = d(y_i, x_i)$ for all $i > 0$, we have that

- $d(x_i, y_i) \leq \epsilon$ for each $i > 0$,
- $$(x_i)_{i\geq 0} = (x, x_1, x_2, x_3, \dots)$$
  is a δ-pseudo orbit of f with $x_i \in C(x)$ for all $i \geq 0$ and so is ϵ-shadowed by some $y \in X$.

It follows that ξ is 2ϵ-shadowed by y. Since $\epsilon > 0$ is arbitrary, we obtain $x \in Sh(f)$, thus (3) $\implies$ (1) has been proved. This completes the proof of Lemma 1.2. □

We give a proof of Theorem 1.1.

Proof of Theorem 1.1 First, we prove the "if" part. Let $C \in \mathcal{C}_{no}(f|_{C(x)})$. Due to Corollary 2.1, since $f|_C : C \to C$ is not chain continuous, there are $p \in C$ and $e > 0$ such that for any $\delta > 0$, there are δ-chains $(x_i)_{i=0}^k$ and $(y_i)_{i=0}^k$ of $f|_C$ with $x_0 = y_0 = p$ and $d(x_k, y_k) > e$. Fix $0 < r < e$ and take any $\epsilon > 0$ with $r + 2\epsilon < e$. Since $x \in Sh(f)$, there is $\delta_0 > 0$ such that every δ_0-pseudo orbit $(x_i)_{i\geq 0}$ of f with $x_0 = x$ is ϵ-shadowed by some $y \in X$. We fix a pair

$$((x_i)_{i=0}^K, (y_i)_{i=0}^K)$$

of δ_0-chains $f|_C$ with $x_0 = y_0 = p$ and $d(x_K, y_K) > e$. Since $C \subset C(x)$, we have $x \to q$ for some $q \in C$. We also fix a δ_0-chain $\alpha = (z_i)_{i=0}^L$ of f with $z_0 = x$ and $z_L = q$. Since $f|_C$ is chain transitive, by compactness of C, there is $M > 0$ such that for any $w \in C$, there is a δ_0-chain $(w_i)_{i=0}^m$ of $f|_C$ with $w_0 = w$, $w_m = p$, and $m \leq M$. It follows that for any $w \in C$, there is a pair

$$(a^w, b^w) = ((a_i^w)_{i=0}^{k_w}, (b_i^w)_{i=0}^{k_w})$$

of δ_0-chains of $f|_C$ with $a_0^w = b_0^w = w$, $d(a_{k_w}^w, b_{k_w}^w) > e$, and $k_w \leq K + M$. Given any $N \geq 1$ and $s = (s_i)_{i=1}^N \in \{a, b\}^N$, we inductively define a family of δ_0-chains

$$\alpha(s, n) = (c(s, n)_i)_{i=0}^{k(s,n)}$$

of $f|_C$, $1 \leq n \leq N$, by $\alpha(s, 1) = s_1^q$ and $\alpha(s, n+1) = s_{n+1}^{c(s,n)_{k(s,n)}}$ for any $1 \leq n \leq N - 1$. Then, we consider a family of δ_0-chains

$$\alpha(s) = (c(s)_i)_{i=0}^{k(s)} = \alpha\alpha(s, 1)\alpha(s, 2)\cdots\alpha(s, N)$$

of $f, s \in \{a, b\}^N$. Note that $c(s)_0 = x$ and $k(s) \leq L + N(K+M)$ for all $s \in \{a, b\}^N$; and for any $s, t \in \{a, b\}^N$ with $s \neq t$, we have $d(c(s)_i, c(t)_i) > e$ for some

$$0 \leq i \leq \min\{k(s), k(t)\} \leq L + N(K + M).$$

By the choice of δ_0, for every $s \in \{a, b\}^N$, there is $x(s) \in X$ such that $d(f^i(x(s)), c(s)_i) \leq \epsilon$ for all $0 \leq i \leq k(s)$. It follows that

$$\{x(s) \colon s \in \{a, b\}^N\}$$

is an $(L + N(K + M), r)$-separated subset of $B_\epsilon(x) = \{y \in X \colon d(x, y) \leq \epsilon\}$. Since $N \geq 1$ is arbitrary, we obtain

$$\begin{aligned} h(f, B_\epsilon(x), r) &= \limsup_{n\to\infty} \frac{1}{n} \log s_n(f, B_\epsilon(x), r) \\ &\geq \limsup_{N\to\infty} \frac{1}{L + N(K + M)} \log s_{L+N(K+M)}(f, B_\epsilon(x), r) \\ &\geq \limsup_{N\to\infty} \frac{1}{L + N(K + M)} \log 2^N \\ &= \frac{1}{K + M} \log 2 > 0. \end{aligned}$$

Since $\epsilon > 0$ with $r + 2\epsilon < e$ is arbitrary, we conclude that $x \in Ent_r(f)$, proving the "if" part.

Next, we prove the "only if" part. Let $x \in Ent_r(f)$ for some $r > 0$. Due to Lemma 2.1, it suffices to show that

$$f|_{CR(f|_{C(x)})} \colon CR(f|_{C(x)}) \to CR(f|_{C(x)})$$

is not chain continuous. For any $\epsilon > 0$, let

$$S_\epsilon = \{y \in C(x) \colon d(y, CR(f|_{C(x)})) \leq \epsilon\}$$

and

$$T_\epsilon = \{y \in C(x) \colon d(y, CR(f|_{C(x)})) \geq \epsilon\}.$$

Since

$$CR(f|_{C(x)}) = C(x) \cap CR(f),$$

we have $T_\epsilon \cap CR(f) = \emptyset$; therefore, for any $p \in T_\epsilon$, we can take a neighborhood U_p of p in X such that

(1) $d(a, b) \leq r$ and $d(f(a), f(b)) \leq \epsilon$ for all $a, b \in U_p$,
(2) $f^i(c) \notin U_p$ for all $c \in U_p$ and $i > 0$.

We take $p_1, p_2, \dots, p_M \in T_\epsilon$ with $T_\epsilon \subset \bigcup_{j=1}^M U_{p_j}$. Let $U = \bigcup_{j=1}^M U_{p_j}$ and take $0 < \Delta \le \epsilon$ such that

$$\{z \in X : d(z, T_\epsilon) \le \Delta\} \subset U.$$

Since $x \in C(x)$ and $C(x)$ is chain stable, we can take a closed neighborhood K of x in X such that

(3) $d(a, b) \le \epsilon$ for all $a, b \in K$,
(4) $d(f^i(c), C(x)) \le \Delta$ for all $c \in K$ and $i \ge 0$.

