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Universal Inequalities for Eigenvalues of a Clamped Plate Problem of the Drifting Laplacian

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Abstract
In this paper, we study the universal inequalities for eigenvalues of a clamped plate problem of the drifting Laplacian in several cases, and establish some universal inequalities that are different from those obtained previously in (Du et al. in Z Angew Math Phys 66(3):703–726, 2015).

Keywords Universal inequalities · Drifting Laplacian · Clamped plate problem · Compact manifolds

Mathematics Subject Classification 35P15 · 53C20 · 53C42 · 58G25

1 Introduction

Let $\left(M, \langle,\rangle\right)$ be an n-dimensional complete Riemannian manifold with a metric $\langle,\rangle$, and the triple $\left(M, \langle,\rangle, e^{-\theta} d\nu\right)$ be a smooth metric measure space, where θ is a smooth real valued function on M (or at least $\theta \in C^2(M)$) and is called the potential function, and $d\nu$ is Riemannian volume element (also called the volume density, or Riemannian volume measure) related to $\langle,\rangle$. Metric measure spaces have been studied widely in geometric analysis (see e.g. Cao and Zhou 2010; Cheng et al. 2014; Munteanu

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and Wang 2012; Wei and Wylie 2009) and so on. In particular, Perelman (2002) established the entropy formulae for Ricci flow on this kind of spaces. On smooth metric measure space $\big(M, \langle , \rangle, e^{-\theta} d\nu\big)$, the so-called drifting Laplacian (also called weighted Laplacian, or the Witten–Laplacian) $\mathbb{L}_\theta$ can be defined as follows:

$$\mathbb{L}_\theta := \Delta - \langle \nabla\theta, \nabla(\cdot)\rangle = e^{\theta}\,\mathrm{div}\big[e^{-\theta}\nabla(\cdot)\big],$$

where Δ and ∇ are the Laplacian and the gradient operator on M, respectively. This operator plays an important role in probability theory and geometric analysis, and has been extensively studied (see Cao and Zhou 2010; Fang et al. 2008; Li and Wei 2015; Munteanu and Wang 2012; Wei and Wylie 2009), etc.. Moreover, when M is a self-shrinker and $\theta = \frac{|x|}{2}$, it becomes the $\mathfrak{L}$ operator introduced by Colding and Minicozzi (2012).

Let $\Omega \subset M$ be a bounded connected domain with smooth boundary $\partial\Omega$. Consider the following eigenvalue problem for the bi-drifting Laplacian $\mathbb{L}_\theta^2$ on Ω:

$$\begin{cases} \mathbb{L}_\theta^2 u = \Lambda u, & \text{in } \Omega, \\ u = \frac{\partial u}{\partial \mathbf{n}} = 0, & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where $\mathbf{n}$ denotes the outward unit normal vector field of $\partial\Omega$. Problem (1.1) is often called clamped plate problem of the drifting Laplacian (cf. Cheng and Yang 2006; Cheng et al. 2010; Wang and Xia 2011). For any $f, g \in C^4(\Omega) \cap C^3(\partial\Omega)$ with

$$f|_{\partial\Omega} = g|_{\partial\Omega} = 0; \quad \left.\frac{\partial f}{\partial \mathbf{n}}\right|_{\partial\Omega} = \left.\frac{\partial g}{\partial \mathbf{n}}\right|_{\partial\Omega} = 0,$$

by using integration by parts we obtain

$$\int_\Omega \langle \nabla f, \nabla g\rangle \mathrm{d}\mu = -\int_\Omega g\mathbb{L}_\theta f d\mu = -\int_\Omega f\mathbb{L}_\theta g d\mu$$

and

$$\int_\Omega g\mathbb{L}_\theta^2 f d\mu = \int_\Omega \mathbb{L}_\theta f \mathbb{L}_\theta g d\mu = \int_\Omega f\mathbb{L}_\theta^2 g d\mu,$$

where $d\mu := e^{-\theta} d\nu$ is often called weighted volume density. This implies that $\mathbb{L}_\theta^2$ is self-adjoint on the space of functions

$$\left\{ f \in C^4(\Omega) \cap C^3(\partial\Omega) : \, f|_{\partial\Omega} = \left.\frac{\partial f}{\partial \mathbf{n}}\right|_{\partial\Omega} = 0 \right\}$$

with respect to the inner product

$$\langle\langle f, g\rangle\rangle := \int_\Omega f g d\mu.$$

So the spectrum of $\mathbb{L}_\theta$ in (1.1) is real and discrete, and all the eigenvalues can be listed in a non-decreasing manner (cf. Du et al. 2015):

$$0 < \Lambda_1 \leq \Lambda_2 \leq \cdots \leq \Lambda_k \leq \cdots \nearrow +\infty,$$

where each eigenvalue Λ_k $(i = 1, 2, \ldots)$ is repeated according to its finite multiplicity.

In particular, when the potential function θ is a constant, $\mathbb{L}_\theta$ is exactly Δ, and problem (1.1) becomes the following eigenvalue problem for the biharmonic operator Δ^2 on Ω:

$$\begin{cases} \Delta^2 u = \Gamma u, & \text{in } \Omega, \\ u = \frac{\partial u}{\partial \mathbf{n}} = 0, & \text{on } \partial\Omega, \end{cases} \tag{1.2}$$

which is often called clamped plate problem of the Laplacian. When $M = \mathbb{R}^n$, it describes characteristic vibrations of a clamped plate in elastic mechanics. Obviously, problem (1.2) also has a real and discrete spectrum (cf. Agmon 1965):

$$0 < \Gamma_1 \leq \Gamma_2 \leq \cdots \leq \Gamma_k \leq \cdots \nearrow +\infty,$$

where each Γ_i $(i = 1, 2, \ldots)$ is also repeated according to its finite multiplicity.

We say that an inequality on a domain Ω is called “universal" if it does not involve geometric quantities of Ω such as volume or area, etc., but only dimension n.

In this paper, our main interest is to derive universal inequalities for eigenvalues of the clamped plate problem of the drifting Laplacian. Therefore, we will focus on that topic and mainly introduce the results on the clamped plate problem of the drifting Laplacian. Interested readers may refer to Ashbaugh (1999), Chen and Cheng (2008), Cheng and Yang (2005), Cheng and Yang (2007), Wang and Xia (2008), Xia (2013) for some results on universal inequalities of the Laplacian eigenvalues, and also refer to Payne et al. (1956), Hile and Yeh (1984), Chen and Qian (1990), Hook (1990), Ashbaugh (1999), Cheng and Yang (2006), Wang and Xia (2007b), Cheng and Yang (2011), Wang and Xia (2011), Cheng et al. (2010), Cheng et al. (2009), Cheng and Yang (2011), Wang and Xia (2007a), Xia (2013), El Soufi et al. (2009) for some results on the clamped plate problem of the Laplacian.

The study of universal inequalities for eigenvalues of the clamped plate problem of the drifting Laplacian, has a long history. It is difficult to describe the complete literature in this topic. Only a few results are summarized below.

Some interesting results concerning the eigenvalues of the drifting Laplacian can be found, for instance, in Batista et al. (2014), Cheng et al. (2014), Du et al. (2015), Futaki et al. (2013), Ma and Du (2010), Ma and Liu (2008), Ma and Liu (2009), Xia and Xu (2014). In particular, it is worth mentioning that Xia and Xu (2014) studied the eigenvalues of Dirichlet problems of the drifting Laplacian on compact manifolds, obtained some universal inequalities for eigenvalues, and also gave a lower bound of the first eigenvalue of the drifting Laplacian on a compact manifold with boundary.

In what follows, we assume that θ is a smooth function on Ω with $C_0 = \max_{\bar{\Omega}} |\nabla\theta|$. Du et al. (2015) studied the clamped plate problem of the drifting Laplacian, and established the following universal inequalities:

(i) If M is isometrically immersed in a Euclidean space $\mathbb{R}^m$ with mean curvature vector $\mathbf{H}$, then

$$\sum_{i=1}^{k}(\Lambda_{k+1}-\Lambda_i)^2 \le \frac{1}{n}\left\{\sum_{i=1}^{k}(\Lambda_{k+1}-\Lambda_i)^2\left[(2n+4)\Lambda_i^{\frac{1}{2}}+4C_0\Lambda_i^{\frac{1}{4}}+n^2H_0^2+C_0^2\right]\right\}^{\frac{1}{2}} \times\left[\sum_{i=1}^{k}(\Lambda_{k+1}-\Lambda_i)\left(4\Lambda_i^{\frac{1}{2}}+4C_0\Lambda_i^{\frac{1}{4}}+n^2H_0^2+C_0^2\right)\right]^{\frac{1}{2}} \tag{1.3}$$

and

$$\sum_{i=1}^{n}(\Lambda_{i+1}-\Lambda_1)^{\frac{1}{2}} \le \left\{\left(4\Lambda_1^{\frac{1}{2}}+4C_0\Lambda_1^{\frac{1}{4}}+n^2H_0^2+C_0^2\right) \times\left[(2n+4)\Lambda_1^{\frac{1}{2}}+4C_0\Lambda_1^{\frac{1}{4}}+n^2H_0^2+C_0^2\right]\right\}^{\frac{1}{2}},$$

where $H_0=\sup_{\Omega}|\mathbf{H}|$.

(ii) If there exist a function $\phi:\Omega\to\mathbb{R}$ and a positive constant A_0 such that

$$|\nabla\phi|=1,\quad |\Delta\phi|\le A_0,$$

then

$$\sum_{i=1}^{k}(\Lambda_{k+1}-\Lambda_i)^2 \le \left\{\sum_{i=1}^{k}(\Lambda_{k+1}-\Lambda_i)^2\left[6\Lambda_i^{\frac{1}{2}}+4\left(A_0+C_0\right)\Lambda_i^{\frac{1}{4}}+(A_0+C_0)^2\right]\right\}^{\frac{1}{2}} \times\left\{\sum_{i=1}^{k}(\Lambda_{k+1}-\Lambda_i)\left[4\Lambda_i^{\frac{1}{2}}+4\left(A_0+C_0\right)\Lambda_i^{\frac{1}{4}}+(A_0+C_0)^2\right]\right\}^{\frac{1}{2}}.$$

(iii) If Ω admits an eigenmap $f=(f_1,f_2,\ldots,f_{m+1}):\Omega\to\mathbb{S}^m$ corresponding to an eigenvalue τ, that is,

$$\Delta f_\alpha=-\tau f_\alpha,\quad \alpha=1,\ldots,m+1,\quad \sum_{\alpha=1}^{m+1}f_\alpha^2=1,$$

then

$$\sum_{i=1}^{k}(\Lambda_{k+1}-\Lambda_i)^2 \leq \left\{\sum_{i=1}^{k}(\Lambda_{k+1}-\Lambda_i)^2\left[6\Lambda_i^{\frac{1}{2}}+4C_0\Lambda_i^{\frac{1}{4}}+\left(C_0^2+\tau\right)\right]\right\}^{\frac{1}{2}} \times\left\{\sum_{i=1}^{k}(\Lambda_{k+1}-\Lambda_i)\left[4\Lambda_i^{\frac{1}{2}}+4C_0\Lambda_i^{\frac{1}{4}}+\left(C_0^2+\tau\right)\right]\right\}^{\frac{1}{2}},$$

where $\mathbb{S}^m$ is the unit sphere of dimension m.

More results on the clamped plate problem of the drifting Laplacian can be found in Du et al. (2015), Xiong et al. (2022), Zeng (2020a, b), Zeng (2022), Zeng and Sun (2022), Li et al. (2022) and the references therein. In conclusion, the study of the universal inequality for eigenvalues of the bi-drifting Laplacian is still a very active research field.

In this paper, our objective is to derive some universal inequalities for eigenvalues of the bi-drifting Laplacian. Under the same assumptions as in Du et al. (2015, Theorem 1.1), we establish some universal inequalities that are different from those in Du et al. (2015). Our main results can be roughly stated as the following theorem.

Theorem 1.1 *Let M be an n-dimensional complete Riemannian manifold and Ω be a bounded domain with smooth boundary in M. Let θ be a smooth function on Ω with* $C_0=\max_{\bar{\Omega}}|\nabla\theta|$. *Denote by Γ_i the i-th eigenvalue of problem (1.1), respectively.*

(i) If M is isometrically immersed in a Euclidean space $\mathbb{R}^m$ with mean curvature vector $\mathbf{H}$, then

$$\sum_{i=1}^{k}(\Lambda_{k+1}-\Lambda_i) \leq \frac{1}{n}\left\{\sum_{i=1}^{k}(\Lambda_{k+1}-\Lambda_i)\left[(2n+4)\Lambda_i^{\frac{1}{2}}+4C_0\Lambda_i^{\frac{1}{4}}+n^2H_0^2+C_0^2\right]\right\}^{\frac{1}{2}} \times\left[\sum_{i=1}^{k}\left(4\Lambda_i^{\frac{1}{2}}+4C_0\Lambda_i^{\frac{1}{4}}+n^2H_0^2+C_0^2\right)\right]^{\frac{1}{2}}, \tag{1.4}$$

where $H_0=\sup_{\Omega}|\mathbf{H}|$.

