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A Unique Continuation Result for a 2D System of Nonlinear Equations for Surface Waves

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Abstract
In this paper, we establish a result of unique continuation for a special two-dimensional nonlinear system that models the evolution of long water waves with small amplitude in the presence of surface tension. More precisely, we will show that if $(\eta, \Phi) = (\eta(x, y, t), \Phi(x, y, t))$ is a solution of the nonlinear system, in a suitable function space, and (η, Φ) vanishes on an open subset Ω of $\mathbb{R}^2 \times [-T, T]$, then $(\eta, \Phi) \equiv 0$ in the horizontal component of Ω. To state such property, we use a Carleman-type estimate for a differential operator $\mathcal{L}$ related to the system. We prove the Carleman estimate using a particular version of the well known Treves' inequality.

Keywords Nonlinear system · Long waves · Carleman estimate · Unique continuation

Mathematics Subject Classification 35B60 · 35Q35

1 Introduction

The focus of the present work is the following two-dimensional system

$$\begin{cases} \left(I - \frac{\mu}{2}\Delta\right)\eta_t + \Delta\Phi - \frac{2\mu}{3}\Delta^2\Phi + \epsilon\nabla\cdot(\eta\nabla\Phi) = 0, \\ \left(I - \frac{\mu}{2}\Delta\right)\Phi_t + \eta - \mu\sigma\Delta\eta + \frac{\epsilon}{2}|\nabla\Phi|^2 = 0, \end{cases} \tag{1}$$

that describes the evolution of long water waves with small amplitude in the presence of surface tension (see Quintero and Montes 2013). Here, ϵ is the amplitude parameter

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(nonlinearity coefficient), μ is the long-wave parameter (dispersion coefficient), σ is the inverse of the Bond number (associated with the surface tension) and the functions $\eta = \eta(x, y, t)$ and $\Phi = \Phi(x, y, t)$ denote the wave elevation and the potential velocity on the bottom $z = 0$, respectively.

As happens in water wave models, there is a Hamiltonian type structure which is clever to find the appropriate space for special solutions (solitary waves for example) and also provide relevant information for the study of the Cauchy problem. For the system (1), the Hamiltonian functional $\mathcal{H} = \mathcal{H}(t)$ is defined as

$$\mathcal{H}\begin{pmatrix}\eta\\ \Phi\end{pmatrix} = \frac{1}{2}\int_{\mathbb{R}^2}\big(\eta^2 + \mu\sigma|\nabla\eta|^2 + |\nabla\Phi|^2 + \frac{2\mu}{3}|\Delta\Phi|^2 + \epsilon\eta|\nabla\Phi|^2\big)dxdy,$$

and the Hamiltonian type structure is given by

$$\begin{pmatrix}\eta_t\\ \Phi_t\end{pmatrix} = \mathcal{J}\mathcal{H}'\begin{pmatrix}\eta\\ \Phi\end{pmatrix}, \quad \mathcal{J} = \Big(I - \frac{\mu}{2}\Delta\Big)^{-1}\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}.$$

We see directly that the functional $\mathcal{H}$ is well defined when $\eta, \nabla\Phi \in H^1(\mathbb{R}^2)$, for t in some interval. These conditions already characterize the natural space for the study of solutions of the system (1). Certainly, in Quintero and Montes (2013) showed for the model (1) the existence of solitary wave solutions which propagate with speed of wave $\theta > 0$,

$$\eta(x, y, t) = \frac{1}{\epsilon}u\left(\frac{x - \theta t}{\sqrt{\mu}}, \frac{y}{\sqrt{\mu}}\right), \quad \Phi(x, y, t) = \frac{\sqrt{\mu}}{\epsilon}v\left(\frac{x - \theta t}{\sqrt{\mu}}, \frac{y}{\sqrt{\mu}}\right), \tag{2}$$

in the energy space $H^1(\mathbb{R}^2) \times \mathcal{V}(\mathbb{R}^2)$, where $H^1(\mathbb{R}^2)$ is the usual Sobolev space of order 1 and the space $\mathcal{V}(\mathbb{R}^2)$ is defined with respect to the norm given by

$$\begin{aligned}\|w\|^2_{\mathcal{V}(\mathbb{R}^2)} &= \int_{\mathbb{R}^2}\left(|\nabla w|^2 + |\Delta w|^2\right)dxdy\\ &= \int_{\mathbb{R}^2}\left(w_x^2 + w_y^2 + w_{xx}^2 + 2w_{xy}^2 + w_{yy}^2\right)dxdy.\end{aligned}$$

In Quintero and Montes (2016), it was proved the local well-posedness for the Cauchy problem associated to the system (1) in the Sobolev type space $H^{s-1}(\mathbb{R}^2) \times \mathcal{V}^s(\mathbb{R}^2)$, $s \geq 2$, where $H^s(\mathbb{R}^2)$ is the usual Sobolev space of order s defined as the completion of the Schwartz class with respect to the norm

$$\|w\|^2_{H^s(\mathbb{R}^2)} = \int_{\mathbb{R}^2}\left(1 + |\xi|^2\right)^s|\widehat{w}(\xi)|^2 d\xi$$

and $\mathcal{V}^s(\mathbb{R}^2)$ denotes the completion of the Schwartz class with respect to the norm

$$\|w\|^2_{\mathcal{V}^s(\mathbb{R}^2)} = \int_{\mathbb{R}^2}\left(1 + |\xi|^2\right)^s|\xi|^2|\widehat{w}(\xi)|^2 d\xi,$$

where $\widehat{w}$ is the Fourier transform of w defined on $\mathbb{R}^2$ by

$$\widehat{w}(\xi) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}^2} e^{-ix\cdot\xi} w(x)dx;$$

and in the work (Montes and Quintero 2015), using a general result established by Grillakis et al. (1987) to analyze the orbital stability of solitary waves for a class of abstract Hamiltonian systems, Quintero and Montes showed the orbital stability of the solutions of the form (2). The existence of x-periodic solitary wave solutions for the system (1) can be seen in Quintero and Montes (2017).

In the present work we will prove a unique continuation result for the system (1). More precisely, we show that if $(\eta, \Phi) = (\eta(x, y, t), \Phi(x, y, t))$ is a solution of the system (1) in a suitable function space,

$$\begin{aligned} &\eta, \eta_t \in L^2\left(-T, T; H^2_{loc}(\mathbb{R}^2)\right), \ \Phi \in L^2\left(-T, T; H^4_{loc}(\mathbb{R}^2)\right), \\ &\Phi_t \in L^2\left(-T, T; H^2_{loc}(\mathbb{R}^2)\right), \end{aligned}$$

and (η, Φ) vanishes on an open subset Ω of $\mathbb{R}^2 \times [-T, T]$, then $(\eta, \Phi) \equiv 0$ in the horizontal component of Ω. The horizontal component Ω_1 of an open subset $\Omega \subseteq \mathbb{R}^2 \times \mathbb{R}$ is the set defined by

$$\Omega_1 = \left\{(x, y, t) \in \mathbb{R}^2 \times [-T, T] \ : \ \exists (x_1, y_1) \in \mathbb{R}^2, \ (x_1, y_1, t) \in \Omega\right\}.$$

The unique continuation property has been intensively studied for a long time. An important work on the subject was done by Saut and Scheurer (1987). They showed a unique continuation result for a general class of dispersive equations including the well known KdV equation,

$$u_t + uu_x + u_{xxx} = 0,$$

and various generalizations. In a similar way, Shang (2007) showed a unique continuation result for the symmetric regularized long wave equation,

$$u_{tt} - u_{xx} + \frac{1}{2}\left(u^2\right)_{xt} - u_{xxtt} = 0.$$

In the previous equations, a Carleman estimate is established to prove that if a solution u vanishes on an open subset Ω, then $u \equiv 0$ in the horizontal component of Ω.

By using the inverse scattering transform and some results from the Hardy function theory, Zhang (1992) established that if u is a solution of the KdV equation, then it cannot have compact support at two different moments unless it vanishes identically. In the work (Bourgain 1997), Bourgain introduced a different approach and prove that if a solution u to the KdV equation has compact support in a nontrivial time interval $I = [t_1, t_2]$, then $u \equiv 0$. His argument is based on an analytic continuation of the Fourier transform via the Paley–Wiener Theorem and the dispersion relation of the linear part of the equation. It also applies to higher order dispersive nonlinear models, and to higher spatial dimensions; in particular, Panthee (2005) showed that if u is a smooth solution of the Kadomtsev–Petviashvili (KP) equation,

$$u_t + u_{xxx} + uu_x + \partial_x^{-1} u_{yy} = 0,$$

such that, for some $B > 0$,

$$\operatorname{supp} u(t) \subset [-B, B] \times [-B, B] \ \ \forall t \in [t_1, t_2],$$

then $u \equiv 0$.