For any $q \in X$ and $n \ge 1$, let

$$A(q, n) = \{0 \le i \le n-1 : f^i(q) \in U\}$$

and take

$$g(q, n) : A(q, n) \to \{U_{p_j} : 1 \le j \le M\}$$

such that $f^i(q) \in g(q, n)(i)$ for every $i \in A(q, n)$. By (2), we have $|A(q, n)| \le M$ for all $q \in X$ and $n \ge 1$. Note that

$$|\{(A(q, n), g(q, n)) : q \in X\}| \le \sum_{k=0}^{\min\{n, M\}} \binom{n}{k} M^k \le (M+1)n^M M^M.$$

for all $n \ge 1$. Since $x \in Ent_r(f)$, we have

$$h(f, K, r) = \limsup_{n\to\infty} \frac{1}{n} \log s_n(f, K, r) > 0;$$

therefore,

$$s_n(f, K, r) > (M+1)n^M M^M$$

for some $n \ge 1$. This implies that there are $u, v \in K$ such that $d_n(u, v) > r$ and

$$(A(u, n), g(u, n)) = (A(v, n), g(v, n)).$$

We fix $0 \le N \le n-1$ with $d(f^N(u), f^N(v)) > r$ and let

$$(A, g) = (A(u, n), g(u, n)) = (A(v, n), g(v, n)).$$

If

$$A \cap \{0 \le l \le N\} = \emptyset,$$

then $f^l(u), f^l(v) \notin U$ for all $0 \le l \le N$. By (4) and the choice of Δ, for any $0 \le l \le N$, we obtain

$$\{f^l(u), f^l(v)\} \subset \{w \in X : d(w, S_\epsilon) \le \Delta\}.$$

It follows that

$$\max\{d(f^l(u), CR(f|_{C(x)})), d(f^l(v), CR(f|_{C(x)}))\} \le \epsilon + \Delta \le 2\epsilon$$

for all $0 \le l \le N$. Moreover, by $u, v \in K$ and (3), we obtain $d(u, v) \le \epsilon$. If

$$A \cap \{0 \le l \le N\} \ne \emptyset,$$

letting

$$L = \max\,[A \cap \{0 \le l \le N\}],$$

we have $f^L(u), f^L(v) \in g(L)$ and $g(L) \in \{U_{p_j} : 1 \le j \le M\}$. By (1), we have $L < N$ and $d(f^{L+1}(u), f^{L+1}(v)) \le \epsilon$. By

$$A \cap \{L + 1 \le l \le N\} = \emptyset,$$

(4), and the choice of Δ, similarly as above, we obtain

$$\max\{d(f^l(u), CR(f|_{C(x)})), d(f^l(v), CR(f|_{C(x)}))\} \le \epsilon + \Delta \le 2\epsilon$$

for all $L + 1 \le l \le N$. Since $\epsilon > 0$ is arbitrary, we conclude that $f|_{CR(f|_{C(x)})}$ is not chain continuous, thus the "only if" part has been proved. This completes the proof of Theorem 1.1. □

For the proof of Theorem 1.2, we need two lemmas.

Lemma 3.1 *Let $f : X \to X$ be a continuous map.*

(1) For any $x, y \in X$ and $r > 0$, if $x \in Sh(f)$ and $y \in C(x) \cap Ent_r(f)$, then $x \in Ent_s(f)$ for all $0 < s < r$.
(2) For any $x, y \in X$ and $r, b > 0$, if $x \in Sh(f)$ and $y \in C(x) \cap Ent_{r,b}(f)$, then $x \in Ent_{s,b}(f)$ for all $0 < s < r$.

Proof Let $x \in Sh(f)$ and $y \in C(x)\backslash\{x\}$. For any $0 < s < r$, we fix $\epsilon > 0$ with $s + 2\epsilon < r$. Since $x \in Sh(f)$, there is $\delta > 0$ such that every δ-pseudo orbit of $(x_i)_{i \ge 0}$ of f with $x_0 = x$ is ϵ-shadowed by some $z \in X$. Since $y \in C(x) \setminus \{x\}$ and so $x \to y$, we have a $\delta/2$-chain $(y_i)_{i=0}^k$ of f with $y_0 = x$ and $y_k = y$. For $K \in \mathcal{K}(y)$, $n \ge 1$, and $r > 0$, we take an (n, r)-separated subset $E(K, n, r)$ of K with $|E(K, n, r)| = s_n(f, K, r)$. If K is sufficiently small, then for any $p \in E(K, n, r)$,

$$(z_i^p)_{i=0}^{k+n-1} = (y_0, y_1, \ldots, y_{k-1}, p, f(p), \ldots, f^{n-1}(p))$$

is a δ-chain of f with $z_0^p = y_0 = x$ and so there is $z_p \in X$ with $d(f^i(z_p), z_i^p) \le \epsilon$ for all $0 \le i \le k+n-1$. It follows that

$$\{z_p \colon p \in E(K, n, r)\}$$

is a $(k+n, s)$-separated subset of $B_\epsilon(x) = \{w \in X \colon d(x, w) \le \epsilon\}$ and so

$$s_{k+n}(f, B_\epsilon(x), s) \ge |E(K, n, r)| = s_n(f, K, r),$$

implying

$$\begin{aligned} h(f, B_\epsilon(x), s) &= \limsup_{n\to\infty} \frac{1}{k+n} \log s_{k+n}(f, B_\epsilon(x), s) \\ &\ge \limsup_{n\to\infty} \frac{1}{k+n} \log s_n(f, K, r) \\ &= h(f, K, r). \end{aligned}$$

Since $\epsilon > 0$ with $s + 2\epsilon < r$ is arbitrary, if $y \in Ent_r(f)$ (resp. $y \in Ent_{r,b}(f)$ for some $b > 0$), we obtain $x \in Ent_s(f)$ (resp. $x \in Ent_{s,b}(f)$). Since $0 < s < r$ is arbitrary, the lemma has been proved. □

Let $f \colon X \to X$ be a continuous map. For $\delta, r > 0$ and $n \ge 1$, we say that two δ-chains $(x_i)_{i=0}^n$ and $(y_i)_{i=0}^n$ of f is (n, r)-*separated* if $d(x_i, y_i) > r$ for some $0 \le i \le n$. Let

$$s_n(f, X, r, \delta)$$

denote the largest cardinality of a set of (n, r)-separated δ-chains of f. The following lemma is from Misiurewicz (1986).