(ii) If there exists a function $\phi:\Omega\to$ R *and a constant A_0 such that*

$$|\nabla\phi|=1,\quad |\Delta\phi|\leq A_0,\quad on\ \ \Omega, \tag{1.5}$$

then

$$\sum_{i=1}^{k}(\Lambda_{k+1}-\Lambda_i) \leq \left\{\sum_{i=1}^{k}(\Lambda_{k+1}-\Lambda_i)\left[6\Lambda_i^{\frac{1}{2}}+4\left(A_0+C_0\right)\Lambda_i^{\frac{1}{4}}+\left(A_0+C_0\right)^2\right]\right\}^{\frac{1}{2}} \times\left[\sum_{i=1}^{k}\left(2\Lambda_i^{\frac{1}{4}}+A_0+C_0\right)^2\right]^{\frac{1}{2}}. \tag{1.6}$$

(iii) If there exists a function $\psi : \Omega \to \mathbb{R}$ *and a constant* B_0 *such that*

$$|\nabla \psi| = 1, \quad \Delta \psi = B_0, \quad on \ \ \Omega, \tag{1.7}$$

then

$$\sum_{i=1}^{k}(\Lambda_{k+1}-\Lambda_i) \le \left\{\sum_{i=1}^{k}(\Lambda_{k+1}-\Lambda_i)\left[(4+2B_0)\Lambda_i^{\frac{1}{2}}+4C_0\Lambda_i^{\frac{1}{4}}+C_0^2-B_0^2\right]\right\}^{\frac{1}{2}} \times \left[\sum_{i=1}^{k}\left(4\Lambda_i^{\frac{1}{2}}+4C_0\Lambda_i^{\frac{1}{4}}+C_0^2-B_0^2\right)\right]^{\frac{1}{2}}. \tag{1.8}$$

(iv) *If there exist l functions* $\phi_p : \Omega \to \mathrm{R}$ *such that*

$$\langle \nabla\phi_p, \nabla\phi_q\rangle = \delta_{pq}, \quad \Delta\phi_p = 0 \ \ on \ \ \Omega, \quad p, q = 1, \ldots, l, \tag{1.9}$$

then

$$\sum_{i=1}^{k}(\Lambda_{k+1}-\Lambda_i) \le \frac{1}{l}\left\{\sum_{i=1}^{k}(\Lambda_{k+1}-\Lambda_i)\left[(4+2l)\Lambda_i^{\frac{1}{2}}+4C_0\Lambda_i^{\frac{1}{4}}+C_0^2\right]\right\}^{\frac{1}{2}} \times \left[\sum_{i=1}^{k}\left(2\Lambda_i^{\frac{1}{4}}+C_0\right)^2\right]^{\frac{1}{2}}. \tag{1.10}$$

(v) *If* Ω *admits an eigenmap* $f = (f_1, f_2, \ldots, f_{m+1}) : \Omega \to \mathbb{S}^{\mathrm{m}}$ *corresponding to an eigenvalue* τ*, that is,*

$$\Delta f_\alpha = -\tau f_\alpha, \quad \alpha = 1, \ldots, m+1, \quad \sum_{\alpha=1}^{m+1} f_\alpha^2 = 1, \tag{1.11}$$

then

$$\sum_{i=1}^{k}(\Lambda_{k+1}-\Lambda_i) \le \left[\sum_{i=1}^{k}(\Lambda_{k+1}-\Lambda_i)\left(6\Lambda_i^{\frac{1}{2}}+4C_0\Lambda_i^{\frac{1}{4}}+\tau+C_0^2\right)\right]^{\frac{1}{2}} \times \left[\sum_{i=1}^{k}\left(4\Lambda_i^{\frac{1}{2}}+4C_0\Lambda_i^{\frac{1}{4}}+\tau+C_0^2\right)\right]^{\frac{1}{2}}, \tag{1.12}$$

where $\mathbb{S}^m$ *is the unit sphere of dimension m.*

Remark 1.1 (*see* Wang and Xia 2011) (i) Any Hadamard manifold with Ricci curvature bounded below admits functions (e.g., its Busemann function) satisfying (1.5) (cf. Ballmann et al. 1985; Heintze and Im Hof 1978; Sakai 1996).

(ii) Let (N, ds_N^2) be a complete Riemannian manifold and define a Riemannian metric on $M = \mathbb{R} \times N$ by

$$ds_M^2 = dt^2 + \eta^2(t) ds_N^2.$$

where η is a positive smooth function defined on $\mathbb{R}$ with $\eta(0) = 1$. The manifold (M, ds_M^2) is called a warped product and denoted by $M = \mathbb{R} \times_\eta N$. It is known that M is a complete Riemannian manifold. Set $\eta = e^{-t}$. The warped product manifold $M = \mathbb{R} \times_{e^{-t}} N$ admits functions satisfying (1.7).

(iii) Let $\mathbb{H}^n$ be the n-dimensional hyperbolic space with constant curvature -1. One can show that $\mathbb{H}^n$ admits a warped product model, $\mathbb{H}^n = \mathbb{R} \times_{e^{-t}} \mathbb{R}^{n-1}$. Therefore, $\mathbb{H}^n$ admits functions satisfying (1.7).

(iv) Let N be any complete Riemannian manifold. Let

$$M = \mathbb{R}^l \times N := \{(x, z) \mid x = (x_1, \ldots, x_l) \in \mathbb{R}^l, \ z \in N\}$$

be the product of $\mathbb{R}^l$ and N endowed with the product metric. One can check that the projection functions defined by

$$\phi_p(x, z) = x_p \qquad \text{for} \quad p = 1, \ldots, l,$$

satisfy (1.9).

(v) Any compact homogeneous Riemannian manifold admits eigenmaps to some unit sphere for the first positive eigenvalue of the Laplacian (cf. Li 1980). In other words, any compact homogeneous Riemannian manifold admits a family of functions $\{f_\alpha\}_{\alpha=1}^{m+1}$ satisfying (1.11), where τ is the first positive eigenvalue of the Laplacian.

The reader may refer to Wang and Xia (2011, Example 2.1–2.4) and also Ballmann et al. (1985), do Carmo et al. (2010), Heintze and Im Hof (1978), Li (1980), Wang and Xia (2011), Xia and Xu (2014), Sakai (1996) for more details of Remark 1.1.

Remark 1.2 (i) Our universal inequalities are very different from the previous available universal inequalities. In particular, our results can reveal the relationship between the $(k + 1)$-th eigenvalue and the first k eigenvalues relatively quickly.

(ii) In some sense, our results may be better than others. So we believe that our results are new. For example, in case (i) of Theorem 1.1, the result of Du et al. (2015) is (1.3), and our result is (1.4). In the following we will try to show that (1.4) may be better than (1.3).

Clearly, inequality (1.3) can be simply rewritten as follows:

$$\sum_{i=1}^{k}(\Lambda_{k+1} - \Lambda_i)^2 \leq \frac{1}{n}\left[\sum_{i=1}^{k}(\Lambda_{k+1} - \Lambda_i)^2 A_i\right]^{\frac{1}{2}} \left(\sum_{i=1}^{k}(\Lambda_{k+1} - \Lambda_i) B_i\right)^{\frac{1}{2}}, \tag{1.13}$$

where

$$A_i = (2n+4)\Lambda_i^{\frac{1}{2}} + 4C_0\Lambda_i^{\frac{1}{4}} + n^2H_0^2 + C_0^2, \quad B_i = 4\Lambda_i^{\frac{1}{2}} + 4C_0\Lambda_i^{\frac{1}{4}} + n^2H_0^2 + C_0^2.$$

From inequality (1.13), and using the weighted Chebyshev inequality (see Lemma 5.1 in Appendix), we infer

$$\sum_{i=1}^{k}(\Lambda_{k+1}-\Lambda_i)^2 \le \frac{1}{n}\left[\sum_{i=1}^{k}(\Lambda_{k+1}-\Lambda_i)^2\right]^{\frac{1}{2}}\left(\sum_{i=1}^{k}(\Lambda_{k+1}-\Lambda_i)A_iB_i\right)^{\frac{1}{2}}. \tag{1.14}$$

Thus we have

$$\sum_{i=1}^{k}(\Lambda_{k+1}-\Lambda_i)^2 \le \frac{1}{n^2}\sum_{i=1}^{k}(\Lambda_{k+1}-\Lambda_i)A_iB_i, \tag{1.15}$$

which implies

$$\begin{aligned}
\Lambda_{k+1} \le\ & \frac{\sum_{i=1}^{k}\Lambda_i}{k} + \frac{1}{2n^2}\frac{\sum_{i=1}^{k}A_iB_i}{k} + \left\{\left(\frac{\sum_{i=1}^{k}\Lambda_i}{k} + \frac{1}{2n^2}\frac{\sum_{i=1}^{k}A_iB_i}{k}\right)^2\right.\\
& \left. - \left(\frac{\sum_{i=1}^{k}\Lambda_i^2}{k} + \frac{1}{n^2}\frac{\sum_{i=1}^{k}A_iB_i\Lambda_i}{k}\right)\right\}^{\frac{1}{2}}\\
=\ & \frac{\sum_{i=1}^{k}\Lambda_i}{k} + \frac{1}{2n^2}\frac{\sum_{i=1}^{k}A_iB_i}{k} + \left\{\left[\left(\frac{\sum_{i=1}^{k}\Lambda_i}{k}\right)^2 - \frac{\sum_{i=1}^{k}\Lambda_i^2}{k}\right]\right.\\
& \left. + \frac{1}{n^2}\left(\frac{\sum_{i=1}^{k}A_iB_i}{k}\frac{\sum_{i=1}^{k}\Lambda_i}{k} - \frac{\sum_{i=1}^{k}A_iB_i\Lambda_i}{k}\right) + \left(\frac{1}{2n^2}\frac{\sum_{i=1}^{k}A_iB_i}{k}\right)^2\right\}^{\frac{1}{2}}\\
\le\ & \frac{\sum_{i=1}^{k}\Lambda_i}{k} + \frac{1}{n^2}\frac{\sum_{i=1}^{k}A_iB_i}{k}.
\end{aligned} \tag{1.16}$$

Here, in obtaining the last inequality, we used twice the Chebyshev sum inequality (see Lemma 5.2 in Appendix).

Similarly, inequality (1.4) can be simply rewritten as follows:

$$\sum_{i=1}^{k}(\Lambda_{k+1}-\Lambda_i) \le \frac{1}{n}\left[\sum_{i=1}^{k}(\Lambda_{k+1}-\Lambda_i)A_i\right]^{\frac{1}{2}}\left(\sum_{i=1}^{k}B_i\right)^{\frac{1}{2}}, \tag{1.17}$$

where A_i and B_i are defined as above.

From inequality (1.17), and using the Chebyshev sum inequality, we can get

$$\begin{aligned}
\Lambda_{k+1} &\leq \frac{\sum_{i=1}^k \Lambda_i}{k} + \frac{1}{2n^2} \frac{\sum_{i=1}^k A_i}{k} \frac{\sum_{i=1}^k B_i}{k} \\
&\quad + \left\{ \left(\frac{\sum_{i=1}^k \Lambda_i}{k} + \frac{1}{2n^2} \frac{\sum_{i=1}^k A_i}{k} \frac{\sum_{i=1}^k B_i}{k} \right)^2 \right. \\
&\quad \left. - \left[\left(\frac{\sum_{i=1}^k \Lambda_i}{k} \right)^2 + \frac{1}{n^2} \frac{\sum_{i=1}^k A_i \Lambda_i}{k} \frac{\sum_{i=1}^k B_i}{k} \right] \right\}^{\frac{1}{2}} \\
&= \frac{\sum_{i=1}^k \Lambda_i}{k} + \frac{1}{2n^2} \frac{\sum_{i=1}^k A_i}{k} \frac{\sum_{i=1}^k B_i}{k} \\
&\quad + \left\{ \frac{1}{n^2} \left(\frac{\sum_{i=1}^k A_i}{k} \frac{\sum_{i=1}^k \Lambda_i}{k} - \frac{\sum_{i=1}^k A_i \Lambda_i}{k} \right) \frac{\sum_{i=1}^k B_i}{k} \right. \\
&\quad \left. + \left(\frac{1}{2n^2} \frac{\sum_{i=1}^k A_i}{k} \frac{\sum_{i=1}^k B_i}{k} \right)^2 \right\}^{\frac{1}{2}} \\
&\leq \frac{\sum_{i=1}^k \Lambda_i}{k} + \frac{1}{n^2} \frac{\sum_{i=1}^k A_i}{k} \frac{\sum_{i=1}^k B_i}{k}. \qquad (1.18)
\end{aligned}$$

According to the Chebyshev sum inequality, we know that

$$\frac{\sum_{i=1}^k A_i}{k} \frac{\sum_{i=1}^k B_i}{k} \leq \frac{\sum_{i=1}^k A_i B_i}{k}.$$

This shows that inequality (1.18) is better than (1.16). Therefore, inequality (1.4) may be better than (1.3) in some sense. So we believe that it is new.