More recently, Kenig et al. (2002) proposed a new method and proved that if a sufficiently smooth solution u to a generalized KdV equation is supported in a half line at two different instants of time, then $u \equiv 0$. Moreover, Escauriaza et al. (2007) established uniqueness properties of solutions of the k-generalized Korteweg–de Vries equation,

$$u_t + u^k u_x + u_{xxx} = 0, \quad k \in \mathbb{Z}^+. \tag{3}$$

They obtained sufficient conditions on the behavior of the difference $u_1 - u_2$ of two solutions u_1, u_2 of (3) at two different times $t_0 = 0$ and $t_1 = 1$ which guarantee that $u_1 \equiv u_2$. This kind of uniqueness results has been deduced under the assumption that the solutions coincide in a large sub-domain of $\mathbb{R}$ at two different times. In a similar fashion, Bustamante et al. (2011) proved that if u is a smooth solution of the Zakharov–Kuznetsov equation,

$$u_t + u_{xxx} + u_{xyy} + uu_x = 0,$$

such that, for some $B > 0$,

$$\operatorname{supp} u(t_2), \operatorname{supp} u(t_1) \ \subset [-B, B] \times [-B, B],$$

then $u \equiv 0$. Moreover, in Bustamante et al. (2013) it was proved that if the difference of two sufficiently smooth solutions of the Zakharov–Kuznetsov equation decays as $e^{-a(x^2+y^2)^{3/4}}$ at two different times, for some $a > 0$ large enough, then both solutions coincide. More unique continuation results can be seen in Carvajal and Panthee (2005), Carvajal and Panthee (2006), Iório (2003a, b) and Kenig et al. (2003).

Following from close the works of Saut and Scheurer (1987), we base our analysis in finding an appropriate Carleman-type estimate for the linear operator $\mathcal{L}$ associated to the system (1). In order to do this we use a particular version of the well known Treves' inequality. For the operator $\mathcal{L}$ we also prove that if a solution vanishes in a ball in the xyt space, which passes through the origin, then it also vanished in a neighborhood of the origin. The paper is organized as follows. In Sect. 2, using a particular version of the Treves inequality, we establish a Carleman estimate for a differential operator $\mathcal{L}$ closely related to our problem. In Sect. 3, first we give some useful technical results. Later, we show the unique continuation result for the system (1).

2 Carleman Estimate

In this section, we will use the notation $D = \big(\partial_x, \partial_y, \partial_t\big)$. If $P = P(\xi_1, \xi_2, \xi_3)$ is a polynomial in three variables, has constant coefficients and degree m, then we consider the differential operator of order m associated to P,

$$P(D) = P\left(\partial_x, \partial_y, \partial_t\right) = \sum_{|\alpha|\le m} a_\alpha D^\alpha,$$

where $D^\alpha = \partial_x^{\alpha_1}\partial_y^{\alpha_2}\partial_t^{\alpha_3}$ and $|\alpha| = \alpha_1 + \alpha_2 + \alpha_3$. By definition

$$P^{(\beta)}(\xi_1, \xi_2, \xi_3) = \partial_{\xi_1}^{\beta_1}\partial_{\xi_2}^{\beta_2}\partial_{\xi_3}^{\beta_3} P(\xi_1, \xi_2, \xi_3), \quad \beta = (\beta_1, \beta_2, \beta_3) \in \mathbb{N}^3.$$

Using a particular version of the Treves' inequality, we will establish a Carleman estimate for the differential operator $\mathcal{L}$ defined as

$$\mathcal{L} = \begin{pmatrix} P_1(\partial_x, \partial_y, \partial_t) + \big(f_1, f_2\big)\cdot\nabla & P_2(\partial_x, \partial_y, \partial_t) + f_3\,\Delta \\ P_3(\partial_x, \partial_y, \partial_t) & P_4(\partial_x, \partial_y, \partial_t) + \big(f_4, f_5\big)\cdot\nabla \end{pmatrix}, \tag{4}$$

where $f_j = f_j(x, y, t)$, for $j = 1, 2, 3$ and the operators P_j, $j = 1, 2, 3, 4$ are defined by

$$P_1(\partial_x, \partial_y, \partial_t) = (I - a\Delta)\partial_t + c_1\partial_x^3 + c_2\partial_y^3 + c_1\partial_y^2\partial_x + c_2\partial_x^2\partial_y,$$

$$P_2(\partial_x, \partial_y, \partial_t) = -b\Delta^2, \quad P_3(\partial_x, \partial_y, \partial_t) = I - c\Delta,$$

and

$$\begin{aligned} P_4(\partial_x, \partial_y, \partial_t) = (I - d\Delta)\partial_t + c_3\partial_x^3 + c_4\partial_y^3 \\ + c_3\partial_y^2\partial_x + c_4\partial_x^2\partial_y. \end{aligned}$$

Theorem 2.1 (Treves' Inequality) *Let $P(D) = P\left(\partial_x, \partial_y, \partial_t\right)$ be a differential operator of order m with constant coefficients. Then for all $\xi = (\xi_1, \xi_2, \xi_3) \in \mathbb{R}^3$, $\alpha = (\alpha_1, \alpha_2, \alpha_3) \in \mathbb{N}^3$ and $\Psi \in C_0^\infty(\mathbb{R}^3)$ we have that*

$$\begin{aligned} &\frac{2^{2|\alpha|}\xi^{2\alpha}}{\alpha!}\int_{\mathbb{R}^3} |P^{(\alpha)}(D)\Psi|^2 e^{\psi((x,y,t),\xi)}dxdydt \\ &\quad\le C(m, \alpha)\int_{\mathbb{R}^3} |P(D)\Psi|^2 e^{\psi((x,y,t),\xi)}dxdydt, \end{aligned} \tag{5}$$

where

$$\psi\left((x,y,t),\xi\right)=x^2\xi_1^2+y^2\xi_2^2+t^2\xi_3^2,\quad \xi^{2\alpha}=\xi_1^{2\alpha_1}\xi_2^{2\alpha_2}\xi_3^{2\alpha_3},$$

$$|\alpha|=\alpha_1+\alpha_2+\alpha_3,\quad \alpha!=\alpha_1!\alpha_2!\alpha_3!,\quad C(m,\alpha)=\begin{cases}\sup_{|r+\alpha|\le m}\binom{r+\alpha}{\alpha}, & \text{if } |\alpha|\le m,\\ 0, & \text{if } |\alpha|>m.\end{cases}$$

Proof See Theorem 2.4 in Treves (1966). □

Corollary 2.2 *Let* $P(D)=P\left(\partial_x,\partial_y,\partial_t\right)$ *be a differential operator of order* m *with constant coefficients. Then for all* $\alpha=(\alpha_1,\alpha_2,\alpha_3)\in\mathbb{N}^3$, $\delta>0$, $\tau>0$, $\Psi\in C_0^\infty(\mathbb{R}^3)$ *and* $\psi(x,y,t)=(x-\delta)^2+(y-\delta)^2+\delta^2t^2$ *we have that*

$$\frac{2^{2|\alpha|}\tau^{|\alpha|}\delta^{2\alpha_3}}{\alpha!}\int_{\mathbb{R}^3}|P^{(\alpha)}(D)\Psi|^2e^{2\tau\psi}\,dxdydt\le C(m,\alpha)\int_{\mathbb{R}^3}|P(D)\Psi|^2e^{2\tau\psi}\,dxdydt \tag{6}$$

with

$$|\alpha|=\alpha_1+\alpha_2+\alpha_3,\quad \alpha!=\alpha_1!\alpha_2!\alpha_3!,\quad C(m,\alpha)=\begin{cases}\sup_{|r+\alpha|\le m}\binom{r+\alpha}{\alpha}, & \text{if } |\alpha|\le m,\\ 0, & \text{if } |\alpha|>m.\end{cases}$$

Proof We will use the above theorem with the differential operator

$$Q(D)=P(D+a)=P\left(\partial_x+2\tau\delta,\partial_y+2\tau\delta,\partial_t\right),$$

where

$$\tau>0,\quad a=(2\tau\delta,2\tau\delta,0),\quad z=(x,y,t),\quad \xi=(\xi_1,\xi_2,\xi_3)=(\sqrt{2\tau},\sqrt{2\tau},\sqrt{2\tau}\delta).$$