Lemma 3.2 (Misiurewicz)

$$h_{\text{top}}(f) = \lim_{r\to 0} \lim_{\delta\to 0} \limsup_{n\to\infty} \frac{1}{n} \log s_n(f, X, r, \delta).$$

We give a proof of Theorem 1.2.

Proof of Theorem 1.2 First, we prove the "if" part. Since

$$h_{\text{top}}(f|_{C(x)}) = h(f, C(x)) > 0,$$

we have $h(f, C(x), r) > 0$ for some $r > 0$. Taking $0 < b \le h(f, C(x), r)$, we obtain $C(x) \cap Ent_{r,b}(f) \ne \emptyset$. Since $x \in Sh(f)$, by Lemma 3.1, this implies $x \in Ent_{s,b}(f)$ for all $0 < s < r$, thus the "if" part has been proved.

Next, we prove the"only if" part. Let $x \in Entr_{r_0,b}(f)$ for some $r_0, b > 0$. Then, we have

$$h(f, K, r_0) = \limsup_{n\to\infty} \frac{1}{n} \log s_n(f, K, r_0) \geq b$$

for all $K \in \mathcal{K}(x)$. Since $C(x)$ is chain stable, taking $0 < s < r_0$, we obtain

$$\limsup_{n\to\infty} \frac{1}{n} \log s_n(f, C(x), s, \delta) \geq b$$

for all $\delta > 0$. From Lemma 3.2, it follows that

$$\begin{aligned} h_{\text{top}}(f|_{C(x)}) &= \lim_{r\to 0} \lim_{\delta\to 0} \limsup_{n\to\infty} \frac{1}{n} \log s_n(f, C(x), r, \delta) \\ &\geq \lim_{\delta\to 0} \limsup_{n\to\infty} \frac{1}{n} \log s_n(f, C(x), s, \delta) \\ &\geq b > 0, \end{aligned}$$

thus the "only if" part has been proved. This completes the proof of Theorem 1.2. □

Next, we prove Theorem 1.3. The proof of the following lemma is left to the reader.

Lemma 3.3 *Let $f : X \to X$ be a continuous map and let $C \in \mathcal{C}(f)$. For any $\epsilon > 0$, there is $\delta > 0$ such that every δ-chain $(x_i)_{i=0}^k$ of $f|_{CR(f)}$ with $x_0 \in C$ satisfies $d(x_i, C) \leq \epsilon$ for all $0 \leq i \leq k$.*

Proof of Theorem 1.3 Due to Corollary 2.1, it is sufficient to show that each of the three conditions implies that $f|_C : C \to C$ is not chain continuous.

(1) Taking $y \in \omega(x, f) \cap Sen(f|_{CR(f)})$, we have

$$y \in C \cap Sen_{e_0}(f|_{CR(f)})$$

for some $e_0 > 0$. Taking $0 < e < e_0$, we obtain that for any $\delta > 0$, there are δ-chains $(x_i)_{i=0}^k$ and $(y_i)_{i=0}^k$ of $f|_{CR(f)}$ with $x_0 = y_0 = x$ and $d(x_k, y_k) > e$. By Lemma 3.3, this implies that $f|_C$ is not chain continuous and thus $C \in \mathcal{C}_{no}(f|_{C(x)})$.

(2) Since (x, y) is a scrambled pair for f, we have

$$\liminf_{i\to\infty} d(f^i(x), f^i(y)) = 0;$$

therefore, there are a sequence $0 \leq i_1 < i_2 < \cdots$ and $z \in X$ such that

$$\lim_{j\to\infty} d(f^{i_j}(x), f^{i_j}(y)) = 0$$

and

$$\lim_{j\to\infty} f^{i_j}(x) = z,$$

which implies $z \in \omega(x, f) \cap \omega(y, f)$ and so $\omega(y, f) \subset C$. By

$$\omega(x, f) \cup \omega(y, f) \subset C,$$

we obtain

$$\lim_{i\to\infty} d(f^i(x), C) = \lim_{i\to\infty} d(f^i(y), C) = 0.$$

On the other hand, since (x, y) is a scrambled pair for f, we have

$$\limsup_{i\to\infty} d(f^i(x), f^i(y)) > 0.$$

These condition clearly imply that $f|_C$ is not chain continuous and thus $C \in \mathcal{C}_{no}(f|_{C(x)})$.

(3) Since $\omega(x, f)$ is not a minimal set for f, we have a closed f-invariant subset S of $\omega(x, f)$ such that

$$\emptyset \neq S \neq \omega(x, f).$$

Fix $p \in S$, $q \in \omega(x, f)$, and $e > 0$ with $d(p, q) > e$. Since $\omega(x, f) \subset C$ and $f|_C\colon C \to C$ is chain transitive, for any $\delta > 0$, there are δ-chains $(x_i)_{i=0}^k$ and $(y_i)_{i=0}^k$ of $f|_C$ with $x_0 = y_0 = y_k = p$ and $x_k = q$. This implies that $f|_C$ is not chain continuous and thus $C \in \mathcal{C}_{no}(f|_{C(x)})$. □

Finally, we give an example mentioned in Remark 1.5. For a continuous map $f\colon X \to X$, $C \in \mathcal{C}(f)$ is said to be *terminal* if C is chain stable. The proof of the following lemma is left to the reader.

Lemma 3.4 *Let $f\colon X \to X$ be a continuous map. For any $x \in X$ and $C \in \mathcal{C}(f)$ with $\omega(x, f) \subset C$, if C is terminal, then*

$$C(x) = \{f^i(x)\colon i \geq 0\} \cup C.$$

Example 3.1 This example is taken from Kawaguchi (2021b). Let $\sigma\colon [-1, 1]^{\mathbb{N}} \to [-1, 1]^{\mathbb{N}}$ be the shift map and let d be the metric on $[-1, 1]^{\mathbb{N}}$ defined by

$$d(x, y) = \sup_{i\geq 1} 2^{-i}|x_i - y_i|$$

for all $x = (x_i)_{i\geq 1}, y = (y_i)_{i\geq 1} \in [-1, 1]^{\mathbb{N}}$. Let $s = (s_k)_{k\geq 1}$ be a sequence of numbers with $1 > s_1 > s_2 > \cdots$ and $\lim_{k\to\infty} s_k = 0$. Put

$$S = \{0\} \cup \{-s_k\colon k \geq 1\} \cup \{s_k\colon k \geq 1\},$$

a closed subset of $[-1, 1]$. We define a closed σ-invariant subset X of $S^{\mathbb{N}}$ by

$$X = \{x = (x_i)_{i\geq 1} \in S^{\mathbb{N}}\colon |x_1| \geq |x_2| \geq \cdots\}.$$