Corollary 1.1 *Under the same assumptions of Theorem 1.1, we have*
(i) *If M is isometrically immersed in $\mathbb{R}^m$ with mean curvature vector $\mathbf{H}$, then*

$$\begin{aligned}
\sum_{i=1}^k (\Lambda_{k+1} - \Lambda_i) &\leq \frac{1}{n^2 k} \sum_{i=1}^k \left[(2n+4)\Lambda_i^{\frac{1}{2}} + 4C_0 \Lambda_i^{\frac{1}{4}} + n^2 H_0^2 + C_0^2 \right] \\
&\quad \times \sum_{i=1}^k \left(4\Lambda_i^{\frac{1}{2}} + 4C_0 \Lambda_i^{\frac{1}{4}} + n^2 H_0^2 + C_0^2 \right),
\end{aligned}$$

where $H_0 = \sup_\Omega |\mathbf{H}|$. Moreover,

$$\Lambda_{k+1} \le \frac{\sum_{i=1}^{k}\Lambda_i}{k} + \frac{1}{n^2}\left[(2n+4)\frac{\sum_{i=1}^{k}\Lambda_i^{\frac{1}{2}}}{k} + 4C_0\frac{\sum_{i=1}^{k}\Lambda_i^{\frac{1}{4}}}{k} + n^2H_0^2 + C_0^2\right]$$
$$\times\left(4\frac{\sum_{i=1}^{k}\Lambda_i^{\frac{1}{2}}}{k} + 4C_0\frac{\sum_{i=1}^{k}\Lambda_i^{\frac{1}{4}}}{k} + n^2H_0^2 + C_0^2\right).$$

In particular,

$$\Lambda_{k+1} \le \frac{1}{n^2}\Big\{(n+4)^2\Lambda_k + 8(n+4)C_0\Lambda_k^{\frac{3}{4}} + 2\big[(n+4)n^2H_0^2 + (n+12)C_0^2\big]\Lambda_k^{\frac{1}{2}}$$
$$+8C_0\big(n^2H_0^2 + C_0^2\big)\Lambda_k^{\frac{1}{4}} + \big(n^2H_0^2 + C_0^2\big)^2\Big\}.$$

(ii) *If there exists a function* $\phi : \Omega \to \mathrm{R}$ *and a constant* A_0 *such that*

$$|\nabla\phi| = 1, \quad |\Delta\phi| \le A_0, \quad on \ \ \Omega,$$

then

$$\sum_{i=1}^{k}(\Lambda_{k+1} - \Lambda_i) \le \frac{1}{k}\sum_{i=1}^{k}\left[6\Lambda_i^{\frac{1}{2}} + 4(A_0 + C_0)\Lambda_i^{\frac{1}{4}} + (A_0 + C_0)^2\right]$$
$$\times\sum_{i=1}^{k}\big(2\Lambda_i^{\frac{1}{4}} + A_0 + C_0\big)^2.$$

Moreover,

$$\Lambda_{k+1} \le \frac{\sum_{i=1}^{k}\Lambda_i}{k} + \left[6\frac{\sum_{i=1}^{k}\Lambda_i^{\frac{1}{2}}}{k} + 4(A_0 + C_0)\frac{\sum_{i=1}^{k}\Lambda_i^{\frac{1}{4}}}{k} + (A_0 + C_0)^2\right]$$
$$\times\left[4\frac{\sum_{i=1}^{k}\Lambda_i^{\frac{1}{2}}}{k} + 4(A_0 + C_0)\frac{\sum_{i=1}^{k}\Lambda_i^{\frac{1}{4}}}{k} + (A_0 + C_0)^2\right].$$

In particular,

$$\Lambda_{k+1} \le 25\Lambda_k + 40(A_0+C_0)\Lambda_k^{\frac{3}{4}} + 26(A_0 + C_0)^2\Lambda_k^{\frac{1}{2}} + 8(A_0+C_0)^3\Lambda_k^{\frac{1}{4}} + (A_0+C_0)^4.$$

(iii) *If there exists a function* $\psi : \Omega \to \mathbb{R}$ *and a constant* B_0 *such that*

$$|\nabla\psi| = 1, \quad \Delta\psi = B_0, \quad on \ \ \Omega,$$

then

$$\sum_{i=1}^{k}(\Lambda_{k+1}-\Lambda_i) \le \frac{1}{k}\left\{\sum_{i=1}^{k}\left[(4+2B_0)\Lambda_i^{\frac{1}{2}}+4C_0\Lambda_i^{\frac{1}{4}}+C_0^2-B_0^2\right]\right\} \times\left[\sum_{i=1}^{k}\left(4\Lambda_i^{\frac{1}{2}}+4C_0\Lambda_i^{\frac{1}{4}}+C_0^2-B_0^2\right)\right].$$

Moreover,

$$\Lambda_{k+1} \le \frac{\sum_{i=1}^{k}\Lambda_i}{k}+\left[(4+2B_0)\frac{\sum_{i=1}^{k}\Lambda_i^{\frac{1}{2}}}{k}+4C_0\frac{\sum_{i=1}^{k}\Lambda_i^{\frac{1}{4}}}{k}+C_0^2-B_0^2\right] \times\left(4\frac{\sum_{i=1}^{k}\Lambda_i^{\frac{1}{2}}}{k}+4C_0\frac{\sum_{i=1}^{k}\Lambda_i^{\frac{1}{4}}}{k}+C_0^2-B_0^2\right).$$

In particular,

$$\Lambda_{k+1} \le (8B_0+17)\Lambda_k+8(B_0+4)C_0\Lambda_k^{\frac{3}{4}}+2\left[(B_0+12)C_0^2-4B_0^2-B_0^3\right]\Lambda_k^{\frac{1}{2}} +8(C_0^2-B_0^2)C_0\Lambda_k^{\frac{1}{4}}+(C_0^2-B_0^2)^2.$$

(iv) *If there exist* l *functions* $\phi_p : \Omega \to \mathrm{R}$ *such that*

$$\langle\nabla\phi_p,\nabla\phi_q\rangle=\delta_{pq},\quad \Delta\phi_p=0,\quad on\ \ \Omega,\quad p,q=1,\ldots,l,$$

then

$$\sum_{i=1}^{k}(\Lambda_{k+1}-\Lambda_i) \le \frac{1}{l^2k}\left\{\sum_{i=1}^{k}\left[(4+2l)\Lambda_i^{\frac{1}{2}}+4C_0\Lambda_i^{\frac{1}{4}}+C_0^2\right]\right\} \times\left[\sum_{i=1}^{k}\left(2\Lambda_i^{\frac{1}{4}}+C_0\right)^2\right].$$

Moreover,

$$\Lambda_{k+1} \le \frac{\sum_{i=1}^{k}\Lambda_i}{k}+\frac{1}{l^2}\left[(4+2l)\frac{\sum_{i=1}^{k}\Lambda_i^{\frac{1}{2}}}{k}+4C_0\frac{\sum_{i=1}^{k}\Lambda_i^{\frac{1}{4}}}{k}+C_0^2\right] \times\left(4\frac{\sum_{i=1}^{k}\Lambda_i^{\frac{1}{2}}}{k}+4C_0\frac{\sum_{i=1}^{k}\Lambda_i^{\frac{1}{4}}}{k}+C_0^2\right).$$

In particular,

$$\Lambda_{k+1} \leq \frac{1}{l^2}\left[(4+l)^2\Lambda_k + 8(4+l)C_0\Lambda_k^{\frac{3}{4}} + 2(12+l)C_0^2\Lambda_k^{\frac{1}{2}} + 8C_0^3\Lambda_k^{\frac{1}{4}} + C_0^4\right].$$

(v) *If* Ω *admits an eigenmap* $f = (f_1, f_2, \ldots, f_{m+1}) : \Omega \to \mathbb{S}^{\mathrm{m}}$ *corresponding to an eigenvalue* τ, *that is,*

$$\Delta f_\alpha = -\tau f_\alpha, \quad \alpha = 1, \ldots, m+1, \quad \sum_{\alpha=1}^{m+1} f_\alpha^2 = 1,$$

where $\mathbb{S}^m$ *is the unit sphere of dimension m, then*

$$\sum_{i=1}^{k}(\Lambda_{k+1} - \Lambda_i) \leq \frac{1}{k}\left[\sum_{i=1}^{k}\left(6\Lambda_i^{\frac{1}{2}} + 4C_0\Lambda_i^{\frac{1}{4}} + \tau + C_0^2\right)\right] \times \left[\sum_{i=1}^{k}\left(4\Lambda_i^{\frac{1}{2}} + 4C_0\Lambda_i^{\frac{1}{4}} + \tau + C_0^2\right)\right].$$

Moreover,

$$\Lambda_{k+1} \leq \frac{\sum_{i=1}^k \Lambda_i}{k} + \left(6\frac{\sum_{i=1}^k \Lambda_i^{\frac{1}{2}}}{k} + 4C_0\frac{\sum_{i=1}^k \Lambda_i^{\frac{1}{4}}}{k} + \tau + C_0^2\right) \times \left(4\frac{\sum_{i=1}^k \Lambda_i^{\frac{1}{2}}}{k} + 4C_0\frac{\sum_{i=1}^k \Lambda_i^{\frac{1}{4}}}{k} + \tau + C_0^2\right).$$

In particular,

$$\Lambda_{k+1} \leq 25\Lambda_k + 40C_0\Lambda_k^{\frac{3}{4}} + 2(5\tau + 13C_0^2)\Lambda_k^{\frac{1}{2}} + 8C_0(\tau + C_0^2)\Lambda_k^{\frac{1}{4}} + (\tau + C_0^2)^2.$$

Remark 1.3 (i) For a complete minimal submanifold M in a Euclidean space, we can infer $H_0 = 0$. As a special case of Theorem 1.1 (i) and Corollary 1.1 (i), we can get the corresponding results.

(ii) For an n-dimensional unit sphere $\mathbb{S}^n$, which can be considered as a hypersurface in $\mathbb{R}^{n+1}$ with $|\mathbf{H}| = 1$, and so we can infer $H_0 = 1$. Again as a special case of Theorem 1.1 (i) and Corollary 1.1 (i), we can get the corresponding results.

The plan of the paper is the following: In Sect. 2, we will establish a key lemma (see Lemma 2.1 below), which is needed to prove Theorem 1.1. With the aid of Lemma 2.1, we will prove Theorem 1.1 in Sect. 3. Finally in Sect. 4, we will use Theorem 1.1 and the reverse Chebyshev inequality to prove Corollary 1.1. For readers' convenience, we will collect three inequalities (used in this paper) in the appendix.

2 Preliminaries

Throughout this paper, we use the notations from Wang and Xia (2011) extensively. In order to achieve our goal, we need establish a key lemma, which plays an important role in the proof of Theorem 1.1. In this process, we modify the standard arguments as in Du et al. (2015), Wang and Xia (2007b), Wang and Xia (2011). For readers' convenience, we present a very detailed proof here.

Lemma 2.1 *Let Λ_i, $i = 1, 2, \ldots$, be the i-th eigenvalue of problem (1.1) and u_i be the orthonormal eigenfunction corresponding to Λ_i, that is,*

$$\begin{cases} \mathbb{L}_\theta^2 u_i = \Lambda_i u_i, & in\ \Omega, \\ u_i = \frac{\partial u_i}{\partial \mathbf{n}} = 0, & on\ \partial\Omega, \\ \int_\Omega u_i u_j = \delta_{ij}, & \forall\ i, j = 1, 2, \ldots. \end{cases} \tag{2.1}$$

Then for any function $h \in C^4(\Omega) \cap C^3(\partial\Omega)$ and any positive integer k, we have

$$\begin{aligned} &\sum_{i=1}^{k}(\Lambda_{k+1} - \Lambda_i) \int_\Omega u_i^2 |\nabla h|^2 d\mu \\ &\leq \delta \sum_{i=1}^{k}(\Lambda_{k+1} - \Lambda_i) \int_\Omega \Big[u_i^2 (\mathbb{L}_\theta h)^2 + 4\big(\langle \nabla h, \nabla u_i \rangle^2 + u_i \mathbb{L}_\theta h \langle \nabla h, \nabla u_i \rangle\big) \\ &\quad -2|\nabla h|^2 u_i \mathbb{L}_\theta u_i \Big] d\mu + \frac{1}{4\delta} \sum_{i=1}^{k} \int_\Omega \Big[u_i^2 (\mathbb{L}_\theta h)^2 + 4\big(\langle \nabla h, \nabla u_i \rangle^2 \\ &\quad + u_i \mathbb{L}_\theta h \langle \nabla h, \nabla u_i \rangle\big)\Big] d\mu \end{aligned} \tag{2.2}$$

where δ is any positive constant and $\langle \cdot, \cdot \rangle$ stands for the inner product of two vector fields.