Then, using inequality (5) we have that

$$\begin{aligned}\frac{2^{2|\alpha|}\xi^{2\alpha}}{\alpha!}&\int_{\mathbb{R}^3}|P^{(\alpha)}(D+a)\Psi|^2e^{\psi(z,\xi)}dxdydt\\&=\frac{2^{2|\alpha|}\tau^{|\alpha|}\delta^{2\alpha_3}}{\alpha!}\int_{\mathbb{R}^3}|P^{(\alpha)}(D+a)\Psi|^2e^{2\tau\left(x^2+y^2+\delta^2t^2\right)}dxdydt\\&\le C(m,\alpha)\int_{\mathbb{R}^3}|P(D+a)\Psi|^2e^{2\tau\left(x^2+y^2+\delta^2t^2\right)}dxdydt\end{aligned}$$

for all $\Psi \in C_0^\infty(\mathbb{R}^3)$, $\alpha = (\alpha_1, \alpha_2, \alpha_3) \in \mathbb{N}^3$ and any $\tau > 0$. Now, multiply both sides of the previous inequality by $e^{4\tau\delta^2}$ we obtain

$$\frac{2^{2|\alpha|}\tau^{|\alpha|}\delta^{2\alpha_3}}{\alpha!}\int_{\mathbb{R}^3}|e^{2\tau\delta(x+y)}P^{(\alpha)}(D+a)\Psi|^2 e^{2\tau\psi(x,y,t)}dxdydt$$
$$\le C(m,\alpha)\int_{\mathbb{R}^3}|e^{2\tau\delta(x+y)}P(D+a)\Psi|^2 e^{2\tau\psi(x,y,t)}dxdydt.$$

In particular, we can choose $\Psi = \widetilde{\Psi}e^{-2\tau\delta(x+y)}$ where $\widetilde{\Psi} \in C_0^\infty(\mathbb{R}^3)$. Observing that

$$P^{(\alpha)}(D)(\widetilde{\Psi}) = e^{2\tau\delta(x+y)}P^{(\alpha)}(D+a)(\widetilde{\Psi}e^{-2\tau\delta(x+y)})$$

and also that

$$P(D)(\widetilde{\Psi}) = e^{2\tau\delta(x+y)}P(D+a)(\widetilde{\Psi}e^{-2\tau\delta(x+y)})$$

we obtain

$$\frac{2^{2|\alpha|}\tau^{|\alpha|}\delta^{2\alpha_3}}{\alpha!}\int_{\mathbb{R}^3}|P^{(\alpha)}(D)\Psi|^2 e^{2\tau\psi}dxdydt \le C(m,\alpha)\int_{\mathbb{R}^3}|P(D)\Psi|^2 e^{2\tau\psi}dxdydt.$$

□

Now we present the Carleman estimate for the differential operator $\mathcal{L}$.

Theorem 2.3 *Let $\mathcal{L}$ the differential operator defined in (4), where c_1, c_2, c_3, c_4 are real constants and $f_1, f_2, f_3, f_4, f_5 \in L^\infty_{loc}(\mathbb{R}^3)$. Let $\delta > 0$ and*

$$B_\delta := \{(x,y,t) \in \mathbb{R}^3 \ : \ x^2 + y^2 + t^2 < \delta^2\}, \quad \psi(x,y,t)$$
$$= (x-\delta)^2 + (y-\delta)^2 + \delta^2 t^2.$$

Then, there exists $C > 0$ such that for all $\Psi = (\Psi_1, \Psi_2) \in C_0^\infty(B_\delta) \times C_0^\infty(B_\delta)$ and $\tau > 0$ with

$$\frac{\|f_1\|^2_{L^\infty(B_\delta)} + \|f_2\|^2_{L^\infty(B_\delta)}}{\tau^2\delta^2 a^2} \le \frac{1}{8}, \quad \frac{1}{d^2}\left(\frac{1}{\tau\delta^2} + \frac{1}{\tau^3 c_4^2}\right)\|f_3\|^2_{L^\infty(B_\delta)} \le \frac{1}{8}$$

and

$$\frac{\|f_4\|^2_{L^\infty(B_\delta)} + \|f_5\|^2_{L^\infty(B_\delta)}}{\tau^2\delta^2 d^2} \le \frac{1}{16},$$

we have that

$$\tau^3 c_1^2 \int_{B_\delta} |\Psi_1|^2 e^{2\tau\psi} dxdydt + \tau^2\delta^2 a^2 \int_{B_\delta} |\nabla\Psi_1|^2 e^{2\tau\psi} dxdydt$$
$$+ (\tau^3 c_4^2 + \tau^4 b^2) \int_{B_\delta} |\Psi_2|^2 e^{2\tau\psi} dxdydt + (\tau^2\delta^2 d^2 + \tau^3 b^2) \int_{B_\delta} |\nabla\Psi_2|^2 e^{2\tau\psi} dxdydt$$
$$+ \tau\delta^2 \int_{B_\delta} |(I - d\Delta)\Psi_2|^2 e^{2\tau\psi} dxdydt \leq C \int_{B_\delta} |\mathcal{L}\Psi|^2 e^{2\tau\psi} dxdydt. \tag{7}$$

Proof Let $\Psi = (\Psi_1, \Psi_2) \in C_0^\infty(B_\delta) \times C_0^\infty(B_\delta)$. Consider the polynomial

$$P_1(\xi_1, \xi_2, \xi_3) = \xi_3 - a(\xi_1^2 + \xi_2^2)\xi_3 + c_1\xi_1^3 + c_2\xi_2^3 + c_1\xi_2^2\xi_1 + c_2\xi_1^2\xi_2$$

and

$$P_1(D) = P_1(\partial_x, \partial_y, \partial_t) = (I - a\Delta)\partial_t + c_1\partial_x^3 + c_2\partial_y^3 + c_1\partial_{yyx} + c_2\partial_{xxy}$$

the differential operator associated to P_1. Then, if $\alpha = (1, 0, 1)$ we have that

$$P_1^{(\alpha)}(\xi_1, \xi_2, \xi_3) = P_1^{(1,0,1)}(\xi_1, \xi_2, \xi_3) = -2a\xi_1, \quad P_1^{(\alpha)}(D)\Psi_1 = -2a\partial_x\Psi_1$$

and

$$C(3, \alpha) = \sup_{|r+\alpha|\leq 3} \binom{r+\alpha}{\alpha} = 2.$$

Thus, using Theorem 2.1 we see that

$$\begin{aligned}\tau^2\delta^2 a^2 \int_{B_\delta} |\partial_x\Psi_1|^2 e^{2\tau\psi} dxdydt &\leq 32\tau^2\delta^2 a^2 \int_{B_\delta} |\partial_x\Psi_1|^2 e^{2\tau\psi} dxdydt \\ &= \frac{2^{2|\alpha|}\tau^{|\alpha|}\delta^{2\alpha_2}}{\alpha!} \int_{B_\delta} |P_1^{(\alpha)}(D)\Psi_1|^2 e^{2\tau\psi} dxdydt \\ &\leq \int_{B_\delta} |P_1(D)\Psi_1|^2 e^{2\tau\psi} dxdydt.\end{aligned} \tag{8}$$

Now, if $\alpha = (0, 1, 1)$ we have that

$$P_1^{(\alpha)}(\xi_1, \xi_2, \xi_3) = P_1^{(0,1,1)}(\xi_1, \xi_2, \xi_3) = -2a\xi_2, \quad P_1^{(\alpha)}(D)\Psi_1 = -2a\partial_y\Psi_1$$

and also

$$C(3, \alpha) = \sup_{|r+\alpha|\leq 3} \binom{r+\alpha}{\alpha} = 2.$$

So, using Theorem 2.1 we see that

$$\begin{aligned}\tau^2\delta^2 a^2\int_{B_\delta}|\partial_y\Psi_1|^2 e^{2\tau\psi}\,dxdydt &\le 32\tau^2\delta^2 a^2\int_{B_\delta}|\partial_y\Psi_1|^2 e^{2\tau\psi}\,dxdydt\\ &= \frac{2^{2|\alpha|}\tau^{|\alpha|}\delta^{2\alpha_2}}{\alpha!}\int_{B_\delta}|P_1^{(\alpha)}(D)\Psi_1|^2 e^{2\tau\psi}\,dxdydt\\ &\le \int_{B_\delta}|P_1(D)\Psi_1|^2 e^{2\tau\psi}\,dxdydt. \qquad (9)\end{aligned}$$

Moreover,

$$P_1^{(3,0,0)}(\xi_1,\xi_2,\xi_3) = 6c_1, \quad P_1^{(3,0,0)}(D)\Psi_1 = 6c_1\Psi_1, \quad C(3,(3,0,0)) = 1.$$

Then, using again the Theorem 2.1 we obtain that

$$\begin{aligned}\tau^3 c_1^2\int_{B_\delta}|\Psi_1|^2 e^{2\tau\psi}\,dxdydt &\le \frac{2^6\tau^3}{6}\int_{B_\delta}|P_1^{(3,0,0)}(D)\Psi_1|^2 e^{2\tau\psi}\,dxdydt\\ &\le \int_{B_\delta}|P_1(D)\Psi_1|^2 e^{2\tau\psi}\,dxdydt. \qquad (10)\end{aligned}$$