Let $f = \sigma|_X : X \to X$, $X_k = \{-s_k, s_k\}^{\mathbb{N}}$ for each $k \geq 1$, and let $X_0 = \{0^\infty\}$. Then, we have

$$CR(f) = \{x = (x_i)_{i\geq 1} \in X : |x_1| = |x_2| = \cdots\} = X_0 \cup \bigcup_{k\geq 1} X_k$$

and

$$\mathcal{C}(f) = \{X_0\} \cup \{X_k : k \geq 1\}.$$

Note that X_0 is terminal. For a rapidly decreasing sequence $s = (s_k)_{k\geq 1}$, we can show that f satisfies the shadowing property and so $X = Sh(f)$. Let $x = (s_1, s_2, s_3, \dots)$ and note that $x \in X$. Since $\omega(x, f) = X_0$ and X_0 is terminal, by Lemma 3.4, we obtain

$$C(x) = \{f^i(x) : i \geq 0\} \cup X_0.$$

By Theorem 1.1, we see that

$$x \notin \bigcup_{r>0} Ent_r(f).$$

We shall show that $x \in Ent(f)$. Let

$$x_k = (s_1, s_2, \dots, s_k, s_k, s_k, \dots),$$

$k \geq 1$, and note that $x_k \in X$ for each $k \geq 1$. For any $k \geq 1$, since $h(f, X_k) \geq \log 2 > 0$, we have $h(f, X_k, r_k) > 0$ and so $X_k \cap Ent_{r_k}(f) \neq \emptyset$ for some $r_k > 0$. For every $k \geq 1$, since $X_k \subset C(x_k)$, we obtain $C(x_k) \cap Ent_{r_k}(f) \neq \emptyset$; therefore, Lemma 3.1 implies that $x_k \in Ent_{s_k}(f)$ for all $0 < s_k < r_k$. In particular, we have $x_k \in Ent(f)$ for all $k \geq 1$. Since $\lim_{k\to\infty} x_k = x$, we conclude that $x \in Ent(f)$.

4 Proof of Theorem 1.4 and an example

In this section, we prove Theorem 1.4 and give an example mentioned in Sect. 1.

Proof of Theorem 1.4 First, we prove the implication (1) $\implies$ (2). If f is not h-expansive, then

$$h^*_f(\epsilon) = \sup_{x\in X} h(f, \Phi_\epsilon(x)) > 0$$

for all $\epsilon > 0$. Given any $\epsilon > 0$, there exists $x \in X$ such that $h(f, \Phi_{\epsilon/2}(x)) > 0$ and so we have $h(f, \Phi_{\epsilon/2}(x), r) > 0$ for some $r > 0$. For any $0 < \Delta \leq r$, we take an open cover $\mathcal{U} = \{U_i : 1 \le i \le m\}$ of X such that $d(a, b) \le \Delta$ for all $1 \le i \le m$ and $a, b \in U_i$. Since

$$h(f, \Phi_{\epsilon/2}(x), r) = \limsup_{n\to\infty} \frac{1}{n} \log s_n(f, \Phi_{\epsilon/2}(x), r) > 0,$$

we have

$$s_n(f, \Phi_{\epsilon/2}(x), r) > m^2$$

for some $n \ge 1$. Then, there are $u, v \in \Phi_{\epsilon/2}(x)$ such that $d_n(u, v) > r$, $u, v \in U_{i_0}$, and $f^{n-1}(u), f^{n-1}(v) \in U_{i_{n-1}}$ for some $U_{i_0}, U_{i_{n-1}} \in \mathcal{U}$. It follows that

$$\max\{d(u, v), d(f^{n-1}(u), f^{n-1}(v))\} \le \Delta \le r$$

and

$$r < \max_{0\le i\le n-1} d(f^i(u), f^i(v)) \le \epsilon.$$

Since $0 < \Delta \le r$ is arbitrary, this implies the existence of $r > 0$ as in (2). Since $\epsilon > 0$ is arbitrary, (1) $\Longrightarrow$ (2) has been proved.

Next, we prove the implication (2) $\Longrightarrow$ (3). The proof is similar to the proof of Lemma 3.1 in Artigue et al. (2020). Given any $\epsilon > 0$, we choose $r > 0$ as in (2). We fix $0 < \gamma < \min\{\epsilon, r/2\}$. Since f has the shadowing property, there is $\delta > 0$ such that every δ-pseudo orbit of f is γ-shadowed by some point of X. By the choice of r, we obtain a pair

$$(\alpha(0), \alpha(1)) = ((x_i)_{i=0}^k, (y_i)_{i=0}^k)$$

of δ-chains of f with $(x_0, x_k) = (y_0, y_k)$ and

$$r \le \max_{0\le i\le k} d(x_i, y_i) \le \epsilon.$$

Then, the chain transitivity of f gives a δ-chain $\beta = (z_i)_{i=0}^l$ of f with $z_0 = x_k = y_k$ and $z_l = x_0 = y_0$. For any $s = (s_n)_{n\ge1} \in \{0, 1\}^{\mathbb{N}}$, we consider a δ-pseudo orbit

$$\Gamma(s) = \alpha(s_1)\beta\alpha(s_2)\beta\alpha(s_3)\beta\cdots$$

of f. Let $m = k + l$,

$$Y = \{y \in X : \Gamma(s) \text{ is } \gamma\text{-shadowed by } y \text{ for some } s \in \{0, 1\}^{\mathbb{N}}\},$$

and define a map $\pi : Y \to \{0, 1\}^{\mathbb{N}}$ so that $\Gamma(\pi(y))$ is γ-shadowed by y for all $y \in Y$. By a standard argument, we can show that the following conditions are satisfied

- Y is a closed subset of X,

- $f^m(Y) \subset Y$,
- π is well-defined,
- π is surjective,
- π is continuous,
- $\pi \circ f^m = \sigma \circ \pi$, where $\sigma \colon \{0,1\}^{\mathbb{N}} \to \{0,1\}^{\mathbb{N}}$ is the shift map.

It follows that Y is a closed f^m-invariant subset of Y and

$$\pi : (Y, f^m) \to (\{0,1\}^{\mathbb{N}}, \sigma)$$

is a factor map. By the definition of $\Gamma(s)$, $s \in \{0,1\}^{\mathbb{N}}$, we see that

$$\sup_{i\geq 0} d(f^i(x), f^i(y)) \leq 3\epsilon$$

for all $x, y \in Y$. Since $\epsilon > 0$ is arbitrary, (2) $\implies$ (3) has been proved.