Proof For $i = 1, \ldots, k$, consider the functions $\varphi_i : \Omega \to \mathbb{R}$ given by

$$\varphi_i = h u_i - \sum_{j=1}^{k} r_{ij} u_j,$$

where

$$r_{ij} = \int_\Omega h u_i u_j d\mu.$$

It is easy to verify that

$$\varphi_i|_{\partial\Omega} = \frac{\partial \varphi_i}{\partial \mathbf{n}}\bigg|_{\partial\Omega} = 0 \quad \text{and} \quad \int_\Omega u_j \varphi_i = 0, \quad \forall\ i, j = 1, \ldots, k,$$

According to the Rayleigh–Ritz inequality, we have

$$\Lambda_{k+1} \int_\Omega \varphi_i^2 d\mu \le \int_\Omega \varphi_i \mathbb{L}_\theta^2 \varphi_i d\mu. \tag{2.3}$$

By direct computation, we get

$$\mathbb{L}_\theta (hu_i) = h\mathbb{L}_\theta u_i + 2\langle \nabla h, \nabla u_i\rangle + u_i \mathbb{L}_\theta h,$$

and

$$\begin{aligned}\mathbb{L}_\theta^2 (hu_i) &= \mathbb{L}_\theta \big(h\mathbb{L}_\theta u_i + 2\langle \nabla h, \nabla u_i\rangle + u_i \mathbb{L}_\theta h\big)\\ &= h\mathbb{L}_\theta^2 u_i + 2\big\langle \nabla h, \nabla (\mathbb{L}_\theta u_i)\big\rangle + \mathbb{L}_\theta h \mathbb{L}_\theta u_i + 2\mathbb{L}_\theta (\langle \nabla h, \nabla u_i\rangle) + \mathbb{L}_\theta (u_i \mathbb{L}_\theta h)\\ &= \Lambda_i h u_i + p_i,\end{aligned}$$

where

$$p_i := 2\big\langle \nabla h, \nabla (\mathbb{L}_\theta u_i)\big\rangle + \mathbb{L}_\theta h \mathbb{L}_\theta u_i + 2\mathbb{L}_\theta (\langle \nabla h, \nabla u_i\rangle) + \mathbb{L}_\theta (u_i \mathbb{L}_\theta h).$$

This leads to

$$\begin{aligned}\int_\Omega \varphi_i \mathbb{L}_\theta^2 \varphi_i d\mu &= \int_\Omega \varphi_i \left[\mathbb{L}_\theta^2 (hu_i) - \sum_{j=1}^k r_{ij}\Lambda_j u_j\right] d\mu = \int_\Omega \varphi_i \mathbb{L}_\theta^2 (hu_i) d\mu\\ &= \int_\Omega \varphi_i \left[p_i + \Lambda_i h u_i\right] d\mu = \int_\Omega \varphi_i \left[p_i + \Lambda_i \left(hu_i - \sum_{j=1}^k r_{ij} u_j\right)\right] d\mu\\ &= \int_\Omega \varphi_i p_i d\mu + \Lambda_i \|\varphi_i\|^2 = \int_\Omega h u_i p_i d\mu - \sum_{j=1}^k r_{ij} s_{ij} + \Lambda_i \|\varphi_i\|^2,\end{aligned} \tag{2.4}$$

where

$$\|\varphi_i\|^2 = \int_\Omega \varphi_i^2 d\mu$$

and

$$\begin{aligned}s_{ij} := \int_\Omega u_j p_i d\mu = \int_\Omega u_j \big[&\mathbb{L}_\theta (u_i \mathbb{L}_\theta h) + 2\mathbb{L}_\theta \big(\langle \nabla h, \nabla u_i\rangle\big)\\ &+ 2\big\langle \nabla h, \nabla (\mathbb{L}_\theta u_i)\big\rangle + \mathbb{L}_\theta h \mathbb{L}_\theta u_i\big] d\mu.\end{aligned}$$

Multiplying both sides of $\mathbb{L}_\theta^2 u_i = \Lambda_i u_i$ by hu_j we get

$$hu_j \mathbb{L}_\theta^2 u_i = \Lambda_i h u_i u_j. \tag{2.5}$$

Similarly, we can obtain

$$hu_i\mathbb{L}_\theta^2 u_j = \Lambda_j h u_i u_j. \tag{2.6}$$

Subtracting (2.5) from (2.6), integrating the resulted equality on Ω, and using Stokes' formula successively, we get

$$\begin{aligned}
(\Lambda_j - \Lambda_i)r_{ij} &= \int_\Omega (hu_i\mathbb{L}_\theta^2 u_j - hu_j\mathbb{L}_\theta^2 u_i)d\mu = \int_\Omega \left[\mathbb{L}_\theta(hu_i)\mathbb{L}_\theta u_j - \mathbb{L}_\theta(hu_j)\mathbb{L}_\theta u_i\right]d\mu \\
&= \int_\Omega \left\{\left[u_i\mathbb{L}_\theta h + 2\langle\nabla h, \nabla u_i\rangle\right]\mathbb{L}_\theta u_j - \left[u_j\mathbb{L}_\theta h + 2\langle\nabla h, \nabla u_j\rangle\right]\mathbb{L}_\theta u_i\right\} d\mu \\
&= \int_\Omega \Big\{u_j\left[\mathbb{L}_\theta(u_i\mathbb{L}_\theta h) + 2\mathbb{L}_\theta\left(\langle\nabla h, \nabla u_i\rangle\right)\right] - u_j\mathbb{L}_\theta h\mathbb{L}_\theta u_i \\
&\quad +2u_j e^\theta \operatorname{div}\left(e^{-\theta}\mathbb{L}_\theta u_i\nabla h\right)\Big\}d\mu \\
&= \int_\Omega u_j\Big[\mathbb{L}_\theta(u_i\mathbb{L}_\theta h) + 2\mathbb{L}_\theta\left(\langle\nabla h, \nabla u_i\rangle\right) + \mathbb{L}_\theta h\mathbb{L}_\theta u_i + 2\langle\nabla\mathbb{L}_\theta u_i, \nabla h\rangle\Big]d\mu \\
&= s_{ij},
\end{aligned} \tag{2.7}$$

where div is the divergence operator acting on vector fields on Ω. Observe that

$$\begin{aligned}
\int_\Omega hu_i p_i d\mu &= \int_\Omega hu_i \left[\mathbb{L}_\theta\left(u_i\mathbb{L}_\theta h\right) + 2\mathbb{L}_\theta\left(\langle\nabla h, \nabla u_i\rangle\right)\right. \\
&\quad \left.+ 2\left\langle\nabla h, \nabla(\mathbb{L}_\theta u_i)\right\rangle + \mathbb{L}_\theta h\mathbb{L}_\theta u_i\right] d\mu \\
&= \int_\Omega \Big\{\mathbb{L}_\theta(hu_i)u_i\mathbb{L}_\theta h + 2\mathbb{L}_\theta(hu_i)\langle\nabla h, \nabla u_i\rangle \\
&\quad -2\mathbb{L}_\theta u_i\left[e^\theta \operatorname{div}(e^{-\theta}hu_i\nabla h)\right] + hu_i\mathbb{L}_\theta h\mathbb{L}_\theta u_i\Big\}d\mu.
\end{aligned} \tag{2.8}$$

Direct calculation leads to the following equalities:

$$\begin{aligned}
\int_\Omega \mathbb{L}_\theta(hu_i)u_i\mathbb{L}_\theta h d\mu &= \int_\Omega \left(u_i\mathbb{L}_\theta h + 2\langle\nabla h, \nabla u_i\rangle + h\mathbb{L}_\theta u_i\right)u_i\mathbb{L}_\theta h d\mu \\
&= \int_\Omega \left[u_i^2(\mathbb{L}_\theta h)^2 + 2u_i\mathbb{L}_\theta h\langle\nabla h, \nabla u_i\rangle + hu_i\mathbb{L}_\theta u_i\mathbb{L}_\theta h\right] d\mu,
\end{aligned} \tag{2.9}$$

$$\begin{aligned}
\int_\Omega \mathbb{L}_\theta(hu_i)\langle\nabla h, \nabla u_i\rangle d\mu &= \int_\Omega \left(u_i\mathbb{L}_\theta h + 2\langle\nabla h, \nabla u_i\rangle + h\mathbb{L}_\theta u_i\right)\langle\nabla h, \nabla u_i\rangle d\mu \\
&= \int_\Omega \Big(u_i\mathbb{L}_\theta h\langle\nabla h, \nabla u_i\rangle + 2\langle\nabla h, \nabla u_i\rangle^2 \\
&\quad + \langle\nabla h, \nabla u_i\rangle h\mathbb{L}_\theta u_i\Big) d\mu,
\end{aligned} \tag{2.10}$$

and

$$
\begin{aligned}
&\int_\Omega \mathbb{L}_\theta u_i\big[e^\theta \operatorname{div}(e^{-\theta} h u_i \nabla h)\big]d\mu \\
&= \int_\Omega \mathbb{L}_\theta u_i \left[\big\langle\nabla(hu_i), \nabla h\big\rangle + hu_i\mathbb{L}_\theta h\right] d\mu \\
&= \int_\Omega \mathbb{L}_\theta u_i \left(|\nabla h|^2 u_i + h\langle\nabla u_i, \nabla h\rangle + hu_i\mathbb{L}_\theta h\right) d\mu \\
&= \int_\Omega \left(|\nabla h|^2 u_i\mathbb{L}_\theta u_i + h\mathbb{L}_\theta u_i\langle\nabla u_i, \nabla h\rangle + hu_i\mathbb{L}_\theta u_i\mathbb{L}_\theta h\right) d\mu. \qquad (2.11)
\end{aligned}
$$

Thus we can get by combining (2.8)–(2.11) that

$$
\begin{aligned}
&\int_\Omega hu_i \left[\mathbb{L}_\theta(u_i\mathbb{L}_\theta h) + 2\mathbb{L}_\theta\langle\nabla h, \nabla u_i\rangle + 2\big\langle\nabla h, \nabla(\mathbb{L}_\theta u_i)\big\rangle + \mathbb{L}_\theta h\mathbb{L}_\theta u_i\right] d\mu \\
&= \int_\Omega \left[u_i^2(\mathbb{L}_\theta h)^2 + 4\big(\langle\nabla h, \nabla u_i\rangle^2 + u_i\mathbb{L}_\theta h\langle\nabla h, \nabla u_i\rangle\big) - 2|\nabla h|^2 u_i\mathbb{L}_\theta u_i\right] d\mu \\
&\qquad (2.12)
\end{aligned}
$$

So, it follows from (2.3), (2.4), (2.7) and (2.12) that

$$
\begin{aligned}
(\Lambda_{k+1} - \Lambda_i)\,\|\varphi_i\|^2 \le \int_\Omega &\Big[u_i^2(\mathbb{L}_\theta h)^2 + 4\big(\langle\nabla h, \nabla u_i\rangle^2 + u_i\mathbb{L}_\theta h\langle\nabla h, \nabla u_i\rangle\big) \\
&- 2|\nabla h|^2 u_i\mathbb{L}_\theta u_i\Big] d\mu - \sum_{j=1}^k r_{ij}s_{ij}. \qquad (2.13)
\end{aligned}
$$

In order to prove (2.2), let us set

$$
t_{ij} := \int_\Omega u_j \left(\langle\nabla h, \nabla u_i\rangle + \frac{u_i\mathbb{L}_\theta h}{2}\right) d\mu. \qquad (2.14)
$$