Now, by defining

$$P_2(\xi_1,\xi_2,\xi_3) = -b(\xi_1^4 + 2\xi_1^2\xi_2^2 + \xi_2^4), \quad P_2(D) = -b\Delta^2,$$

we have that

$$P_2^{(4,0,0)}(\xi_1,\xi_2,\xi_3) = -24b, \quad P_2^{(4,0,0)}(D)\,\Psi_2 = -24b\Psi_2, \quad C(4,(4,0,0)) = 1$$

and

$$\begin{aligned}\tau^4 b^2\int_{B_\delta}|\Psi_2|^2 e^{2\tau\psi}\,dxdydt &\le \frac{2^8\tau^4}{24}\int_{B_\delta}|P_2^{(4,0,0)}(D)\Psi_2|^2 e^{2\tau\psi}\,dxdydt\\ &\le \int_{B_\delta}|P_2(D)\Psi_2|^2 e^{2\tau\psi}\,dxdydt. \qquad (11)\end{aligned}$$

In a similar fashion

$$\begin{aligned}P_2^{(3,0,0)}(D)\Psi_2 &= -24b\partial_x\Psi_2, \quad P_2^{(0,3,0)}(D)\Psi_2\\ &= -24b\partial_y\Psi_2, \quad C(4,(3,0,0)) = C(4,(0,3,0)) = 4.\end{aligned}$$

Hence, we see that

$$\tau^3 b^2 \int_{B_\delta} |\partial_x \Psi_2|^2 e^{2\tau\psi} dxdydt \le \frac{2^6\tau^3}{24} \int_{B_\delta} |P_2^{(3,0,0)}(D)\Psi_2|^2 e^{2\tau\psi} dxdydt$$
$$\le \int_{B_\delta} |P_2(D)\Psi_2|^2 e^{2\tau\psi} dxdydt \tag{12}$$

and also that

$$\tau^3 b^2 \int_{B_\delta} |\partial_y \Psi_2|^2 e^{2\tau\psi} dxdydt \le \frac{2^6\tau^3}{24} \int_{B_\delta} |P_2^{(0,3,0)}(D)\Psi_2|^2 e^{2\tau\psi} dxdydt$$
$$\le \int_{B_\delta} |P_2(D)\Psi_2|^2 e^{2\tau\psi} dxdydt. \tag{13}$$

By considering

$$P_4(\xi_1, \xi_2, \xi_3) = \xi_3 - d(\xi_1^2 + \xi_2^2)\xi_3 + c_3\xi_1^3 + c_4\xi_2^3 + c_3\xi_2^2\xi_1 + c_4\xi_1^2\xi_2,$$

and

$$P_4(D) = P(\partial_x, \partial_y, \partial_t) = (I - d\Delta)\partial_t + c_3\partial_x^3 + c_4\partial_y^3 + c_3\partial_{yyx} + c_4\partial_{xxy}$$

we have that

$$P_4^{(3,0,0)}(D)\Psi_2 = 6c_2\Psi_2, \quad P_4^{(1,0,1)}(D)\Psi_2 = -2d\partial_x\Psi_2, \quad P_4^{(0,1,1)}(D)\Psi_2 = -2d\partial_y\Psi_2,$$

$$C(3, (3, 0, 0)) = 1, \quad C(3, (1, 0, 1)) = C(3, (0, 1, 1)) = 2.$$

Then, using Theorem 2.1 we obtain that

$$\tau^3 c_4^2 \int_{B_\delta} |\Psi_2|^2 e^{2\tau\psi} dxdydt \le \frac{2^6\tau^3}{6} \int_{B_\delta} |P_4^{(3,0,0)}(D)\Psi_2|^2 e^{2\tau\psi} dxdydt$$
$$\le \int_{B_\delta} |P_4(D)\Psi_2|^2 e^{2\tau\psi} dxdydt \tag{14}$$

and

$$\tau^2\delta^2 d^2 \int_{B_\delta} |\partial_x \Psi_2|^2 e^{2\tau\psi} dxdydt \le 2^4\tau^2\delta^2 \int_{B_\delta} |P_4^{(1,0,1)}(D)\Psi_2|^2 e^{2\tau\psi} dxdydt$$
$$\le \int_{B_\delta} |P_4(D)\Psi_2|^2 e^{2\tau\psi} dxdydt \tag{15}$$

and also that

$$\tau^2\delta^2 d^2 \int_{B_\delta} |\partial_y \Psi_2|^2 e^{2\tau\psi} dxdydt \le 2^4\tau^2\delta^2 \int_{B_\delta} |P_4^{(0,1,1)}(D)\,\Psi_2|^2 e^{2\tau\psi} dxdydt$$
$$\le \int_{B_\delta} |P_4(D)\Psi_2|^2 e^{2\tau\psi} dxdydt. \tag{16}$$

Finally, we see that

$$P_4^{(0,0,1)}(D)\,\Psi_2 = (I - d\Delta)\Psi_2, \quad C(3,(0,0,1)) = 1.$$

Then, using Theorem 2.1 we obtain that

$$\tau\delta^2 \int_{B_\delta} |\Psi_2 - d\Delta\Psi_2|^2 e^{2\tau\psi} dxdydt \le 2^2\tau\delta^2 \int_{B_\delta} |P_4^{(0,0,1)}(D)\,\Psi_2|^2 e^{2\tau\psi} dxdydt$$
$$\le \int_{B_\delta} |P_4(D)\Psi_2|^2 e^{2\tau\psi} dxdydt. \tag{17}$$

From (8)–(17), there is $C > 0$ such that

$$\begin{aligned}
&\tau^3 c_1^2 \int_{B_\delta} |\Psi_1|^2 e^{2\tau\psi} dxdydt + \tau^2\delta^2 a^2 \int_{B_\delta} |\nabla\Psi_1|^2 e^{2\tau\psi} dxdydt + (\tau^3 c_4^2 \\
&\quad + \tau^4 b^2) \int_{B_\delta} |\Psi_2|^2 e^{2\tau\psi} dxdydt \\
&\quad + (\tau^2\delta^2 d^2 + \tau^3 b^2) \int_{B_\delta} |\nabla\Psi_2|^2 e^{2\tau\psi} dxdydt + \tau\delta^2 \int_{B_\delta} |(I - d\Delta)\Psi_2|^2 e^{2\tau\psi} dxdydt \\
&\le C \int_{B_\delta} \left(|P_1(D)\Psi_1|^2 + |P_2(D)\Psi_2|^2 + |P_4(D)\Psi_2|^2\right) e^{2\tau\psi} dxdydt.
\end{aligned} \tag{18}$$

Now, we note that

$$\mathcal{L}_1 = (I - a\Delta)\partial_t + c_1\partial_x^3 + c_2\partial_y^3 + c_1\partial_{yyx} + c_2\partial_{xxy} + \big(f_1(x,y,t), f_2(x,y,t)\big)\cdot\nabla$$

implies that

$$P_1(D)\Psi_1 = \mathcal{L}_1\Psi_1 - \big(f_1(x,y,t), f_2(x,y,t)\big)\cdot\nabla\Psi_1.$$

Then, using inequalities (8)–(9), we have that

$$\begin{aligned}
&\int_{B_\delta} |\big(f_1(x,y,t), f_2(x,y,t)\big)\cdot\nabla\Psi_1|^2 e^{2\tau\psi} dxdydt \\
&\quad \le 2\big(\|f_1\|^2_{L^\infty(B_\delta)} + \|f_2\|^2_{L^\infty(B_\delta)}\big) \int_{B_\delta} \left(|\partial_x\Psi_1|^2 + |\partial_y\Psi_1|^2\right) e^{2\tau\psi} dxdydt \\
&\quad = 2\big(\|f_1\|^2_{L^\infty(B_\delta)} + \|f_2\|^2_{L^\infty(B_\delta)}\big) \int_{B_\delta} |\nabla\Psi_1|^2 e^{2\tau\psi} dxdydt
\end{aligned}$$

$$
\begin{aligned}
&\leq 2\frac{\left(\|f_1\|^2_{L^\infty(B_\delta)} + \|f_2\|^2_{L^\infty(B_\delta)}\right)}{\tau^2\delta^2 a^2}\int_{B_\delta} |P_1(D)\Psi_1|^2 e^{2\tau\psi}\,dxdydt\\
&\leq \frac{4\left(\|f_1\|^2_{L^\infty(B_\delta)} + \|f_2\|^2_{L^\infty(B_\delta)}\right)}{\tau^2\delta^2 a^2}\int_{B_\delta} \left(|\mathcal{L}_1\Psi_1|^2 + |\left(f_1, f_2\right)\cdot\nabla\Psi_1|^2\right) e^{2\tau\psi}\,dxdydt.
\end{aligned} \tag{19}
$$