We shall prove the implication (3) $\implies$ (1). Given any $\epsilon > 0$, we take $m \geq 1$ and Y as in (3). Take $p \in Y$ and note that $Y \subset \Phi_\epsilon(p)$. It follows that

$$h_f^*(\epsilon) \geq h(f, \Phi_\epsilon(p)) \geq h(f, Y) \geq \frac{1}{m} h(f^m, Y) \geq \frac{1}{m} h(\sigma, \{0,1\}^{\mathbb{N}}) = \frac{1}{m}\log 2 > 0.$$

Since $\epsilon > 0$ is arbitrary, (3) $\implies$ (1) has been proved.

The implication (4) $\implies$ (2) is obvious from the definitions. It remains to prove the implication (3) $\implies$ (4). Given any $\epsilon > 0$, we take $m \geq 1$ and Y as in (3). Since

$$h(f^m, Y) \geq h(\sigma, \{0,1\}^{\mathbb{N}}) = \log 2 > 0,$$

by Corollary 2.4 of Blanchard et al. (2002), there is a scrambled pair $(x, y) \in Y^2$ for f^m. Then, (x, y) is also a scrambled pair for f and satisfies

$$\sup_{i\geq 0} d(f^i(x), f^i(y)) \leq \epsilon$$

because $x, y \in Y$. Since $\epsilon > 0$ is arbitrary, (4) $\implies$ (3) has been proved. This completes the proof of Theorem 1.4. □

We use Theorem 1.4 to obtain a counter-example for a question in Artigue et al. (2022). The example will be given as an inverse limit of the full-shift $(\{0,1\}^{\mathbb{Z}}, \sigma)$ with respect to a factor map

$$F \colon (\{0,1\}^{\mathbb{Z}}, \sigma) \to (\{0,1\}^{\mathbb{Z}}, \sigma).$$

We need three auxiliary lemmas. A homeomorphism $f \colon X \to X$ is said to be expansive if there is $e > 0$ such that

$$\Gamma_e(x) = \{x\}$$

for all $x \in X$ and such e is called an *expansive constant* for f. It is known that for a homeomorphism $f: X \to X$ with an expansive constant $e > 0$ and $x, y \in X$, if

$$\sup_{i \geq 0} d(f^i(x), f^i(y)) \leq e,$$

then (x, y) is an asymptotic pair for f. The following lemma gives a sufficient condition for an inverse limit system to be h-expansive.

Lemma 4.1 *Let $\pi = (\pi_n: (X_{n+1}, f_{n+1}) \to (X_n, f_n))_{n \geq 1}$ be a sequence of factor maps such that for every $n \geq 1$, $f_n: X_n \to X_n$ is an expansive transitive homeomorphism with the shadowing property. Let*

$$(Y, g) = \lim_{\pi}(X_n, f_n)$$

and note that g is a transitive homeomorphism with the shadowing property. If g is not h-expansive, then for any $N \geq 1$, there are $M \geq N$ and a scrambled pair $(x_{M+1}, y_{M+1}) \in X_{M+1}^2$ for f_{M+1} such that $(\pi_M(x_{M+1}), \pi_M(y_{M+1}))$ is an asymptotic pair for f_M.

Proof Let D be a metric on Y. Let d_n be a metric on X_n and $e_n > 0$ be an expansive constant for f_n for each $n \geq 1$. Given any $N \geq 1$, we take $\epsilon_N > 0$ such that for any $p = (p_n)_{n \geq 1}, q = (q_n)_{n \geq 1} \in Y$, $D(p, q) \leq \epsilon_N$ implies $d_n(p_n, q_n) \leq e_n$ for all $1 \leq n \leq N$. Since g is not h-expansive, by Theorem 1.4, there is a scrambled pair

$$(x, y) = ((x_n)_{n \geq 1}, (y_n)_{n \geq 1}) \in Y^2$$

for g with

$$\sup_{i \geq 0} D(g^i(x), g^i(y)) \leq \epsilon_N.$$

Then, for every $1 \leq n \leq N$, since

$$\sup_{i \geq 0} d_n(f_n^i(x_n), f_n^i(y_n)) = \sup_{i \geq 0} d_n(g^i(x)_n, g^i(y)_n) \leq e_n,$$

(x_n, y_n) is an asymptotic pair for f_n. Since (x, y) is a scrambled pair for g and so a proximal for g, (x_n, y_n) is a proximal pair for f_n for all $n \geq 1$. If (x_n, y_n) is an asymptotic pair for f_n for all $n \geq 1$, then (x, y) is an asymptotic pair for g, which is a contradiction. Thus, there is $m \geq 1$ such that (x_{m+1}, y_{m+1}) is a scrambled pair for f_{m+1}. Letting

$$M = \min\{m \geq 1: (x_{m+1}, y_{m+1}) \text{ is a scrambled pair for } f_{m+1}\},$$

we see that $M \geq N$, (x_{M+1}, y_{M+1}) is a scrambled pair for f_{M+1}, and (x_M, y_M) is an asymptotic pair for f_M, thus the lemma has been proved. □

A map $F\colon X \to X$ is said to be an *open map* if for any open subset U of X, $f(U)$ is an open subset of X. Any continuous open map $F\colon X \to X$ satisfies the following property: for every $r > 0$, there is $\delta > 0$ such that for any $s, t \in X$ with $d(s, t) \le \delta$ and $u \in F^{-1}(s)$, we have $d(u, v) \le r$ for some $v \in F^{-1}(t)$.

For a continuous map $f\colon X \to X$, a sequence $(x_i)_{i\ge 0}$ of points in X is called a *limit-pseudo orbit* of f if

$$\lim_{i\to\infty} d(f(x_i), x_{i+1}) = 0,$$

and said to be *limit shadowed* by $x \in X$ if

$$\lim_{i\to\infty} d(f^i(x), x_i) = 0.$$

The next lemma is needed for the proof of Lemma 4.3.

Lemma 4.2 *Let $f\colon X \to X$ be a homeomorphism and let $F\colon (X, f) \to (X, f)$ be a factor map such that*

(1) F is an open map,
(2) $d(v, v') \ge 1$ for all $t \in X$ and $v, v' \in F^{-1}(t)$ with $v \ne v'$.

Suppose that

(3) $(x_i)_{i\ge 0}$ is a limit-pseudo orbit of f and limit-shadowed by $x \in X$,
(4) $(z_i)_{i\ge 0}$ is a limit-pseudo orbit of f with $z_i \in F^{-1}(x_i)$ for all $i \ge 0$.

Then, there is $z \in F^{-1}(x)$ such that $(z_i)_{i\ge 0}$ is limit-shadowed by z.