Then we get by Stokes' formula that

$$
\begin{aligned}
t_{ij} + t_{ji} &= \int_\Omega u_j \left(\langle\nabla h, \nabla u_i\rangle + \frac{u_i\mathbb{L}_\theta h}{2}\right) d\mu + \int_\Omega u_i \left(\langle\nabla h, \nabla u_j\rangle + \frac{u_j\mathbb{L}_\theta h}{2}\right) d\mu \\
&= \int_\Omega \left[\langle\nabla h, u_j\nabla u_i + u_i\nabla u_j\rangle + u_iu_j\mathbb{L}_\theta h\right] d\mu \\
&= \int_\Omega \left[\big\langle\nabla h, \nabla(u_iu_j)\big\rangle + u_iu_j\mathbb{L}_\theta h\right] d\mu \\
&= \int_\Omega (-u_iu_j\mathbb{L}_\theta h + u_iu_j\mathbb{L}_\theta h) d\mu = 0 \qquad (2.15)
\end{aligned}
$$

and

$$
\begin{aligned}
&\int_\Omega (-2)\varphi_i \left(\langle \nabla h, \nabla u_i \rangle + \frac{u_i \mathbb{L}_\theta h}{2} \right) d\mu \\
&= \int_\Omega \big[-2hu_i \langle \nabla h, \nabla u_i \rangle - u_i^2 h \mathbb{L}_\theta h \big] d\mu + 2\sum_{j=1}^{k} r_{ij} t_{ij} \\
&= \int_\Omega \left[-\frac{1}{2} \big\langle \nabla (h^2), \nabla (u_i^2) \big\rangle - u_i^2 h \mathbb{L}_\theta h \right] d\mu + 2\sum_{j=1}^{k} r_{ij} t_{ij} \\
&= \int_\Omega \left[\frac{1}{2} u_i^2 \mathbb{L}_\theta (h^2) - u_i^2 h \mathbb{L}_\theta h \right] d\mu + 2\sum_{j=1}^{k} r_{ij} t_{ij} \\
&= \int_\Omega \big[u_i^2 (h \mathbb{L}_\theta h + |\nabla h|^2) - u_i^2 h \mathbb{L}_\theta h \big] d\mu + 2\sum_{j=1}^{k} r_{ij} t_{ij} \\
&= \int_\Omega u_i^2 |\nabla h|^2 d\mu + 2\sum_{j=1}^{k} r_{ij} t_{ij}. \qquad (2.16)
\end{aligned}
$$

Multiplying (2.16) by $(\Lambda_{k+1} - \Lambda_i)$, and using the Cauchy–Schwarz inequality and (2.13), we have

$$
\begin{aligned}
&(\Lambda_{k+1} - \Lambda_i) \left(\int_\Omega u_i^2 |\nabla h|^2 d\mu + 2\sum_{j=1}^{k} r_{ij} t_{ij} \right) \\
&= (\Lambda_{k+1} - \Lambda_i) \int_\Omega (-2)\varphi_i \left(\langle \nabla h, \nabla_i \rangle + \frac{u_i \mathbb{L}_\theta h}{2} \right) d\mu \\
&= (\Lambda_{k+1} - \Lambda_i) \int_\Omega (-2)\varphi_i \left(\langle \nabla h, \nabla u_i \rangle + \frac{u_i \mathbb{L}_\theta h}{2} - \sum_{j=1}^{k} t_{ij} u_j \right) d\mu \\
&\leq \delta (\Lambda_{k+1} - \Lambda_i)^2 \|\varphi_i\|^2 + \frac{1}{\delta} \int_\Omega \left(\langle \nabla h, \nabla u_i \rangle + \frac{u_i \mathbb{L}_\theta h}{2} - \sum_{j=1}^{k} t_{ij} u_j \right)^2 d\mu \\
&= \delta (\Lambda_{k+1} - \Lambda_i)^2 \|\varphi_i\|^2 + \frac{1}{\delta} \left[\int_\Omega \left(\langle \nabla h, \nabla u_i \rangle + \frac{u_i \mathbb{L}_\theta h}{2} \right)^2 d\mu - \sum_{j=1}^{k} t_{ij}^2 \right] \\
&\leq \delta (\Lambda_{k+1} - \Lambda_i) \Bigg\{ \int_\Omega \big[u_i^2 (\mathbb{L}_\theta h)^2 + 4\big(\langle \nabla h, \nabla u_i \rangle^2 + u_i \mathbb{L}_\theta h \langle \nabla h, \nabla u_i \rangle \big) \\
&\quad - 2|\nabla h|^2 u_i \mathbb{L}_\theta u_i \big] d\mu - \sum_{j=1}^{k} r_{ij} s_{ij} \Bigg\} + \frac{1}{\delta} \left[\int_\Omega \left(\langle \nabla h, \nabla u_i \rangle + \frac{u_i \mathbb{L}_\theta h}{2} \right)^2 d\mu - \sum_{j=1}^{k} t_{ij}^2 \right]. \qquad (2.17)
\end{aligned}
$$

Summing over i from 1 to k for (2.17), we get

$$
\begin{aligned}
&\sum_{i=1}^{k}(\Lambda_{k+1}-\Lambda_i)\left(\int_\Omega u_i^2|\nabla h|^2+2\sum_{j=1}^{k}r_{ij}t_{ij}\right)d\mu\\
&\le \delta\sum_{i=1}^{k}(\Lambda_{k+1}-\Lambda_i)\Big\{\int_\Omega\Big[u_i^2(\mathbb{L}_\theta h)^2+4\big(\langle\nabla h,\nabla u_i\rangle^2+u_i\mathbb{L}_\theta h\langle\nabla h,\nabla u_i\rangle\big)\\
&\quad -2|\nabla h|^2u_i\mathbb{L}_\theta u_i\Big]d\mu+\sum_{j=1}^{k}(\Lambda_i-\Lambda_j)r_{ij}^2\Big\}\\
&\quad +\frac{1}{\delta}\sum_{i=1}^{k}\left[\int_\Omega\left(\langle\nabla h,\nabla u_i\rangle+\frac{u_i\mathbb{L}_\theta h}{2}\right)^2d\mu-\sum_{j=1}^{k}t_{ij}^2\right]\\
&\le \delta\sum_{i=1}^{k}(\Lambda_{k+1}-\Lambda_i)\int_\Omega\Big[u_i^2(\mathbb{L}_\theta h)^2+4\big(\langle\nabla h,\nabla u_i\rangle^2+u_i\mathbb{L}_\theta h\langle\nabla h,\nabla u_i\rangle\big)\\
&\quad -2|\nabla h|^2u_i\mathbb{L}_\theta u_i\Big]d\mu-\delta\sum_{i,j=1}^{k}(\Lambda_{k+1}-\Lambda_i)r_{ij}s_{ij}\\
&\quad +\frac{1}{\delta}\sum_{i=1}^{k}\int_\Omega\left(\langle\nabla h,\nabla u_i\rangle+\frac{u_i\mathbb{L}_\theta h}{2}\right)^2d\mu-\frac{1}{\delta}\sum_{i,j=1}^{k}t_{ij}^2. \qquad (2.18)
\end{aligned}
$$

Clearly, the left-hand side of (2.18) can be rewritten as follows.

$$
\begin{aligned}
&\sum_{i=1}^{k}(\Lambda_{k+1}-\Lambda_i)\left(\int_\Omega u_i^2|\nabla h|^2d\mu+2\sum_{j=1}^{k}r_{ij}t_{ij}\right)\\
&\quad=\sum_{i=1}^{k}(\Lambda_{k+1}-\Lambda_i)\int_\Omega u_i^2|\nabla h|^2d\mu+2\sum_{i,j=1}^{k}(\Lambda_{k+1}-\Lambda_i)r_{ij}t_{ij}. \qquad (2.19)
\end{aligned}
$$

Let us set

$$
I:=\sum_{i,j=1}^{k}(\Lambda_{k+1}-\Lambda_i)r_{ij}t_{ij},\qquad J:=\sum_{i,j=1}^{k}(\Lambda_{k+1}-\Lambda_i)r_{ij}s_{ij}.
$$

By exchanging the summation order of i and j in the definition of I, and noticing the following equalities:

$$
r_{ij}=r_{ji},\qquad t_{ij}=-t_{ji},\qquad (\Lambda_j-\Lambda_i)r_{ij}=s_{ij},
$$

we can carry out the following calculations:

$$\begin{aligned} I &= \sum_{i,j=1}^{k} (\Lambda_{k+1} - \Lambda_j) r_{ij} t_{ij} + \sum_{i,j=1}^{k} (\Lambda_j - \Lambda_i) r_{ij} t_{ij} \\ &= \sum_{j,i=1}^{k} (\Lambda_{k+1} - \Lambda_i) r_{ji} t_{ji} + \sum_{i,j=1}^{k} s_{ij} t_{ij} \\ &= -\sum_{j,i=1}^{k} (\Lambda_{k+1} - \Lambda_i) r_{ij} t_{ij} + \sum_{i,j=1}^{k} s_{ij} t_{ij} \\ &= -I + \sum_{i,j=1}^{k} s_{ij} t_{ij}, \end{aligned}$$

which implies

$$I = \frac{1}{2} \sum_{i,j=1}^{k} s_{ij} t_{ij}. \tag{2.20}$$

Similarly, we also get

$$\begin{aligned} J &= \sum_{i,j=1}^{k} \left[(\Lambda_{k+1} - \Lambda_j) + (\Lambda_j - \Lambda_i) \right] r_{ij} s_{ij} \\ &= \sum_{i,j=1}^{k} (\Lambda_{k+1} - \Lambda_j) r_{ij} s_{ij} + \sum_{i,j=1}^{k} s_{ij}^2 \\ &= \sum_{j,i=1}^{k} (\Lambda_{k+1} - \Lambda_i) r_{ji} s_{ji} + \sum_{i,j=1}^{k} s_{ij}^2 \\ &= -\sum_{i,j=1}^{k} (\Lambda_{k+1} - \Lambda_i) r_{ij} s_{ij} + \sum_{i,j=1}^{k} s_{ij}^2 \\ &= -J + \sum_{i,j=1}^{k} s_{ij}^2, \end{aligned}$$

which implies

$$J = \frac{1}{2} \sum_{i,j=1}^{k} s_{ij}^2. \tag{2.21}$$

Hence, it follows from (2.18)–(2.21) that

$$\begin{aligned}&\sum_{i=1}^{k}(\Lambda_{k+1}-\Lambda_i)\int_\Omega u_i^2|\nabla h|^2 d\mu+\sum_{i,j=1}^{k}s_{ij}t_{ij}\\&\le \delta\sum_{i=1}^{k}(\Lambda_{k+1}-\Lambda_i)\int_\Omega\Big[u_i^2(\mathbb{L}_\theta h)^2+4\big(\langle\nabla h,\nabla u_i\rangle^2+u_i\mathbb{L}_\theta h\langle\nabla h,\nabla u_i\rangle\big)\\&\quad-2|\nabla h|^2u_i\mathbb{L}_\theta u_i\Big]d\mu-\frac{\delta}{2}\sum_{i,j=1}^{k}s_{ij}^2\\&\quad+\frac{1}{\delta}\sum_{i=1}^{k}\int_\Omega\left(\langle\nabla h,\nabla u_i\rangle+\frac{u_i\mathbb{L}_\theta h}{2}\right)^2d\mu-\frac{1}{\delta}\sum_{i,j=1}^{k}t_{ij}^2,\end{aligned}$$

or equivalently,

$$\begin{aligned}&\sum_{i=1}^{k}(\Lambda_{k+1}-\Lambda_i)\int_\Omega u_i^2|\nabla h|^2 d\mu+\left(\frac{\delta}{2}\sum_{i,j=1}^{k}s_{ij}^2+\sum_{i,j=1}^{k}s_{ij}t_{ij}+\frac{1}{\delta}\sum_{i,j=1}^{k}t_{ij}^2\right)\\&\le \delta\sum_{i=1}^{k}(\Lambda_{k+1}-\Lambda_i)\int_\Omega\Big[u_i^2(\mathbb{L}_\theta h)^2+4\big(\langle\nabla h,\nabla u_i\rangle^2+u_i\mathbb{L}_\theta h\langle\nabla h,\nabla u_i\rangle\big)\\&\quad-2|\nabla h|^2u_i\mathbb{L}_\theta u_i\Big]d\mu+\frac{1}{\delta}\sum_{i=1}^{k}\int_\Omega\left(\langle\nabla h,\nabla u_i\rangle+\frac{u_i\mathbb{L}_\theta h}{2}\right)^2d\mu.\end{aligned}\tag{2.22}$$

Observe that

$$\frac{\delta}{2}\sum_{i,j=1}^{k}s_{ij}^2+\sum_{i,j=1}^{k}s_{ij}t_{ij}+\frac{1}{\delta}\sum_{i,j=1}^{k}t_{ij}^2=\frac{1}{2\delta}\sum_{i,j=1}^{k}\left[(\delta s_{ij}+t_{ij})^2+t_{ij}^2\right]\ge 0\tag{2.23}$$

and

$$\begin{aligned}&\int_\Omega\left(\langle\nabla h,\nabla u_i\rangle+\frac{u_i\mathbb{L}_\theta h}{2}\right)^2d\mu\\&=\frac{1}{4}\int_\Omega\Big[u_i^2(\mathbb{L}_\theta h)^2+4\big(\langle\nabla h,\nabla u_i\rangle^2+u_i\mathbb{L}_\theta h\langle\nabla h,\nabla u_i\rangle\big)\Big]d\mu.\end{aligned}\tag{2.24}$$

Clearly, inequality (2.2) can be easily derived from (2.22)–(2.24). The proof is complete. □

Remark 2.1 In order to derive the conclusion of Lemma 2.1, we introduced a factor $(\Lambda_{k+1}-\Lambda_i)$ in (2.17). In this process, it can be seen that the unwanted terms on both sides of the inequality are perfectly eliminated. In other literature (e.g. Wang and Xia 2007a, b, 2008, 2011, etc.), in order to eliminate unwanted terms, the authors also introduce a factor $(\Lambda_{k+1}-\Lambda_i)^2$. It is pointed out here that our reasoning of the universal inequality for the eigenvalues of a clamped plate problem is roughly similar to that in the previous literature, with the main modification being that the factors multiplied are different.