In a similar way, for

$$
\mathcal{L}_4\Psi_2 = P_4(D)\Psi_2 + \left(f_4(x, y, t), f_5(x, y, t)\right)\cdot\nabla\Psi_2
$$

we obtain, using (14) and (17), that

$$
\begin{aligned}
&\int_{B_\delta} |f_3(x, y, t)\Delta\Psi_2|^2 e^{2\tau\psi}\,dxdydt \leq \|f_3\|^2_{L^\infty(B_\delta)}\int_{B_\delta} |\Delta\Psi_2|^2 e^{2\tau\psi}\,dxdydt\\
&\leq \frac{\|f_3\|^2_{L^\infty(B_\delta)}}{d^2}\int_{B_\delta} \left(|(I - c\Delta)\Psi_2|^2 + |\Psi_2|^2\right) e^{2\tau\psi}\,dxdydt\\
&\leq \frac{1}{d^2}\left(\frac{1}{\tau\delta^2} + \frac{1}{\tau^3 c_2^2}\right)\|f_3\|^2_{L^\infty(B_\delta)}\int_{B_\delta} |P_4(D)\Psi_2|^2 e^{2\tau\psi}\,dxdydt\\
&\leq \frac{2}{d^2}\left(\frac{1}{\tau\delta^2}+\frac{1}{\tau^3 c_2^2}\right)\|f_3\|^2_{L^\infty(B_\delta)}\int_{B_\delta} \left(|\mathcal{L}_4\Psi_2|^2+|\left(f_4, f_5\right)\cdot\nabla\Psi_2|^2\right) e^{2\tau\psi}\,dxdydt
\end{aligned} \tag{20}
$$

and also, using (15)–(16), we have that

$$
\begin{aligned}
&\int_{B_\delta} |\left(f_4(x, y, t), f_5(x, y, t)\right)\cdot\nabla\Psi_2|^2 e^{2\tau\psi}\,dxdydt\\
&\leq 2\left(\|f_4\|^2_{L^\infty(B_\delta)} + \|f_5\|^2_{L^\infty(B_\delta)}\right)\int_{B_\delta} |\nabla\Psi_2|^2 e^{2\tau\psi}\,dxdydt\\
&\leq \frac{2\left(\|f_4\|^2_{L^\infty(B_\delta)} + \|f_5\|^2_{L^\infty(B_\delta)}\right)}{\tau^2\delta^2 d^2}\int_{B_\delta} |P_4(D)\Psi_2|^2 e^{2\tau\psi}\,dxdydt\\
&\leq \frac{4\left(\|f_4\|^2_{L^\infty(B_\delta)} + \|f_5\|^2_{L^\infty(B_\delta)}\right)}{\tau^2\delta^2 d^2}\int_{B_\delta} \left(|\mathcal{L}_4\Psi_2|^2 + |\left(f_4, f_5\right)\cdot\nabla\Psi_2|^2\right) e^{2\tau\psi}\,dxdydt.
\end{aligned} \tag{21}
$$

Next, if we choose $\tau > 0$ large enough such that

$$
\frac{\|f_1\|^2_{L^\infty(B_\delta)} + \|f_2\|^2_{L^\infty(B_\delta)}}{\tau^2\delta^2 a^2} \leq \frac{1}{8}, \quad \frac{1}{d^2}\left(\frac{1}{\tau\delta^2} + \frac{1}{\tau^3 c_4^2}\right)\|f_3\|^2_{L^\infty(B_\delta)} \leq \frac{1}{8},
$$

and

$$\frac{\|f_4\|^2_{L^\infty(B_\delta)} + \|f_5\|^2_{L^\infty(B_\delta)}}{\tau^2\delta^2 d^2} \le \frac{1}{16},$$

then from inequalities (19)–(21) we have that

$$\begin{aligned}
&\int_{B_\delta} \left(|(f_1, f_2)\cdot\nabla\Psi_1|^2 + |f_3\Delta\Psi_2|^2 + |(f_4, f_5)\cdot\nabla\Psi_2|^2\right) e^{2\tau\psi} dxdydt \\
&\quad \le \frac{1}{2}\int_{B_\delta} \left(|\mathcal{L}_1\Psi_1|^2 + |\mathcal{L}_4\Psi_2|^2\right) e^{2\tau\psi} dxdydt \\
&\qquad + \frac{1}{2}\int_{B_\delta} \left(|(f_1, f_2)\cdot\nabla\Psi_1|^2 + |(f_4, f_5)\cdot\nabla\Psi_2|^2\right) e^{2\tau\psi} dxdydt,
\end{aligned}$$

what implies

$$\begin{aligned}
&\int_{B_\delta} \left(|(f_1, f_2)\cdot\nabla\Psi_1|^2 + |f_3\Delta\Psi_2|^2 + |(f_4, f_5)\cdot\nabla\Psi_2|^2\right) e^{2\tau\psi} dxdydt \\
&\quad \le \int_{B_\delta} \left(|\mathcal{L}_1\Psi_1|^2 + |\mathcal{L}_2\Psi_2|^2 + |\mathcal{L}_3\Psi_1|^2 + |\mathcal{L}_4\Psi_2|^2\right) e^{2\tau\psi} dxdydt \\
&\quad = \int_{B_\delta} |\mathcal{L}\Psi|^2 e^{2\tau\psi} dxdydt,
\end{aligned}$$

where

$$\begin{aligned}
&\mathcal{L}_2 = f_3\Delta - b\Delta^2, \quad \mathcal{L}_3 = I - c\Delta, \\
&|\mathcal{L}\Psi| = \left(|\mathcal{L}_1\Psi_1|^2 + |\mathcal{L}_2\Psi_2|^2 + |\mathcal{L}_3\Psi_1|^2 + |\mathcal{L}_4\Psi_2|^2\right)^{1/2}.
\end{aligned}$$

Therefore

$$\begin{aligned}
&\int_{B_\delta} \left(|P_1(D)\Psi_1|^2 + |P_2(D)\Psi_2|^2 + |P_4(D)\Psi_2|^2\right) e^{2\tau\psi} dxdydt \\
&\le 2\int_{B_\delta} \left(|\mathcal{L}_1\Psi_1|^2 + |(f_1, f_2)\cdot\nabla\Psi_1|^2\right) e^{2\tau\psi} dxdydt \\
&\quad + 2\int_{B_\delta} \left(|\mathcal{L}_2\Psi_2|^2 + |f_3\Delta\Psi_2|^2\right) e^{2\tau\psi} dxdydt \\
&\quad + 2\int_{B_\delta} \left(|\mathcal{L}_4\Psi_2|^2 + |(f_3, f_5)\cdot\nabla\Psi_2|^2\right) e^{2\tau\psi} dxdydt \le 4\int_{B_\delta} |\mathcal{L}\Psi|^2 e^{2\tau\psi} dxdydt.
\end{aligned}$$

Hence, from previous inequality and (18) we obtain the estimate (7). □

Remark 2.4 The estimate (7) is invariant under changes of signs on the components of $\mathcal{L}$.

Corollary 2.5 *Let $T > 0$. Assume that in addition to the hypotheses of the Theorem 2.3 we have that*

$$\eta, \eta_t \in L^2\left(-T, T; H^2_{loc}(\mathbb{R}^2)\right), \quad \Phi \in L^2\left(-T, T; H^4_{loc}(\mathbb{R}^2)\right),$$
$$\Phi_t \in L^2\left(-T, T; H^2_{loc}(\mathbb{R}^2)\right)$$

and the support of η and support of Φ are compact contained in B_δ. Then, the inequality (7) holds if we replace $\Psi = (\Psi_1, \Psi_2)$ by $U = (\eta, \Phi)$. Indeed,

$$\begin{aligned}
&\tau^3 c_1^2 \int_{B_\delta} |\eta|^2 e^{2\tau\psi} dxdydt + \tau^2\delta^2 a^2 \int_{B_\delta} |\nabla\eta|^2 e^{2\tau\psi} dxdydt + (\tau^3 c_4^2 \\
&\quad + \tau^4 b^2) \int_{B_\delta} |\Phi|^2 e^{2\tau\psi} dxdydt \\
&\quad + (\tau^2\delta^2 d^2 + \tau^3 b^2) \int_{B_\delta} |\nabla\Phi|^2 e^{2\tau\psi} dxdydt + \tau\delta^2 \int_{B_\delta} |(I - d\Delta)\Phi|^2 e^{2\tau\psi} dxdydt \\
&\leq C \int_{B_\delta} |\mathcal{L}U|^2 e^{2\tau\psi} dxdydt. \qquad (22)
\end{aligned}$$