Proof By (3), letting $\delta_i = d(x_i, f^i(x))$, $i \ge 0$, we have $\lim_{i\to\infty} \delta_i = 0$. By (1), we can take a sequence $r_i > 0$, $i \ge 0$, such that

- $\lim_{i\to\infty} r_i = 0$,
- for any $i \ge 0$, $s, t \in X$ with $d(s, t) \le \delta_i$, and $u \in F^{-1}(s)$, we have $d(u, v) \le r_i$ for some $v \in F^{-1}(t)$.

With use of (4), we fix $N \ge 0$ satisfying the following conditions

- $0 < r_i < 1/2$ for all $i \ge N$,
- $d(u, v) \le r_i$ implies $d(f(u), f(v)) \le 1/4$ for all $i \ge N$ and $u, v \in X$,
- $d(f(z_i), z_{i+1}) \le 1/4$ for all $i \ge N$.

By $\delta_N = d(x_N, f^N(x))$ and $z_N \in F^{-1}(x_N)$, we obtain $w_N \in F^{-1}(f^N(x))$ with $d(z_N, w_N) \le r_N$. Note that

$$F(f^j(w_N)) = f^j(F(w_N)) = f^j(f^N(x)) = f^{N+j}(x)$$

for every $j \ge 0$. By induction on j, we prove that $d(z_{N+j}, f^j(w_N)) \le r_{N+j}$ for all $j \ge 0$. Assume that $d(z_{N+j}, f^j(w_N)) \le r_{N+j}$ for some $j \ge 0$. Then,

$$\begin{aligned} d(z_{N+j+1}, f^{j+1}(w_N)) &\le d(z_{N+j+1}, f(z_{N+j})) + d(f(z_{N+j}), f(f^j(w_N))) \\ &\le 1/4 + 1/4 = 1/2. \end{aligned}$$

Since

$$\delta_{N+j+1} = d(x_{N+j+1}, f^{N+j+1}(x))$$

and $z_{N+j+1} \in F^{-1}(x_{N+j+1})$, we have

$$d(z_{N+j+1}, w) \le r_{N+j+1}$$

for some $w \in F^{-1}(f^{N+j+1}(x))$. Since $f^{j+1}(w_N) \in F^{-1}(f^{N+j+1}(x))$, by (2), we obtain

$$\begin{aligned} d(z_{N+j+1}, w') &\ge d(f^{j+1}(w_N), w') - d(z_{N+j+1}, f^{j+1}(w_N)) \\ &\ge 1 - 1/2 = 1/2 > r_{N+j+1} \end{aligned}$$

for all $w' \in F^{-1}(f^{N+j+1}(x))$ with $w' \ne f^{j+1}(w_N)$. It follows that $w = f^{j+1}(w_N)$ and so

$$d(z_{N+j+1}, f^{j+1}(w_N)) \le r_{N+j+1};$$

therefore, the induction is complete. Let $z = f^{-N}(w_N)$ and note that

$$f^N(F(z)) = F(f^N(z)) = F(w_N) = f^N(x).$$

Since f is a homeomorphism, we have $F(z) = x$, that is, $z \in F^{-1}(x)$. Moreover, we obtain

$$\begin{aligned} &\lim_{i\to\infty} d(z_i, f^i(z)) \\ &\quad = \lim_{j\to\infty} d(z_{N+j}, f^{N+j}(z)) = \lim_{j\to\infty} d(z_{N+j}, f^j(w_N)) = \lim_{j\to\infty} r_{N+j} = 0, \end{aligned}$$

thus the lemma has been proved. □

The following lemma gives a sufficient condition for an inverse limit system to satisfy the s-limit shadowing property.

Lemma 4.3 *Let $f : X \to X$ be a homeomorphism and let $F : (X, f) \to (X, f)$ be a factor map such that*

(1) F is an open map,
(2) $d(v, v') \ge 1$ for all $t \in X$ and $v, v' \in F^{-1}(t)$ with $v \ne v'$.

Let $(X_n, f_n) = (X, f)$ and $\pi_n = F : (X, f) \to (X, f)$ for all $n \ge 1$. Let

$$(Y, g) = \lim_{\pi}(X_n, f_n).$$

If f has the s-limit shadowing property, then g satisfies the s-limit shadowing property.

Proof Let D be a metric on Y. Given any $\epsilon > 0$, we take $N \geq 1$ and $\epsilon_N > 0$ such that for any $p = (p_n)_{n\geq 1}, q = (q_n)_{n\geq 1} \in Y$, $d(p_N, q_N) \leq \epsilon_N$ implies $D(p, q) \leq \epsilon$. Since f has the s-limit shadowing property, there is $\delta_N > 0$ such that every δ_N-limit-pseudo orbit of f is ϵ_N-limit shadowed by some point of X. We take $\delta > 0$ such that $D(p, q) \leq \delta$ implies $d(p_N, q_N) \leq \delta_N$ for all $p = (p_n)_{n\geq 1}, q = (q_n)_{n\geq 1} \in Y$. Let $\xi = (x^{(i)})_{i\geq 0}$ be a δ-limit-pseudo orbit of g. Then, for every $i \geq 0$, since $D(g(x^{(i)}), x^{(i+1)}) \leq \delta$, we have

$$d(f(x_N^{(i)}), x_N^{(i+1)}) = d(g(x^{(i)})_N, x_N^{(i+1)}) \leq \delta_N.$$

Also, since $\lim_{i\to\infty} D(g(x^{(i)}), x^{(i+1)}) = 0$, we have

$$\lim_{i\to\infty} d(f(x_n^{(i)}), x_n^{(i+1)}) = \lim_{i\to\infty} d(g(x^{(i)})_n, x_n^{(i+1)}) = 0$$

for all $n \geq 1$. It follows that $(x_N^{(i)})_{i\geq 0}$ is a δ_N-limit-pseudo of f and so ϵ_N-limit shadowed by some $x_N \in X$. Then, since

$$\lim_{i\to\infty} d(f(x_N^{(i)}), x_N^{(i+1)}) = \lim_{i\to\infty} d(f^i(x_N), x_N^{(i)}) = 0$$

and

$$\lim_{i\to\infty} d(f(x_{N+1}^{(i)}), x_{N+1}^{(i+1)}) = 0,$$

by Lemma 4.2, we have

$$\lim_{i\to\infty} d(f^i(x_{N+1}), x_{N+1}^{(i)}) = 0$$

for some $x_{N+1} \in F^{-1}(x_N)$. Inductively, we obtain $x_{N+k} \in X$, $k \geq 0$, such that

$$\lim_{i\to\infty} d(f^i(x_{N+k}), x_{N+k}^{(i)}) = 0$$

and $x_{N+k+1} \in F^{-1}(x_{N+k})$ for all $k \geq 0$. We define $y = (y_n)_{n\geq 1} \in Y$ by

$$y_n = \begin{cases} F^{N-n}(x_N) & \text{for all } 1 \leq n \leq N \\ x_n & \text{for all } n \geq N \end{cases}.$$