3 Proof of Theorem 1.1

With all the preparation done, we now prove Theorem 1.1 as follows.

Proof of Theorem 1.1 Let $\{u_i\}_{i=1}^{\infty}$ be the orthonormal eigenfunctions corresponding to the eigenvalues $\{\Lambda_i\}_{i=1}^{\infty}$ of problem (1.1).

(i) Let $x_\alpha,\ \alpha = 1,\dots,m$, be the standard coordinate functions of $\mathbb{R}^m$. Taking $h = x_\alpha$ in (2.2) and summing over α from 1 to m successively, we arrive at

$$\begin{aligned}&\sum_{i=1}^{k}(\Lambda_{k+1}-\Lambda_i)\sum_{\alpha=1}^{m}\int_\Omega u_i^2|\nabla x_\alpha|^2 d\mu\\&\quad\leq \delta\sum_{i=1}^{k}(\Lambda_{k+1}-\Lambda_i)\sum_{\alpha=1}^{m}\int_\Omega\Big[u_i^2(\mathbb{L}_\theta x_\alpha)^2+4\Big(\langle\nabla x_\alpha,\nabla u_i\rangle^2+u_i\mathbb{L}_\theta x_\alpha\langle\nabla x_\alpha,\nabla u_i\rangle\Big)\\&\qquad-2|\nabla x_\alpha|^2u_i\mathbb{L}_\theta u_i\Big]d\mu+\frac{1}{4\delta}\sum_{i=1}^{k}\sum_{\alpha=1}^{m}\int_\Omega\Big[u_i^2(\mathbb{L}_\theta x_\alpha)^2+4\Big(\langle\nabla x_\alpha,\nabla u_i\rangle^2\\&\qquad+u_i\mathbb{L}_\theta x_\alpha\langle\nabla x_\alpha,\nabla u_i\rangle)\Big]d\mu\end{aligned}\tag{3.1}$$

Since M is isometrically immersed in $\mathbb{R}^m$, it is easy to see that

$$\sum_{\alpha=1}^{m}|\nabla x_\alpha|^2=n,\tag{3.2}$$

$$\Delta(x_1,\dots,x_m)=(\Delta x_1,\dots,\Delta x_m)=n\mathbf{H},\tag{3.3}$$

$$\sum_{\alpha=1}^{m}\langle\nabla x_\alpha,\nabla u_i\rangle^2=\sum_{\alpha=1}^{m}\big|\nabla u_i(x_\alpha)\big|^2=|\nabla u_i|^2\tag{3.4}$$

and

$$\sum_{\alpha=1}^{m}\Delta x_\alpha\langle\nabla x_\alpha,\nabla u_i\rangle=\sum_{\alpha=1}^{m}\Delta x_\alpha\nabla u_i(x_\alpha)=\langle n\mathbf{H},\nabla u_i\rangle=0.\tag{3.5}$$

From (3.2)–(3.5), we can easily obtain the following equalities:

$$\sum_{\alpha=1}^{m} \mathbb{L}_\theta x_\alpha \langle \nabla x_\alpha, \nabla u_i \rangle = \sum_{\alpha=1}^{m} (\Delta x_\alpha - \langle \nabla x_\alpha, \nabla\theta \rangle) \langle \nabla x_\alpha, \nabla u_i \rangle = - \langle \nabla\theta, \nabla u_i \rangle , \tag{3.6}$$

$$\begin{aligned}\sum_{\alpha=1}^{m} (\mathbb{L}_\theta x_\alpha)^2 &= \sum_{\alpha=1}^{m} (\Delta x_\alpha - \langle \nabla x_\alpha, \nabla\theta \rangle)^2 \\ &= \sum_{\alpha=1}^{m} \left[(\Delta x_\alpha)^2 - 2\Delta x_\alpha \langle \nabla x_\alpha, \nabla\theta \rangle + \langle \nabla x_\alpha, \nabla\theta \rangle^2 \right] \\ &= n^2 |\mathbf{H}|^2 + |\nabla\theta|^2 \end{aligned} \tag{3.7}$$

and

$$\begin{aligned}&\sum_{\alpha=1}^{m} \left[u_i^2 (\mathbb{L}_\theta x_\alpha)^2 + 4 \left(\langle \nabla x_\alpha, \nabla u_i \rangle^2 + u_i \mathbb{L}_\theta x_\alpha \langle \nabla x_\alpha, \nabla u_i \rangle \right) \right] \\ &\quad = u_i^2 \left(n^2 |\mathbf{H}|^2 + |\nabla\theta|^2 \right) + 4 |\nabla u_i|^2 - 4u_i \langle \nabla\theta, \nabla u_i \rangle . \end{aligned} \tag{3.8}$$

Substituting (3.6)–(3.8) into (3.1), we deduce that

$$\begin{aligned}&n \sum_{i=1}^{k} (\Lambda_{k+1} - \Lambda_i) \\ &\quad \le \delta \sum_{i=1}^{k} (\Lambda_{k+1} - \Lambda_i) \int_\Omega \left[u_i^2 \left(n^2 |\mathbf{H}|^2 + |\nabla\theta|^2 \right) + 4 |\nabla u_i|^2 - 4u_i \langle \nabla\theta, \nabla u_i \rangle \right. \\ &\qquad \left. -2nu_i \mathbb{L}_\theta u_i \right] d\mu + \frac{1}{4\delta} \sum_{i=1}^{k} \int_\Omega \left[u_i^2 \left(n^2 |\mathbf{H}|^2 + |\nabla\theta|^2 \right) \right. \\ &\qquad \left. +4 |\nabla u_i|^2 - 4u_i \langle \nabla\theta, \nabla u_i \rangle \right] d\mu. \end{aligned} \tag{3.9}$$

Since $|\nabla\theta| \le C_0$, we easily infer

$$\int_\Omega |\nabla u_i|^2 \, d\mu = - \int_\Omega u_i \mathbb{L}_\theta u_i d\mu \le \left\{ \int_M u_i^2 d\mu \right\}^{\frac{1}{2}} \left\{ \int_M (\mathbb{L}_\theta u_i)^2 \, d\mu \right\}^{\frac{1}{2}} = \Lambda_i^{\frac{1}{2}} \tag{3.10}$$

and

$$\begin{aligned}- \int_\Omega u_i \langle \nabla\theta, \nabla u_i \rangle \, d\mu &\le \int_\Omega |\nabla\theta| |u_i| |\nabla u_i| d\mu \le C_0 \left\{ \int_\Omega u_i^2 d\mu \right\}^{\frac{1}{2}} \left\{ \int_\Omega |\nabla u_i|^2 \, d\mu \right\}^{\frac{1}{2}} \\ &\le C_0 \Lambda_i^{\frac{1}{4}} . \end{aligned} \tag{3.11}$$

Substituting (3.10) and (3.11) into (3.9), and also using the facts that $|\mathbf{H}|^2 \le H_0^2$ and $|\nabla\theta| \le C_0$, we get

$$n\sum_{i=1}^{k}(\Lambda_{k+1}-\Lambda_i) \le \delta\sum_{i=1}^{k}(\Lambda_{k+1}-\Lambda_i)\left[(2n+4)\Lambda_i^{\frac{1}{2}}+4C_0\Lambda_i^{\frac{1}{4}}+\left(n^2H_0^2+C_0^2\right)\right]$$
$$+\frac{1}{4\delta}\sum_{i=1}^{k}\left[4\Lambda_i^{\frac{1}{2}}+4C_0\Lambda_i^{\frac{1}{4}}+\left(n^2H_0^2+C_0^2\right)\right]. \tag{3.12}$$

Taking

$$\delta=\frac{1}{2}\left\{\frac{\sum_{i=1}^{k}\left[4\Lambda_i^{\frac{1}{2}}+4C_0\Lambda_i^{\frac{1}{4}}+\left(n^2H_0^2+C_0^2\right)\right]}{\sum_{i=1}^{k}(\Lambda_{k+1}-\Lambda_i)\left[(2n+4)\Lambda_i^{\frac{1}{2}}+4C_0\Lambda_i^{\frac{1}{4}}+\left(n^2H_0^2+C_0^2\right)\right]}\right\}^{\frac{1}{2}}$$

in (3.12), one can obtain (1.4).

(ii) Substituting $h=\phi$ into (2.2), we have

$$\sum_{i=1}^{k}(\Lambda_{k+1}-\Lambda_i)$$
$$\le \delta\sum_{i=1}^{k}(\Lambda_{k+1}-\Lambda_i)\int_\Omega\left[u_i^2(\mathbb{L}_\theta\phi)^2+4\left(\langle\nabla\phi,\nabla u_i\rangle^2+u_i\mathbb{L}_\theta\phi\langle\nabla\phi,\nabla u_i\rangle\right)\right.$$
$$\left.-2|\nabla\phi|^2u_i\mathbb{L}_\theta u_i\right]$$
$$+\frac{1}{4\delta}\sum_{i=1}^{k}\int_\Omega\left[u_i^2(\mathbb{L}_\theta\phi)^2+4\left(\langle\nabla\phi,\nabla u_i\rangle^2+u_i\mathbb{L}_\theta\phi\langle\nabla\phi,\nabla u_i\rangle\right)\right]^2. \tag{3.13}$$

By using (1.5) and the Cauchy–Schwarz inequality we obtain

$$|\mathbb{L}_\theta\phi|=|\Delta\phi-\langle\nabla\theta,\nabla\phi\rangle|\le|\Delta\phi|+|\nabla\theta|\cdot|\nabla\phi|\le A_0+C_0 \tag{3.14}$$

and

$$|\langle\nabla\phi,\nabla u_i\rangle|\le|\nabla\phi|\cdot|\nabla u_i|=|\nabla u_i|. \tag{3.15}$$

Combining (3.13), (3.14) and (3.15), and also using (1.5), we infer that

$$
\begin{aligned}
&\sum_{i=1}^{k}(\Lambda_{k+1}-\Lambda_i)\\
&\quad\le \delta\sum_{i=1}^{k}(\Lambda_{k+1}-\Lambda_i)\int_\Omega\Big\{(A_0+C_0)^2u_i^2+4\big[|\nabla u_i|^2+(A_0+C_0)|u_i|\cdot|\nabla u_i|\big]\\
&\qquad-2u_i\mathbb{L}_\theta u_i\Big\}\\
&\quad+\frac{1}{4\delta}\sum_{i=1}^{k}\int_\Omega\Big\{(A_0+C_0)^2u_i^2+4\big[|\nabla u_i|^2+(A_0+C_0)|u_i|\cdot|\nabla u_i|\big]\Big\}. \qquad (3.16)
\end{aligned}
$$

By using the Hölder inequality together with (3.10) we obtain

$$
\int_\Omega |u_i|\cdot|\nabla u_i| \le \left(\int_\Omega u_i^2\right)^{\frac{1}{2}}\left(\int_\Omega|\nabla u_i|^2\right)^{\frac{1}{2}} = \Lambda_i^{\frac{1}{4}}. \qquad (3.17)
$$

Combining (3.10), (3.16) and (3.17), we get

$$
\begin{aligned}
\sum_{i=1}^{k}(\Lambda_{k+1}-\Lambda_i) &\le \delta\sum_{i=1}^{k}(\Lambda_{k+1}-\Lambda_i)\left[6\Lambda_i^{\frac{1}{2}}+4\left(A_0+C_0\right)\Lambda_i^{\frac{1}{4}}+(A_0+C_0)^2\right]\\
&\quad+\frac{1}{4\delta}\sum_{i=1}^{k}\left(2\Lambda_i^{\frac{1}{4}}+A_0+C_0\right)^2. \qquad (3.18)
\end{aligned}
$$

Taking

$$
\delta := \frac{1}{2}\left\{\frac{\sum_{i=1}^{k}\left(2\Lambda_i^{\frac{1}{4}}+A_0+C_0\right)^2}{\sum_{i=1}^{k}(\Lambda_{k+1}-\Lambda_i)\left[6\Lambda_i^{\frac{1}{2}}+4\left(A_0+C_0\right)\Lambda_i^{\frac{1}{4}}+(A_0+C_0)^2\right]}\right\}^{\frac{1}{2}}
$$

in (3.18), we can get (1.6).