Proof Let $\{\rho_\epsilon\}_{\epsilon>0}$ be a regularizing sequence (in three variables) and consider

$$U_\epsilon = (\rho_\epsilon * \eta, \rho_\epsilon * \Phi),$$

where $*$ denotes the usual convolution. Then we have that $U_\epsilon \in C_0^\infty(B_\delta) \times C_0^\infty(B_\delta)$ and the inequality (7) holds for U_ϵ, that is,

$$\begin{aligned}
&\tau^3 c_1^2 \int_{B_\delta} |\rho_\epsilon * \eta|^2 e^{2\tau\psi} dxdydt + \tau^2\delta^2 a^2 \int_{B_\delta} |\nabla(\rho_\epsilon * \eta)|^2 e^{2\tau\psi} dxdydt \\
&\quad + (\tau^3 c_4^2 + \tau^4 b^2) \int_{B_\delta} |\rho_\epsilon * \Phi|^2 e^{2\tau\psi} dxdydt \\
&\quad + (\tau^2\delta^2 d^2 + \tau^3 b^2) \int_{B_\delta} |\nabla(\rho_\epsilon * \Phi)|^2 e^{2\tau\psi} dxdydt \\
&\quad + \tau\delta^2 \int_{B_\delta} |(I - d\Delta)(\rho_\epsilon * \Phi)|^2 e^{2\tau\psi} dxdydt \leq C \int_{B_\delta} |\mathcal{L}U_\epsilon|^2 e^{2\tau\psi} dxdydt. \qquad (23)
\end{aligned}$$

Now, for $n = 0, 1$ and $m = 0, 1, 2$ we have that

$$\begin{aligned}
\|\partial_x^n(\rho_\epsilon * \eta)e^{\tau\psi} - \partial_x^n\eta\, e^{\tau\psi}\|_{L^2(B_\delta)} &= \|(\rho_\epsilon * \partial_x^n\eta)e^{\tau\psi} - \partial_x^n\eta\, e^{\tau\psi}\|_{L^2(B_\delta)} \\
&\leq C\|\partial_x^n(\rho_\epsilon * \eta) - \partial_x^n\eta\|_{L^2(B_\delta)} \to 0,
\end{aligned}$$

$$\|\partial_y^n(\rho_\epsilon * \eta)e^{\tau\psi} - \partial_y^n\eta\, e^{\tau\psi}\|_{L^2(B_\delta)} \leq C\|\partial_y^n(\rho_\epsilon * \eta) - \partial_y^n\eta\|_{L^2(B_\delta)} \to 0,$$

and

$$\|\partial_x^m(\rho_\epsilon * \Phi)e^{\tau\psi} - \partial_x^m\Phi\, e^{\tau\psi}\|_{L^2(B_\delta)} \le C\|\partial_x^m(\rho_\epsilon * \Phi) - \partial_x^m\Phi\|_{L^2(B_\delta)} \to 0,$$

$$\|\partial_y^m(\rho_\epsilon * \Phi)e^{\tau\psi} - \partial_y^m\Phi\, e^{\tau\psi}\|_{L^2(B_\delta)} \le C\|\partial_y^m(\rho_\epsilon * \Phi) - \partial_y^m\Phi\|_{L^2(B_\delta)} \to 0, \quad \text{as} \quad \epsilon \to 0^+,$$

where C is a positive constant depending only on τ and δ. Similarly we have that

$$\int_{B_\delta} \left(|\mathcal{L}U_\epsilon|^2\, e^{2\tau\psi} - |\mathcal{L}U|^2\, e^{2\tau\psi}\right) dxdydt \to 0, \quad \text{as} \quad \epsilon \to 0^+,$$

which allows us to pass to the limit in (23) to conclude the proof of Corollary 2.5. □

3 Unique Continuation

In this section, we prove the unique continuation result for the system (1). Before to do the proof, we establish the following results.

Lemma 3.1 *Let* $T > 0$ *and* $f_1, f_2, f_3, f_4, f_5 \in L^\infty_{loc}(\mathbb{R}^2 \times (-T, T))$. *Let* $U = (\eta, \Phi)$ *with*

$$\eta, \eta_t \in L^2(-T, T; H^2_{loc}(\mathbb{R}^2)), \quad \Phi \in L^2(-T, T; H^4_{loc}(\mathbb{R}^2)), \quad \Phi_t \in L^2(-T, T; H^2_{loc}(\mathbb{R}^2))$$

be a solution of $\mathcal{L}U = 0$ *in* $\mathbb{R}^2 \times (-T, T)$ *where* $\mathcal{L}$ *is the differential operator defined in* (4). *Let*

$$\widetilde{U} = \begin{cases} U & \text{if} \quad t \ge 0 \\ 0 & \text{if} \quad t < 0. \end{cases}$$

Suppose that $\widetilde{U} \equiv 0$ *in the region* $\{(x, y, t) : x < t,\ y < t\}$ *intercepted with a neighborhood of* $(0, 0, 0)$. *Then there exists a neighborhood* $\mathcal{O}_1$ *of* $(0, 0, 0)$ *(in the space* xyt*) such that* $\widetilde{U} \equiv 0$ *in* $\mathcal{O}_1$.

Proof By hypotheses there is $0 < \delta < 1$ such that $\widetilde{U} \equiv 0$ in $R_\delta = R_1 \cup R_2$, where

$$R_1 = \{(x, y, t) : x < t,\ y < t\} \cap B_\delta, \quad R_2 = \{(x, y, t) : t < 0\} \cap B_\delta,$$

$$B_\delta = \{(x, y, t) : x^2 + y^2 + t^2 < \delta^2\}.$$

Next, consider $\chi \in C_0^\infty(B_\delta)$ such that $\chi = 1$ in a neighborhood $\mathcal{O}$ of $(0, 0, 0)$ and define

$$\Psi = (\Psi_1, \Psi_2) = \chi\widetilde{U}.$$

Then we have that

$$\Psi_1, \partial_t \Psi_1 \in L^2(-T, T; H^2_{loc}(\mathbb{R}^2)),$$
$$\Psi_2 \in L^2(-T, T; H^4_{loc}(\mathbb{R}^2)), \quad \partial_t \Psi_2 \in L^2(-T, T; H^2_{loc}(\mathbb{R}^2))$$

and

$$supp\, \Psi \subset B_\delta.$$

By using the definition of χ, we note that $\mathcal{L}\Psi = 0$ in $\mathcal{O}$. Thus, using the Corollary 2.5, we have for $\psi(x, y, t) = (x-\delta)^2 + (y-\delta)^2 + \delta^2 t^2$ and $\tau > 0$ large enough that

$$\begin{aligned}
&\tau^3 c_1^2 \int_{B_\delta} |\Psi_1|^2 e^{2\tau\psi} dxdydt + \tau^2\delta^2 a^2 \int_{B_\delta} |\nabla\Psi_1|^2 e^{2\tau\psi} dxdydt \\
&\quad + (\tau^3 c_4^2 + \tau^4 b^2) \int_{B_\delta} |\Psi_2|^2 e^{2\tau\psi} dxdydt + (\tau^2\delta^2 d^2 + \tau^3 b^2) \int_{B_\delta} |\nabla\Psi_2|^2 e^{2\tau\psi} dxdydt \\
&\quad + \tau\delta^2 \int_{B_\delta} |(I - d\Delta)\Psi_2|^2 e^{2\tau\psi} dxdydt \\
&\leq C \int_{B_\delta} |\mathcal{L}\Psi|^2 e^{2\tau\psi} dxdydt = C \int_{B_\delta \setminus \mathcal{O}} |\mathcal{L}\Psi|^2 e^{2\tau\psi} dxdydt.
\end{aligned} \tag{24}$$

Now, using again the definition of χ and the fact that $\widetilde{U} \equiv 0$ in R_δ, we see that

$$supp\, \Psi \subset D, \quad supp\, \mathcal{L}\Psi \subset D \cap (B_\delta \setminus \mathcal{O}), \quad D = \{(x, y, t) : 0 \leq t \leq x, y < \delta < 1\}.$$

It follows that if $(x, y, t) \neq (0, 0, 0)$ and $(x, y, t) \in D$ then

$$\begin{aligned}
\psi(x, y, t) &= (x-\delta)^2 + (y-\delta)^2 + \delta^2 t^2 \leq (t-\delta)^2 + \delta^2 t^2 \\
&= t^2(2+\delta^2) - 4t\delta + \delta^2 < 2\delta^2.
\end{aligned}$$

Thus, there exists $0 < \epsilon < 2\delta^2$ such that

$$\psi(x, y, t) \leq 2\delta^2 - \epsilon, \quad (x, y, t) \in D \cap (B_\delta \setminus \mathcal{O}).$$