Given any $i \geq 0$, by

$$d(g^i(y)_N, x_N^{(i)}) = d(f^i(y_N), x_N^{(i)}) = d(f^i(x_N), x_N^{(i)}) \leq \epsilon_N,$$

we obtain

$$D(g^i(y), x^{(i)}) \leq \epsilon.$$

Moreover, since

$$\begin{aligned}\lim_{i\to\infty} d(g^i(y)_n, x_n^{(i)}) &= \lim_{i\to\infty} d(f^i(y_n), x_n^{(i)})\\ &= \lim_{i\to\infty} d(f^i(F^{N-n}(x_N)), x_n^{(i)})\\ &= \lim_{i\to\infty} d(F^{N-n}(f^i(x_N)), F^{N-n}(x_N^{(i)})) = 0\end{aligned}$$

for all $1 \le n \le N$; and

$$\begin{aligned}\lim_{i\to\infty} d(g^i(y)_{N+k}, x_{N+k}^{(i)}) &= \lim_{i\to\infty} d(f^i(y_{N+k}), x_{N+k}^{(i)})\\ &= \lim_{i\to\infty} d(f^i(x_{N+k}), x_{N+k}^{(i)}) = 0\end{aligned}$$

for all $k \ge 0$, we obtain

$$\lim_{i\to\infty} D(g^i(y), x^{(i)}) = 0.$$

In other words, ξ is ϵ-limit shadowed by y. Since $\epsilon > 0$ is arbitrary, we conclude that g satisfies the s-limit shadowing property, completing the proof of the lemma. □

Finally, we give the example.

Example 4.1 Let $\mathbb{Z}_2 = \{0, 1\}$. We define a metric d on $\{0, 1\}^{\mathbb{Z}}$ by

$$d(x, y) = \sup_{n\in\mathbb{Z}} 2^{-|n|}|x_n - y_n|$$

for all $x = (x_n)_{n\in\mathbb{Z}}, y = (y_n)_{n\in\mathbb{Z}} \in \{0, 1\}^{\mathbb{Z}}$. Note that the shift map

$$\sigma: \{0, 1\}^{\mathbb{Z}} \to \{0, 1\}^{\mathbb{Z}}$$

is an expansive mixing homeomorphism with the shadowing property and so satisfies the s-limit shadowing property (see, e.g. Barwell et al. 2012). We define a map $F: \{0, 1\}^{\mathbb{Z}} \to \{0, 1\}^{\mathbb{Z}}$ by for any $x = (x_n)_{n\in\mathbb{Z}}, y = (y_n)_{n\in\mathbb{Z}} \in \{0, 1\}^{\mathbb{Z}}$, $y = F(x)$ if and only if

$$y_n = x_n + x_{n+1}$$

for all $n \in \mathbb{Z}$. Note that F gives a factor map

$$F: (\{0, 1\}^{\mathbb{Z}}, \sigma) \to (\{0, 1\}^{\mathbb{Z}}, \sigma).$$

Given any $x = (x_n)_{n\in\mathbb{Z}}, y = (y_n)_{n\in\mathbb{Z}}, z = (z_n)_{n\in\mathbb{Z}}, w = (w_n)_{n\in\mathbb{Z}} \in \{0, 1\}^{\mathbb{Z}}$, assume that

- (x, y) is an asymptotic pair for σ,

- $(F(z), F(w)) = (x, y)$.

Then, there is $N \geq 0$ such that $x_n = y_n$ for all $n \geq N$. If $z_N = w_N$, we have $z_n = w_n$ for all $n \geq N$ and so (z, w) is an asymptotic pair for σ. If $z_N \neq w_N$, we have $z_n \neq w_n$ for all $n \geq N$ and so

$$\liminf_{i\to\infty} d(\sigma^i(z), \sigma^i(w)) = 1 > 0,$$

thus (z, w) is a distal pair for σ. In both cases, (z, w) is not a scrambled pair for σ.

For any $m \geq 1$ and $a = (a_n)_{n=-m}^m \in \{0, 1\}^{2m+1}$, we define $b = (b_n)_{n=-m}^{m-1} \in \{0, 1\}^{2m}$ by

$$b_n = a_n + a_{n+1}$$

for all $-m \leq n \leq m - 1$. Letting

$$S(a) = \{x = (x_n)_{n\in\mathbb{Z}} \colon x_n = a_n \text{ for all } -m \leq n \leq m\}$$

and

$$T(b) = \{x = (x_n)_{n\in\mathbb{Z}} \colon x_n = b_n \text{ for all } -m \leq n \leq m - 1\},$$

we obtain $F(S(a)) = T(b)$, an open subset of $\{0, 1\}^{\mathbb{Z}}$. Since $m \geq 1$ and $a = (a_n)_{n=-m}^m \in \{0, 1\}^{2m+1}$ are arbitrary, it follows that F is an open map. Given any $y = (y_n)_{n\in\mathbb{Z}} \in \{0, 1\}^{\mathbb{Z}}$, we define $\hat{y} = (\hat{y}_n)_{n\in\mathbb{Z}} \in \{0, 1\}^{\mathbb{Z}}$ by $\hat{y}_n = y_n + 1$ for all $n \in \mathbb{Z}$. Then, for any $x \in \{0, 1\}^{\mathbb{Z}}$, taking $y \in F^{-1}(x)$, we have $F^{-1}(x) = \{y, \hat{y}\}$. Note that $d(y, \hat{y}) = 1$ for all $y \in \{0, 1\}^{\mathbb{Z}}$.

Let $(X_n, f_n) = (\{0, 1\}^{\mathbb{Z}}, \sigma)$ and $\pi_n = F \colon (\{0, 1\}^{\mathbb{Z}}, \sigma) \to (\{0, 1\}^{\mathbb{Z}}, \sigma)$ for all $n \geq 1$. Let

$$(Y, g) = \lim_{\pi}(X_n, f_n)$$

and let D be a metric on Y. Since $\{0, 1\}^{\mathbb{Z}}$ is totally disconnected and $\sigma \colon \{0, 1\}^{\mathbb{Z}} \to \{0, 1\}^{\mathbb{Z}}$ is a mixing homeomorphism,

- Y is totally disconnected,
- g is a mixing homeomorphism.