(iii) Substituting $h=\psi$ into (2.2), we have

$$
\begin{aligned}
&\sum_{i=1}^{k}(\Lambda_{k+1}-\Lambda_i)\int_\Omega u_i^2|\nabla\psi|^2d\mu\\
&\quad\le\delta\sum_{i=1}^{k}(\Lambda_{k+1}-\Lambda_i)\int_\Omega\Big[u_i^2(\mathbb{L}_\theta\psi)^2+4\left(\langle\nabla\psi,\nabla u_i\rangle^2+u_i\mathbb{L}_\theta\psi\langle\nabla\psi,\nabla u_i\rangle\right)\\
&\qquad-2|\nabla\psi|^2u_i\mathbb{L}_\theta u_i\Big]d\mu
\end{aligned}
$$

$$+\frac{1}{4\delta}\sum_{i=1}^{k}\int_\Omega\left[u_i^2(\mathbb{L}_\theta\psi)^2+4\left(\langle\nabla\psi,\nabla u_i\rangle^2+u_i\mathbb{L}_\theta\psi\langle\nabla\psi,\nabla u_i\rangle\right)\right]d\mu. \tag{3.19}$$

Using (1.7), it is not difficult to get the following equality:

$$\begin{aligned}\int_\Omega u_i^2(\mathbb{L}_\theta\psi)^2d\mu &= \int_\Omega u_i^2\left(B_0-\langle\nabla\theta,\nabla\psi\rangle\right)^2d\mu\\ &= \int_\Omega u_i^2\left(B_0^2-2B_0\langle\nabla\theta,\nabla\psi\rangle+\langle\nabla\theta,\nabla\psi\rangle^2\right)d\mu\\ &= B_0^2-2B_0\int_\Omega u_i^2\langle\nabla\theta,\nabla\psi\rangle+\int_\Omega u_i^2\langle\nabla\theta,\nabla\psi\rangle^2d\mu.\end{aligned} \tag{3.20}$$

Using Stokes' formula,(1.7) and (3.17), we get

$$\begin{aligned}\int_\Omega u_i\langle\nabla\psi,\nabla u_i\rangle d\mu &= \frac{1}{2}\int_\Omega\left\langle\nabla\psi,\nabla(u_i^2)\right\rangle d\mu\\ &= -\frac{1}{2}\int_\Omega u_i^2\mathbb{L}_\theta\psi\,d\mu\\ &= -\frac{1}{2}\int_\Omega u_i^2\left(B_0-\langle\nabla\theta,\nabla\psi\rangle\right)d\mu\\ &= -\frac{B_0}{2}+\frac{1}{2}\int_\Omega u_i^2\langle\nabla\theta,\nabla\psi\rangle d\mu.\end{aligned} \tag{3.21}$$

Using (1.7) and (3.21), and by direct calculation, we have

$$\begin{aligned}&4\int_\Omega u_i\mathbb{L}_\theta\psi\langle\nabla\psi,\nabla u_i\rangle d\mu\\ &= 4\int_\Omega u_i\left(B_0-\langle\nabla\theta,\nabla\psi\rangle\right)\langle\nabla\psi,\nabla u_i\rangle d\mu\\ &= 4B_0\int_\Omega u_i\langle\nabla\psi,\nabla u_i\rangle d\mu-4\int_\Omega u_i\langle\nabla\theta,\nabla\psi\rangle\langle\nabla\psi,\nabla u_i\rangle d\mu\\ &= -2B_0^2+2B_0\int_\Omega u_i^2\langle\nabla\theta,\nabla\psi\rangle d\mu-4\int_\Omega u_i\langle\nabla\theta,\nabla\psi\rangle\langle\nabla\psi,\nabla u_i\rangle d\mu.\end{aligned} \tag{3.22}$$

By combining (3.20) and (3.22), and using the Cauchy–Schwarz inequality, (1.7), (3.10) and (3.17) we obtain

$$\begin{aligned}&\int_\Omega\left[u_i^2(\mathbb{L}_\theta\psi)^2+4\left(\langle\nabla\psi,\nabla u_i\rangle^2+u_i\mathbb{L}_\theta\psi\langle\nabla\psi,\nabla u_i\rangle\right)\right]d\mu\\ &\le -B_0^2+\int_\Omega\left(u_i\langle\nabla\theta,\nabla\psi\rangle-2\langle\nabla\psi,\nabla u_i\rangle\right)^2d\mu\end{aligned}$$

$$
\begin{aligned}
&\leq -B_0^2 + \int_\Omega \big(C_0|u_i| + 2|\nabla u_i|\big)^2 d\mu \\
&\leq -B_0^2 + \int_\Omega \big(C_0^2|u_i|^2 + 4C_0|u_i||\nabla u_i| + 4|\nabla u_i|^2\big) d\mu \\
&\leq -B_0^2 + C_0^2 + 4C_0\Lambda_i^{\frac{1}{4}} + 4\Lambda_i^{\frac{1}{2}}.
\end{aligned} \tag{3.23}
$$

By (1.7) and (3.10) again, we also get

$$
-2\int_\Omega |\nabla\psi|^2 u_i \mathbb{L}_\theta u_i d\mu = -2\int_\Omega u_i \mathbb{L}_\theta u_i d\mu \leq 2\Lambda_i^{\frac{1}{2}} \tag{3.24}
$$

Substituting (3.23) and (3.24) into (3.19), we arrive at

$$
\begin{aligned}
\sum_{i=1}^k (\Lambda_{k+1} - \Lambda_i) \leq \delta \sum_{i=1}^k (\Lambda_{k+1} - \Lambda_i)\left[6\Lambda_i^{\frac{1}{2}} + 4C_0\Lambda_i^{\frac{1}{4}} + C_0^2 - B_0^2\right] \\
+ \frac{1}{4\delta}\sum_{i=1}^k \left(4\Lambda_i^{\frac{1}{2}} + 4C_0\Lambda_i^{\frac{1}{4}} + C_0^2 - B_0^2\right).
\end{aligned} \tag{3.25}
$$

Taking

$$
\delta := \frac{1}{2}\left\{\frac{\sum_{i=1}^k \left(4\Lambda_i^{\frac{1}{2}} + 4C_0\Lambda_i^{\frac{1}{4}} + C_0^2 - B_0^2\right)}{\sum_{i=1}^k (\Lambda_{k+1} - \Lambda_i)\left[6\Lambda_i^{\frac{1}{2}} + 4C_0\Lambda_i^{\frac{1}{4}} + C_0^2 - B_0^2\right]}\right\}^{\frac{1}{2}}
$$

in (3.25), we can get (1.8).

(iv) Substituting $h = \phi_p$ into (2.2) and summing over p from 1 to l successively, we get

$$
\begin{aligned}
&\sum_{i=1}^k (\Lambda_{k+1} - \Lambda_i)\int_\Omega u_i^2 \sum_{p=1}^l |\nabla\phi_p|^2 d\mu \\
&\leq \delta \sum_{i=1}^k (\Lambda_{k+1} - \Lambda_i)\int_\Omega \sum_{p=1}^l \Big[u_i^2(\mathbb{L}_\theta\phi_p)^2 + 4\big(\langle\nabla\phi_p, \nabla u_i\rangle^2 + u_i\mathbb{L}_\theta\phi_p\langle\nabla\phi_p, \nabla u_i\rangle\big) \\
&\quad -2|\nabla\phi_p|^2 u_i\mathbb{L}_\theta u_i\Big] d\mu \\
&\quad + \frac{1}{4\delta}\sum_{i=1}^k \int_\Omega \sum_{p=1}^l \Big[u_i^2(\mathbb{L}_\theta\phi_p)^2 + 4\big(\langle\nabla\phi_p, \nabla u_i\rangle^2 + u_i\mathbb{L}_\theta\phi_p\langle\nabla\phi_p, \nabla u_i\rangle\big)\Big] d\mu
\end{aligned} \tag{3.26}
$$

We can easily obtain from (1.9) that

$$
\mathbb{L}_\theta\phi_p = \Delta\phi_p - \langle\nabla\theta, \nabla\phi_p\rangle = -\langle\nabla\theta, \nabla\phi_p\rangle.
$$

Since $\{\nabla\phi_p\}_{p=1}^l$ is a set of orthonormal vector fields, we have

$$\sum_{p=1}^{l}(\mathbb{L}_\theta\phi_p)^2 = \sum_{p=1}^{l}\langle\nabla\theta, \nabla\phi_p\rangle^2 \leq |\nabla\theta|^2 \leq C_0^2, \tag{3.27}$$

and

$$\sum_{p=1}^{l}\langle\nabla\phi_p, \nabla u_i\rangle^2 \leq |\nabla u_i|^2. \tag{3.28}$$

By the Cauchy–Schwarz inequality, we also get

$$\begin{aligned}\sum_{p=1}^{l}\mathbb{L}_\theta\phi_p\langle\nabla\phi_p, \nabla u_i\rangle &= -\sum_{p=1}^{l}\langle\nabla\theta, \nabla\phi_p\rangle\langle\nabla\phi_p, \nabla u_i\rangle\\ &\leq \left(\sum_{p=1}^{l}\langle\nabla\theta, \nabla\phi_p\rangle^2\right)^{\frac{1}{2}}\left(\sum_{p=1}^{l}\langle\nabla\phi_p, \nabla u_i\rangle^2\right)^{\frac{1}{2}}\\ &\leq |\nabla\theta|\cdot|\nabla u_i|\\ &\leq C_0|\nabla u_i|.\end{aligned} \tag{3.29}$$

Substituting (3.27)–(3.29) into (3.26), we obtain

$$\begin{aligned}l\sum_{i=1}^{k}(\Lambda_{k+1}-\Lambda_i) \leq \delta\sum_{i=1}^{k}(\Lambda_{k+1}-\Lambda_i)\int_\Omega\Big[C_0^2u_i^2 + 4\big(|\nabla u_i|^2 + C_0|u_i|\cdot|\nabla u_i|\big)\\ -2lu_i\mathbb{L}_\theta u_i\Big]\,d\mu\\ +\frac{1}{4\delta}\sum_{i=1}^{k}\int_\Omega\Big[C_0^2u_i^2 + 4\big(|\nabla u_i|^2 + C_0|u_i|\cdot|\nabla u_i|\big)\Big]\,d\mu.\end{aligned} \tag{3.30}$$

Using (3.10) and (3.17), we derive from (3.30) that

$$\begin{aligned}l\sum_{i=1}^{k}(\Lambda_{k+1}-\Lambda_i) \leq \delta\sum_{i=1}^{k}(\Lambda_{k+1}-\Lambda_i)\left[(4+2l)\Lambda_i^{\frac{1}{2}} + 4C_0\Lambda_i^{\frac{1}{4}} + C_0^2\right]\\ +\frac{1}{4\delta}\sum_{i=1}^{k}\left(4\Lambda_i^{\frac{1}{2}} + 4C_0\Lambda_i^{\frac{1}{4}} + C_0^2\right).\end{aligned} \tag{3.31}$$

Taking

$$\delta := \frac{1}{2}\left\{\frac{\sum_{i=1}^{k}\left(2\Lambda_i^{\frac{1}{4}}+C_0\right)^2}{\sum_{i=1}^{k}(\Lambda_{k+1}-\Lambda_i)\left[(4+2l)\Lambda_i^{\frac{1}{2}}+4C_0\Lambda_i^{\frac{1}{4}}+C_0^2\right]}\right\}^{\frac{1}{2}}$$

in (3.31), we can easily get (1.10).