Moreover, since $\psi(0, 0, 0) = 2\delta^2$, we can choose $\mathcal{O}_1 \subset \mathcal{O}$ a neighborhood of $(0, 0, 0)$ such that

$$\psi(x, y, t) > 2\delta^2 - \epsilon, \quad (x, y, t) \in \mathcal{O}_1.$$

From the above construction and inequality (24), we have that there exists $C_1 > 0$ such that

$$
\begin{aligned}
\tau^3 e^{2\tau(2\delta^2-\epsilon)} \int_{\mathcal{O}_1} \left(|\Psi_1|^2+|\Psi_2|^2\right) dxdydt &\le \tau^3 \int_{\mathcal{O}_1} \left(|\Psi_1|^2+|\Psi_2|^2\right) e^{2\tau\psi} dxdydt \\
&\le \tau^3 \int_{B_\delta} \left(|\Psi_1|^2+|\Psi_2|^2\right) e^{2\tau\psi} dxdydt \\
&\le C_1 \int_{B_\delta\setminus\mathcal{O}} |\mathcal{L}\Psi|^2 e^{2\tau\psi} dxdydt \\
&\le C_1 e^{2\tau(2\delta^2-\epsilon)} \int_{B_\delta\setminus\mathcal{O}} |\mathcal{L}\Psi|^2 dxdydt.
\end{aligned}
$$

Therefore

$$
\int_{\mathcal{O}_1} \left(|\Psi_1|^2+|\Psi_2|^2\right) dxdydt \le \frac{C_1}{\tau^3} \int_{B_\delta\setminus\mathcal{O}} |\mathcal{L}\Psi|^2 dxdydt.
$$

Then, passing to the limit as $\tau \to +\infty$, we have that $\Psi \equiv 0$ in $\mathcal{O}_1$. Since $\widetilde{U} = \Psi$ in $\mathcal{O}$ and $\mathcal{O}_1 \subset \mathcal{O}$, we see that $\widetilde{U} = 0$ in $\mathcal{O}_1$. □

Similarly, we also have the following result.

Lemma 3.2 *Let $T > 0$ and $f_1, f_2, f_3, f_4, f_5 \in L^\infty_{loc}(\mathbb{R}^2 \times (-T, T))$. Let $U = (\eta, \Phi)$ with*

$$
\eta, \eta_t \in L^2(-T, T; H^2_{loc}(\mathbb{R}^2)), \quad \Phi \in L^2(-T, T; H^4_{loc}(\mathbb{R}^2)), \quad \Phi_t \in L^2(-T, T; H^2_{loc}(\mathbb{R}^2))
$$

be a solution of $\mathcal{L}U = 0$ in $\mathbb{R}^2 \times (-T, T)$ where $\mathcal{L}$ is the differential operator defined in (4). Let

$$
\widetilde{U} = \begin{cases} 0 & \text{if } \ t \ge 0 \\ U & \text{if } \ t < 0. \end{cases}
$$

Suppose that $\widetilde{U} \equiv 0$ in the region $\{\,(x, y, t) \ : \ x < -t, \ y < -t\,\}$ intercepted with a neighborhood of $(0, 0, 0)$. Then there exists a neighborhood $\mathcal{O}_2$ of $(0, 0, 0)$ (in the space xyt) such that $\widetilde{U} \equiv 0$ in $\mathcal{O}_2$.

Corollary 3.3 *Let $T > 0$ and $F_1, F_2, F_3, F_4, F_5 \in L^\infty_{loc}(\mathbb{R}^2 \times (-T, T))$. Let $U = (\eta, \Phi)$ with*

$$
\begin{aligned}
&\eta, \eta_t \in L^2(-T, T; H^2_{loc}(\mathbb{R}^2)), \quad \Phi \in L^2(-T, T; H^4_{loc}(\mathbb{R}^2)), \\
&\Phi_t \in L^2(-T, T; H^2_{loc}(\mathbb{R}^2))
\end{aligned}
$$

be a solution in $\mathbb{R}^2 \times (-T, T)$ of the system

$$
\begin{cases}
(I - a\Delta)\eta_t - b\Delta^2\Phi + \big(F_1(x, y, t), F_2(x, y, t)\big) \cdot \nabla\eta + F_3(x, y, t)\Delta\Phi = 0, \\
\qquad (I - c\Delta)\Phi_t + \eta - d\Delta\eta + \big(F_4(x, y, t), F_5(x, y, t)\big) \cdot \nabla\Phi = 0.
\end{cases}
$$

Let γ be a sphere passing through the origin $(0, 0, 0)$*. Suppose that* $U \equiv 0$ *in the interior of* γ *in a neighborhood of* $(0, 0, 0)$*. Then, there exists a neighborhood of* $(0, 0, 0)$ *where* $U \equiv 0$*.*

Proof Let us assume that the sphere (a piece of it) γ is given by $(x, y) = (g_1(t), g_2(t))$. By using the hypotheses, we have that $U \equiv 0$ in the region $\{(x, y, t) : x < g_1(t),\ y < g_2(t)\}$ intercepted with a neighborhood of $(0, 0, 0)$. Then, we can to see that there exists $\omega_1, \omega_2 \in \mathbb{R} \setminus \{0, 1\}$ such that $U \equiv 0$ in a neighborhood of $(0, 0, 0)$ intercepted with the region $\{(x, y, t) : x < h_1(t),\ y < h_2(t)\}$ where

$$h_j(t) = \begin{cases} \omega_j t & \text{if} \quad t \geq 0, \quad j = 1, 2 \\ -\frac{1}{\omega_j} t & \text{if} \quad t < 0, \quad j = 1, 2. \end{cases}$$

Now, we consider the following change of variables $(x, y, t) \to (X, Y, T)$ with

$$\begin{aligned} X &= x - h_1(t) + |t| \\ Y &= y - h_2(t) + |t| \\ T &= t. \end{aligned}$$

Notice that in the new variables, if $T \geq 0$ then the function

$$U = U(X, Y, T) = (\eta(X, Y, T), \Phi(X, Y, T))$$

is a solution of the system

$$\begin{cases} (I - a\Delta)\eta_T - b\Delta^2\Phi + a\big((\omega_1 - 1)\partial_X^3\eta + (\omega_2 - 1)\partial_Y^3\eta + (\omega_2 - 1)\partial_{XXY}\eta \\ +(\omega_1 - 1)\partial_{YYX}\eta\big) + (1 - \omega_1 + F_1, 1 - \omega_1 + F_2) \cdot \nabla\eta + F_3\Delta\Phi = 0, \\ \\ (I - c\Delta)\Phi_T + \eta - d\Delta\eta + c\big((\omega_1 - 1)\partial_X^3\Phi + (\omega_2 - 1)\partial_Y^3\Phi + (\omega_2 - 1)\partial_{XXY}\Phi \\ +(\omega_1 - 1)\partial_{YYX}\Phi\big) + (1 - \omega_1 + F_4, 1 - \omega_2 + F_5) \cdot \nabla\Phi = 0. \end{cases}$$

Then, $U \equiv 0$ in the region $\{(X, Y, T) : X < T,\ T < T\ T \geq 0\}$ intercepted with a neighborhood of $(0, 0, 0)$ and U satisfies

$$\mathcal{L}U = 0 \quad \text{if} \quad T \geq 0,$$

where

$$\mathcal{L} = \begin{pmatrix} P_1(\partial_X, \partial_Y, \partial_T) + \big(f_1, f_2\big) \cdot \nabla & P_2(\partial_X, \partial_Y, \partial_T) + f_3\Delta \\ \\ P_3(\partial_X, \partial_Y, \partial_T) & P_4(\partial_X, \partial_Y, \partial_T) + \big(f_4, f_5\big) \cdot \nabla \end{pmatrix}$$

with

$$P_1(\partial_X, \partial_Y, \partial_T) = (I - a\Delta)\partial_T + c_1\partial_X^3 + c_2\partial_Y^3 + c_1\partial_{YYX} + c_2\partial_{XXY},$$

$$P_2(\partial_X, \partial_Y, \partial_T) = -b\Delta^2, \quad P_3(\partial_X, \partial_Y, \partial_T) = I - d\Delta,$$

and

$$P_4(\partial_X, \partial_Y, \partial_T) = (I - c\Delta)\partial_T + c_3\partial_X^3 + c_4\partial_Y^3 + c_3\partial_{YYX} + c_4\partial_{XXY},$$

and also

$$c_1 = a(\omega_1 - 1), c_2 = a(\omega_2 - 1), c_3 = c(\omega_1 - 1), c_4 = c(\omega_2 - 1),$$

$$f_1 = 1 - \omega_1 + F_1, f_2 = 1 - \omega_2 + F_2, \quad f_3 = F_3, f_4 = 1 - \omega_1 + F_4, f_5 = 1 - \omega_2 + F_5.$$

So, using Lemma 3.1 with the previous differential operator $\mathcal{L}$, we obtain that there exists a neighborhood $\mathcal{O}_1$ of $(0, 0, 0)$ in the space XYT where $U \equiv 0$.