By Lemmas 4.1 and 4.3, we obtain the following properties

- g is h-expansive,
- g has the s-limit shadowing property.

We shall show that

- g is not countably-expansive,
- g satisfies $Y_e = \emptyset$, where

$$Y_e = \{q \in Y \colon \Gamma_\epsilon(q) = \{q\} \text{ for some } \epsilon > 0\}.$$

Let $q = (q_n)_{n\geq 1} \in Y$ and $N \geq 1$. Let $F^{-1}(x) = \{x^a, x^b\}$ for all $x \in \{0,1\}^{\mathbb{Z}}$. Then, for all $c = (c_k)_{k\geq 1} \in \{a,b\}^{\mathbb{N}}$, we define $q(c) = (q(c)_n)_{n\geq 1} \in Y$ by $q(c)_n = q_n$ for all $1 \leq n \leq N$; and

$$q(c)_{N+k} = q(c)_{N+k-1}^{c_k}$$

for all $k \geq 1$. Given any $\epsilon > 0$, if $N \geq 1$ is large enough, $q(c)$, $c \in \{a,b\}^{\mathbb{N}}$, satisfies $q(c) \in \Gamma_\epsilon(q)$ for all $c \in \{a,b\}^{\mathbb{N}}$. Since

$$\{q(c) \colon c \in \{a,b\}^{\mathbb{N}}\}$$

is an uncountable set, it follows that g is not countably-expansive. Since $q \in Y$ and $\epsilon > 0$ are arbitrary, it also follows that $Y_e = \emptyset$.

Data availability Data sharing not applicable to this article as no datasets were generated or analysed during the current study.

References

Anosov, D.V.: Geodesic flows on closed Riemann manifolds with negative curvature. Proc. Steklov Inst. Math. **90**, 235 (1967)

Aoki, N., Hiraide, K.: Topological Theory of Dynamical Systems. Recent Advances. North-Holland Mathematical Library, 52, North-Holland Publishing Co., Amsterdam (1994)

Aponte, J., Villavicencio, H.: Shadowable points for flows. J. Dyn. Control Syst. **24**, 701–719 (2018)

Arbieto, A., Rego, E.: Positive entropy through pointwise dynamics. Proc. Am. Math. Soc. **148**, 263–271 (2020)

Arbieto, A., Rego, E.: On the entropy of continuous flows with uniformly expansive points and the globalness of shadowable points with gaps. Bull. Braz. Math. Soc. (N.S.) **53**, 853–872 (2022)

Artigue, A., Carvalho, B., Cordeiro, W., Vieitez, J.: Beyond topological hyperbolicity: the L-shadowing property. J. Differ. Equ. **268**, 3057–3080 (2020)

Artigue, A., Carvalho, B., Cordeiro, W., Vieitez, J.: Countably and entropy expansive homeomorphisms with the shadowing property. Proc. Am. Math. Soc. **150**, 3369–3378 (2022)

Auslander, J., Glasner, E., Weiss, B.: On recurrence in zero dimensional flows. Forum Math. **19**, 107–114 (2007)

Barwell, A.D., Good, C., Oprocha, P.: Shadowing and expansivity in subspaces. Fund. Math. **219**, 223–243 (2012)

Blanchard, F., Glasner, E., Kolyada, S., Maass, A.: On Li–Yorke pairs. J. Reine Angew. Math. **547**, 51–68 (2002)

Bowen, R.: Entropy-expansive maps. Trans. Am. Math. Soc. **164**, 323–331 (1972)

Bowen, R.: Equilibrium States and the Ergodic Theory of Anosov Diffeomorphisms. Lecture Notes in Mathematics, vol. 470. Springer, Berlin (1975)

Das, T., Khan, A.G.: Average shadowing and persistence in pointwise dynamics. Topol. Appl. **292**, 107629 (2021)

Das, T., Khan, A.G.: Stability theorems in pointwise dynamics. Topol. Appl. **320**, 108218 (2022)

Das, P., Das, T., Khan, A.G.: Measure expansivity and specification for pointwise dynamics. Bull. Braz. Math. Soc. (N.S.) **50**, 933–948 (2019)

Das, P.K., Das, T., Khan, A.G.: Pointwise dynamics under orbital convergence. Bull. Braz. Math. Soc. (N.S.) **51**, 1001–1016 (2020)

Dong, M., Lee, K., Morales, C.: Pointwise topological stability and persistence. J. Math. Anal. Appl. **480**, 123334 (2019)

Dong, M., Jung, W., Morales, C.: Eventually shadowable points. Qual. Theory Dyn. Syst. **19**, 16 (2020)

Furstenberg, H.: Recurrence in Ergodic Theory and Combinatorial Number Theory. Princeton University Press, Princeton (1981)

Jung, W., Lee, K., Morales, C.A.: Pointwise persistence and shadowing. Monatsh. Math. **192**, 111–123 (2020)

Kawaguchi, N.: Quantitative shadowable points. Dyn. Syst. **32**, 504–518 (2017a)

Kawaguchi, N.: Properties of shadowable points: chaos and equicontinuity. Bull. Braz. Math. Soc. (N.S.) **48**, 599–622 (2017b)

Kawaguchi, N.: Maximal chain continuous factor. Discrete Contin. Dyn. Syst. **41**, 5915–5942 (2021a)

Kawaguchi, N.: Generic and dense distributional chaos with shadowing. J. Differ. Equ. Appl. **27**, 1456–1481 (2021b)

Koo, N., Lee, K., Morales, C.A.: Pointwise topological stability. Proc. Edinb. Math. Soc. **61**, 1179–1191 (2018)

Luo, X.F., Nie, X.X., Yin, J.D.: On the shadowing property and shadowable point of set-valued dynamical systems. Acta Math. Sin. (Engl. Ser.) **36**, 1384–1394 (2020)

Misiurewicz, M.: Remark on the Definition of Topological Entropy. Dynamical Systems and Partial Differential Equations (Caracas, 1984), Univ. Simon Bolivar, Caracas, 65–67 (1986)

Morales, C.A.: Shadowable points. Dyn. Syst. **31**, 347–356 (2016)

Pilyugin, SYu.: Shadowing in Dynamical Systems. Lecture Notes in Mathematics, vol. 1706. Springer, Berlin (1999)

Shin, B.: On the set of shadowable measures. J. Math. Anal. Appl. **469**, 872–881 (2019)

Ye, X., Zhang, G.: Entropy points and applications. Trans. Am. Math. Soc. **359**, 6167–6186 (2007)