(v) Taking $h = f_\alpha$ in (2.2) and summing over α from 1 to $m+1$ successively, we have

$$\begin{aligned}
&\tau\sum_{i=1}^{k}(\Lambda_{k+1}-\Lambda_i)\\
&\leq \delta\sum_{i=1}^{k}(\Lambda_{k+1}-\Lambda_i)\int_\Omega\sum_{\alpha=1}^{m+1}\Big[u_i^2(\mathbb{L}_\theta f_\alpha)^2+4\big(\langle\nabla f_\alpha,\nabla u_i\rangle^2+u_i\mathbb{L}_\theta f_\alpha\langle\nabla f_\alpha,\nabla u_i\rangle\big)\\
&\quad -2|\nabla f_\alpha|^2u_i\mathbb{L}_\theta u_i\Big]d\mu+\frac{1}{4\delta}\sum_{i=1}^{k}\int_\Omega\sum_{\alpha=1}^{m+1}\Big[u_i^2(\mathbb{L}_\theta f_\alpha)^2\\
&\quad +4\big(\langle\nabla f_\alpha,\nabla u_i\rangle^2+u_i\mathbb{L}_\theta f_\alpha\langle\nabla f_\alpha,\nabla u_i\rangle\big)\Big]d\mu.
\end{aligned}\tag{3.32}$$

By direct calculations and applying (1.11) we obtain

$$\sum_{\alpha=1}^{m+1}|\nabla f_\alpha|^2=\frac{1}{2}\Delta\left(\sum_{\alpha=1}^{m+1}f_\alpha^2\right)-\sum_{\alpha=1}^{m+1}f_\alpha\Delta f_\alpha=\tau\sum_{\alpha=1}^{m+1}f_\alpha^2=\tau.\tag{3.33}$$

Note that since $\sum_{\alpha=1}^{m+1}f_\alpha^2=1$, we also have

$$\sum_{\alpha=1}^{m+1}f_\alpha\nabla f_\alpha=\frac{1}{2}\nabla\left(\sum_{\alpha=1}^{m+1}f_\alpha^2\right)=0.\tag{3.34}$$

By using (1.11), (3.10), (3.33), and the Cauchy–Schwarz inequality we obtain

$$\begin{aligned}
&\int_\Omega\sum_{\alpha=1}^{m+1}\langle\nabla f_\alpha,\nabla u_i\rangle^2d\mu\leq\int_\Omega\left(\sum_{\alpha=1}^{m+1}|\nabla f_\alpha|^2\right)|\nabla u_i|^2d\mu\\
&\quad=\tau\int_\Omega|\nabla u_i|^2d\mu\leq\tau\Lambda_i^{\frac{1}{2}},
\end{aligned}\tag{3.35}$$

$$\begin{aligned}
&\sum_{\alpha=1}^{m+1}\mathbb{L}_\theta f_\alpha\langle\nabla f_\alpha,\nabla u_i\rangle=\sum_{\alpha=1}^{m+1}\big(\Delta f_\alpha-\langle\nabla\theta,\nabla f_\alpha\rangle\big)\langle\nabla f_\alpha,\nabla u_i\rangle\\
&\quad\leq\sum_{\alpha=1}^{m+1}\big(-\tau f_\alpha\langle\nabla f_\alpha,\nabla u_i\rangle+|\nabla f_\alpha|^2|\nabla\theta||\nabla u_i|\big)
\end{aligned}$$

$$\leq \tau C_0 \left|\nabla u_i\right|, \tag{3.36}$$

and

$$\begin{aligned}
\sum_{\alpha=1}^{m+1} (\mathbb{L}_\theta f_\alpha)^2 &= \sum_{\alpha=1}^{m+1} \left(\Delta f_\alpha - \langle \nabla\theta, \nabla f_\alpha \rangle\right)^2 \\
&= \sum_{\alpha=1}^{m+1} \left[(\Delta f_\alpha)^2 - 2\Delta f_\alpha \langle \nabla\theta, \nabla f_\alpha\rangle + \langle \nabla\theta, \nabla f_\alpha\rangle^2\right] \\
&\leq \sum_{\alpha=1}^{m+1} \left[\tau^2 f_\alpha^2 + 2\tau f_\alpha \langle \nabla\theta, \nabla f_\alpha\rangle + |\nabla\theta|^2 |\nabla f_\alpha|^2\right] \\
&\leq \tau^2 + \tau C_0^2,
\end{aligned} \tag{3.37}$$

Using (1.11), (3.10), (3.17), (3.33), (3.35) and (3.36), it follows from (3.32) that

$$\begin{aligned}
\tau \sum_{i=1}^{k} (\Lambda_{k+1} - \Lambda_i) &\leq \delta \sum_{i=1}^{k} (\Lambda_{k+1} - \Lambda_i)\Big[(\tau^2 + \tau C_0^2) \int_\Omega u_i^2 d\mu \\
&\quad + 4\big(\tau \Lambda_i^{\frac{1}{2}} + \tau C_0 \int_\Omega |u_i| \cdot |\nabla u_i| d\mu\big) - 2\tau \int_\Omega u_i \mathbb{L}_\theta u_i d\mu\Big] \\
&\quad + \frac{1}{4\delta} \sum_{i=1}^{k} \Big[(\tau^2 + \tau C_0^2) \int_\Omega u_i^2 d\mu + 4\big(\tau \Lambda_i^{\frac{1}{2}} + \tau C_0 \int_\Omega |u_i| \cdot |\nabla u_i| d\mu\big)\Big] \\
&\leq \delta \sum_{i=1}^{k} \left(\Lambda_{k+1} - \Lambda_i\right) \left(6\tau \Lambda_i^{\frac{1}{2}} + 4\tau C_0 \Lambda_i^{\frac{1}{4}} + \tau^2 + \tau C_0^2\right) \\
&\quad + \frac{1}{4\delta} \sum_{i=1}^{k} \left(4\tau \Lambda_i^{\frac{1}{2}} + 4\tau C_0 \Lambda_i^{\frac{1}{4}} + \tau^2 + \tau C_0^2\right).
\end{aligned}$$

or equivalently,

$$\begin{aligned}
\sum_{i=1}^{k} (\Lambda_{k+1} - \Lambda_i) &\leq \delta \sum_{i=1}^{k} \left(\Lambda_{k+1} - \Lambda_i\right) \left(6\Lambda_i^{\frac{1}{2}} + 4C_0 \Lambda_i^{\frac{1}{4}} + \tau + C_0^2\right) \\
&\quad + \frac{1}{\delta} \sum_{i=1}^{k} \left(4\tau \Lambda_i^{\frac{1}{2}} + 4\tau C_0 \Lambda_i^{\frac{1}{4}} + \tau^2 + \tau C_0^2\right).
\end{aligned} \tag{3.38}$$

Taking

$$\delta := \frac{1}{2} \left\{ \frac{\sum_{i=1}^{k} \left(4\tau \Lambda_i^{\frac{1}{2}} + 4\tau C_0 \Lambda_i^{\frac{1}{4}} + \tau^2 + \tau C_0^2\right)}{\sum_{i=1}^{k} \left(\Lambda_{k+1} - \Lambda_i\right)\left(6\Lambda_i^{\frac{1}{2}} + 4C_0 \Lambda_i^{\frac{1}{4}} + \tau + C_0^2\right)} \right\}^{\frac{1}{2}}$$

in (3.38), we can get (1.12).

The proof is now complete. □

4 Proof of Corollary 1.1

We now use Theorem 1.1 and the reverse Chebyshev inequality (see Lemma 5.3 in Appendix) to prove Corollary 1.1 as follows.

Proof of Corollary 1.1 (i) Since $\{\Lambda_{k+1}-\Lambda_i\}_{i=1}^k$ is decreasing and

$$\left\{(2n+4)\Lambda_i^{\frac{1}{2}}+4C_0\Lambda_i^{\frac{1}{4}}+n^2H_0^2+C_0^2\right\}_{i=1}^k$$

is increasing, we get by the reverse Chebyshev inequality that

$$\begin{aligned}&\sum_{i=1}^k(\Lambda_{k+1}-\Lambda_i)\left[(2n+4)\Lambda_i^{\frac{1}{2}}+4C_0\Lambda_i^{\frac{1}{4}}+n^2H_0^2+C_0^2\right]\\&\quad\le\frac{1}{k}\left[\sum_{i=1}^k(\Lambda_{k+1}-\Lambda_i)\right]\left\{\sum_{i=1}^k\left[(2n+4)\Lambda_i^{\frac{1}{2}}+4C_0\Lambda_i^{\frac{1}{4}}+n^2H_0^2+C_0^2\right]\right\}.\end{aligned}$$

Substituting this into (1.4) and simplifying the resulted inequality successively, we get

$$\begin{aligned}\sum_{i=1}^k(\Lambda_{k+1}-\Lambda_i)\le{}&\frac{1}{n^2k}\sum_{i=1}^k\left[(2n+4)\Lambda_i^{\frac{1}{2}}+4C_0\Lambda_i^{\frac{1}{4}}+n^2H_0^2+C_0^2\right]\\&\times\sum_{i=1}^k\left(4\Lambda_i^{\frac{1}{2}}+4C_0\Lambda_i^{\frac{1}{4}}+n^2H_0^2+C_0^2\right).\end{aligned}$$

From this inequality, we immediately obtain

$$\begin{aligned}\Lambda_{k+1}\le{}&\frac{\sum_{i=1}^k\Lambda_i}{k}+\frac{1}{n^2}\left[(2n+4)\frac{\sum_{i=1}^k\Lambda_i^{\frac{1}{2}}}{k}+4C_0\frac{\sum_{i=1}^k\Lambda_i^{\frac{1}{4}}}{k}+n^2H_0^2+C_0^2\right]\\&\times\left(4\frac{\sum_{i=1}^k\Lambda_i^{\frac{1}{2}}}{k}+4C_0\frac{\sum_{i=1}^k\Lambda_i^{\frac{1}{4}}}{k}+n^2H_0^2+C_0^2\right).\end{aligned}$$

It then follows from the last inequality and $\Lambda_i\le\Lambda_k$ $(i=1,\ldots,k)$ that

$$\Lambda_{k+1}\le\Lambda_k+\frac{1}{n^2}\left[(2n+4)\Lambda_k^{\frac{1}{2}}+4C_0\Lambda_k^{\frac{1}{4}}+n^2H_0^2+C_0^2\right]$$

$$\times \left(4\Lambda_k^{\frac{1}{2}} + 4C_0\Lambda_k^{\frac{1}{4}} + n^2H_0^2 + C_0^2\right).$$

Simplifying the above inequality, we can see that

$$\Lambda_{k+1} \leq \frac{1}{n^2}\Big\{(n+4)^2\Lambda_k + 8(n+4)C_0\Lambda_k^{\frac{3}{4}} + 2\big[(n+4)n^2H_0^2 + (n+12)C_0^2\big]\Lambda_k^{\frac{1}{2}}$$
$$+8C_0\big(n^2H_0^2 + C_0^2\big)\Lambda_k^{\frac{1}{4}} + \big(n^2H_0^2 + C_0^2\big)^2\Big\}.$$

This completes the proof of (i).

The rest of the proof is similar to (i). So we omit it. The proof is complete. □

Data availability There are no data availability issues.

Declarations

Conflict of interest The authors have not disclosed any competing interests.

Appendix

To prove some results of this paper, we need the following inequalities:

Lemma 5.1 (Weighted Chebyshev inequality, see Hardy et al. 1988) *Let* $\{a_i\}_{i=1}^k$, $\{b_i\}_{i=1}^k$ *and* $\{c_i\}_{i=1}^k$ *be three sequences of non-negative real numbers with* $\{a_i\}_{i=1}^k$ *decreasing;* $\{b_i\}_{i=1}^k$ *and* $\{c_i\}_{i=1}^k$ *increasing. Then the following inequality holds*

$$\left(\sum_{i=1}^k a_i^2 b_i\right)\left(\sum_{i=1}^k a_i c_i\right) \leq \left(\sum_{i=1}^k a_i^2\right)\left(\sum_{i=1}^k a_i b_i c_i\right).$$

Lemma 5.2 (Chebyshev sum inequality, see Hardy et al. 1988) *Let* $\{a_i\}_{i=1}^k$ *and* $\{b_i\}_{i=1}^k$ *be two sequences of real numbers with* $\{a_i\}_{i=1}^k$ *and* $\{b_i\}_{i=1}^k$ *increasing or decreasing. Then the following inequality holds*

$$\frac{1}{k}\left(\sum_{i=1}^k a_i\right)\left(\sum_{i=1}^k b_i\right) \leq \sum_{i=1}^k a_i b_i.$$

with equality if and only if

$$a_1 = \cdots = a_k, \quad or \quad b_1 = \cdots = b_k.$$

Lemma 5.3 (Reverse Chebyshev inequality, see Hardy et al. 1988) *Suppose* $\{a_i\}_{i=1}^k$ *and* $\{b_i\}_{i=1}^k$ *are two real sequences with* $\{a_i\}_{i=1}^k$ *increasing and* $\{b_i\}_{i=1}^k$ *decreasing. Then the following inequality holds:*

$$\frac{1}{k}\left(\sum_{i=1}^k a_i\right)\left(\sum_{i=1}^k b_i\right) \geq \sum_{i=1}^k a_i b_i,$$

with equality if and only if

$$a_1 = \cdots = a_k \quad or \quad b_1 = \cdots = b_k.$$

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