In a similar fashion, $U \equiv 0$ in the region $\{(X, Y, T) : X < -T, \ Y < -T, \ T < 0\}$ intercepted with a neighborhood of $(0, 0, 0)$ and U satisfies

$$\mathcal{L}U = 0 \quad \text{if} \quad T < 0,$$

where

$$c_1 = a\left(1 - \frac{1}{\omega_1}\right), \ c_2 = a\left(1 - \frac{1}{\omega_2}\right), c_3 = c\left(1 - \frac{1}{\omega_1}\right), \quad c_4 = c\left(1 - \frac{1}{\omega_2}\right),$$

$$f_1 = \frac{1}{\omega_1} - 1 + F_1, \quad f_2 = \frac{1}{\omega_1} - 1 + F_2, \quad f_3 = F_3, \quad f_4 = \frac{1}{\omega_1} - 1 + F_4, \quad f_5 = \frac{1}{\omega_2} - 1 + F_5.$$

Then, from Lemma 3.2 we have that there exists a neighborhood $\mathcal{O}_2$ of $(0, 0, 0)$ in the space XYT where $U \equiv 0$. Thus, returning to the original variables (x, y, t) we have the result. □

Now we have the main result on the unique continuation property for the system (1).

Theorem 3.4 *Let $T > 0$ and $(\eta, \Phi) = (\eta(x, y, t), \Phi(x, y, t))$ with*

$$\eta, \eta_t \in L^2(-T, T; H^2_{loc}(\mathbb{R}^2)), \quad \Phi \in L^2(-T, T; H^4_{loc}(\mathbb{R}^2)), \quad \Phi_t \in L^2(-T, T; H^2_{loc}(\mathbb{R}^2))$$

be a solution in $\mathbb{R}^2 \times (-T, T)$ of the system (1). If $(\eta, \Phi) \equiv 0$ in an open subset Ω of $\mathbb{R}^2 \times (-T, T)$, then $(\eta, \Phi) \equiv 0$ in the horizontal component of Ω.

Proof By defining the functions

$$F_1(x,y,t)=\epsilon\partial_x\Phi,\quad F_2(x,y,t)=\epsilon\partial_y\Phi,\quad F_3(x,y,t)=1+\epsilon\eta,$$

and

$$F_4(x,y,t)=\tfrac{\epsilon}{2}\partial_x\Phi,\quad F_5(x,y,t)=\tfrac{\epsilon}{2}\partial_y\Phi,$$

the system (1) takes the form

$$\begin{cases}(I-a\Delta)\eta_t-b\Delta^2\Phi+\big(F_1,F_2\big)\cdot\nabla\eta+F_3\Delta\Phi=0,\\[2ex] \quad (I-c\Delta)\Phi_t+\eta-d\Delta\eta+\big(F_4,F_5\big)\cdot\nabla\Phi=0,\end{cases}\tag{25}$$

with $F_1, F_2, F_3, F_4, F_5 \in L^{\infty}_{loc}(\mathbb{R}^2\times(-T,T))$ and $a=c=\frac{\mu}{2}$, $b=\frac{2\mu}{3}$, $d=\mu\sigma$. Then, we will show the result for the system (25).

Denote by Ω_1 the horizontal component of Ω and let

$$\Lambda=\{(x,y,t)\in\Omega_1\ :\ (\eta,\Phi)\equiv 0\quad\text{in a neighborhood of}\quad(x,y,t)\}.$$

Let $Q\in\Omega_1$ arbitrary. Choose $P\in\Lambda$ and let Γ be a continuous curve contained in Ω_1 joining P to Q, parametrized by a continuous function $f:[0,1]\to\Omega_1$ with $f(0)=P$ and $f(1)=Q$. Since $P\in\Lambda$, there exists $r>0$ such that

$$(\eta,\Phi)\equiv 0\quad\text{in}\quad B_r(P).\tag{26}$$

Taking $0<r_0<\min\{r, dist(\Gamma,\partial\Omega_1)\}$, where $\partial\Omega_1$ denotes the boundary of Ω_1, we have that

$$B_{r_0}(P)\subset\Lambda.$$

Now, if $r_1<\frac{r_0}{4}$ we see that

$$B_{2r_1}(f(s))\subset\Omega_1,\qquad\text{for all } s\in[0,1];\tag{27}$$

in fact, if $w\in B_{2r_1}(f(s))$ and $w\notin\Omega_1$ then

$$\|w-f(s)\|<2r_1<r_0<dist(\Gamma,\partial\Omega_1)\le\|w-f(s)\|,$$

which is a contradiction.

Next, let

$$\Lambda_1=\{(x,y,t)\in\Lambda\ :\ (\eta,\Phi)\equiv 0\quad\text{in}\quad B_{r_1}(x,y,t)\cap\Omega_1\}$$

and

$$S=\{0\le \ell\le 1\ :\ f(s)\in\Lambda_1\quad\text{whenever}\quad 0\le s\le \ell\},\quad \ell_0=\sup S.$$

We will prove that $f(\ell_0)\in\Lambda_1$. If $w\in B_{r_1}(f(\ell_0))$ and $r_2=\|w-f(\ell_0)\|$ then there exists $0<\delta<\ell_0$ such that $\|f(\ell_0)-f(\ell_0-\delta)\|<r_1-r_2$. Therefore

$$\|w-f(\ell_0-\delta)\|\le\|w-f(\ell_0)\|+\|f(\ell_0)-f(\ell_0-\delta)\|<r_1,$$

and so $w\in B_{r_1}(f(\ell_0-\delta))$. Now, from the definition of ℓ_0 there exists $\ell_\delta\in S$ such that $\ell_0-\delta<\ell_\delta\le\ell_0$, what implies $f(\ell_0-\delta)\in\Lambda_1$. Then, using (27) we see that

$$(\eta,\Phi)\equiv 0\quad\text{in}\quad B_{r_1}(f(\ell_0-\delta))\cap\Omega_1=B_{r_1}(f(\ell_0-\delta)).\tag{28}$$

Consequently we obtain that $(\eta(w),\Phi(w))=0$ and then

$$(\eta,\Phi)\equiv 0\quad\text{in}\quad B_{r_1}(f(\ell_0)).\tag{29}$$

Hence, we have showed $f(\ell_0)\in\Lambda_1$.

If $\ell_0=1$ then from previous analysis we have that $Q=f(1)\in\Lambda_1\subset\Lambda$. Thus, since Q was arbitrarily chosen we obtain that $(\eta,\Phi)\equiv 0$ in Ω_1, which proves Theorem 3.4. Then to finish the proof of Theorem 3.4 remains to prove that $\ell_0=1$. In fact, let us suppose that $\ell_0<1$ and let

$$G=\{Z\in\Omega_1\ :\ \|Z-f(\ell_0)\|=r_1\}.$$

For $w=(x_1,y_1,t_1)\in G$ fixed, we consider the change of variable $(x,y,t)\to(X,Y,T)$ where

$$\begin{aligned}X&=x-x_1,\\ Y&=y-y_1,\\ T&=t-t_1.\end{aligned}$$

Notice that $(0,0,0)\in G^*=\{Z=(X,Y,T)\ :\ \|Z-(f(\ell_0)-w)\|=r_1\}$. Moreover, from (29) we see that

$$(\eta(X,Y,T),\Phi(X,Y,T))=0,\quad (X,Y,T)\in B_{r_1}(f(\ell_0)-w).$$

So that, by using Corollary 3.3, there exists $r_w^*>0$ such that

$$(\eta(X,Y,T),\Phi(X,Y,T))=0,\quad (X,Y,T)\in B_{r_w^*}(0,0,0).$$

Returning to the original variables we have that for each $w\in G$ there exists $r_w^*>0$ such that

$$(\eta,\Phi)\equiv 0\quad\text{in}\quad B_{r_w^*}(w).$$

Then, using (29) and the compactness of G, we have that there is $\epsilon_1 > 0$ such that

$$(\eta, \Phi) \equiv 0 \quad \text{in} \quad B_{r_1+\epsilon_1}(f(\ell_0)). \tag{30}$$

Now, we note that there exists $0 < \delta_1 < 1 - \ell_0$ such that if $w \in B_{r_1}(f(\ell_0 + \delta_1))$ then

$$\|w - f(\ell_0)\| \leq \|w - f(\ell_0 + \delta_1)\| + \|f(\ell_0 + \delta_1) - f(\ell_0)\| < r_1 + \epsilon_1.$$

Thus, $w \in B_{r_1+\epsilon_1}(f(\ell_0))$ and so $B_{r_1}(f(\ell_0 + \delta_1)) \subset B_{r_1+\epsilon_1}(f(\ell_0))$. Therefore, using (30) we have that $(\eta, \Phi) \equiv 0$ in $B_{r_1}(f(\ell_0 + \delta_1))$. Consequently $f(\ell_0 + \delta_1) \in \Lambda_1$, which contradicts the definition of ℓ_0. So, $\ell_0 = 1$ and the proof of Theorem 3.4 is complete. □

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Declarations

Conflict of interest The authors declare that they have no conflict of interest.

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