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On the Location of Roots of the Characteristic Polynomial of (p, q)-Distance Fibonacci Sequences

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Abstract

Let p, q, k and ℓ be positive integers. The (p, q, k, ℓ)-Fibonacci sequence $(F_{k,\ell,p,q})_{n\geq 0}$ is the four-parameter sequence defined by the following recurrence

$$F_{k,\ell,p,q}(n) = kF_{k,\ell,p,q}(n-p) + \ell F_{k,\ell,p,q}(n-q),$$

with appropriate initial conditions. In this paper, we study the geometric, algebraic, and analytic aspects of the roots of the characteristic polynomial of this sequence, namely, $f(x) = x^q - kx^{q-p} - \ell$.

Keywords Generalized Fibonacci sequence · Linear recurrence sequence · Characteristic polynomial · Eneström–Kakeya theorem · Descartes' sign rule · Rouché's theorem.

1 Introduction

The Fibonacci sequence $(F_n)_n$ (which is defined by the recurrence $F_n = F_{n-1} + F_{n-2}$, with $F_0 = 0$ and $F_1 = 1$) is probably the most known example of a recurrence sequence. Many generalizations (in many directions) of this sequence have appeared in the literature. For example, for integers a and b, the $U_n(a, b)$ *Lucas sequence* is defined by the recurrence $U_n(a, b) = aU_{n-1}(a, b) - bU_{n-2}(a, b)$. Despite being another

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sequence, any term of a Lucas sequence is still a linear combination of the preceding two ones. In Włoch et al. (2013), Wloch et al. provided another generalization by considering linear combination of two previous more "distant" terms. In fact, they defined the $(2,k)$-*Fibonacci numbers* (or $(2,k)$-*distance Fibonacci numbers*) by the recurrence relation $F_2(k,n) = F_2(k,n-2)+F_2(k,n-k)$, for $n \geq k$, with $F_2(k,n) = 1$ for $n \in [0, k-1]$.

In this work, we are interested in the following four-parameter recurrence which was defined in da Silva et al. (2018): let p, q, k and ℓ be positive integers, with $q > p$, the linear recurrence $F_{k,\ell,p,q} := (F_{k,\ell,p,q})_{n\geq 1}$ is defined by

$$F_{k,\ell,p,q}(n) = kF_{k,\ell,p,q}(n-p) + \ell F_{k,\ell,p,q}(n-q). \tag{1}$$

We note that for any multiset $\sigma = \{a_1, \ldots, a_q\}$ of integers, there is a unique linear recurrence sequence, say $F^{(\sigma)}_{k,\ell,p,q}(n)$, which satisfies (1) and with initial values $F^{(\sigma)}_{k,\ell,p,q}(i) = a_i$, for $i \in [1,q]$ (we call $F^{(\sigma)}_{k,\ell,p,q}(n)$ the sequence of (p,q,k,ℓ)-*Fibonacci numbers with initial values in* σ).

This is a q-order recurrence sequence and it generalizes some famous sequences such as the *Fibonacci*, *Lucas*, *Pell*, *Jacobsthal*, *Padovan*, and *Narayana sequences*. However, the most notable example (in the sense in which its recurrence is not "complete") is the sequence of *Perrin numbers* $(P_n)_n$ (OEIS A001608), defined by the recurrence

$$P_n = P_{n-2} + P_{n-3},$$

with initial conditions $P_0 = 3$, $P_1 = 0$ and $P_2 = 2$. We refer the reader to da Silva et al. (2018) (and references therein) for more information about these sequences.

In a general vein, a sequence $(u_n)_n$ is an *s order homogeneous linear recurrence sequence* if

$$u_{n+s} = c_{s-1}u_{n+s-1} + \cdots + c_0 u_n,$$

for some complex numbers $c_0, \ldots, c_{s-1}$, with $c_0 \neq 0$ (the recurrence is said to be *complete* if all these numbers are non-zero). We call the polynomial

$$x^s - c_{s-1}x^{s-1} - \cdots - c_1 x - c_0$$

the *characteristic polynomial* of $(u_n)_{n\geq 0}$ and its roots

$$\alpha_1, \ldots, \alpha_s, \text{ numbered such that } |\alpha_1| \geq \cdots \geq |\alpha_s|,$$

the *roots* of $(u_n)_n$. We say that $(u_n)_n$ has a *dominant root* if $|\alpha_1| > |\alpha_2|$. Also, this polynomial depends only on the recurrence, for example, the characteristic polynomial of the Fibonacci and Lucas sequences is the same, namely, $x^2 - x - 1$. For this reason, we suppress the explicit dependence on σ in the notation of $F^{(\sigma)}_{k,\ell,p,q}(n)$.

A classical result on linear recurrence sequences asserts that (u_n) has the "non-recurrent" formula:

$$u_n = g_1(n)\alpha_1^n + g_2(n)\alpha_2^n + \cdots + g_\ell(n)\alpha_\ell^n, \quad \text{for all } n, \tag{2}$$

where $g_j(n)$ is a polynomial with degree at most $m_j - 1$ (see Shorey and Tijdeman 1986, Theorem C.1). For the Fibonacci sequence, one has the Binet's formula $F_n = (\alpha^n - \beta^n)/\sqrt{5}$ (where $\alpha := (1+\sqrt{5})/2$ and $\beta = -1/\alpha$).

The study of behavior of the roots of the characteristic polynomial of a recurrence (which gives information about its asymptotic behavior) has a very long history, and it became more popular after the seminal works of Baker on effective lower bounds for linear forms in logarithms. For example, as a consequence of the *Baker's method* (or *transcendental method*) we have:

Theorem 1 *Suppose that* $(u_n))n$ *is a sequence of integers of the form*

$$u_n = a\alpha^n + O(|\alpha|^{n\theta}), \quad \textit{with } \theta \in (0,1), \tag{3}$$

where a *and* α *are non-zero algebraic numbers, with* $|\alpha| > 1$ *and such that* $u_n - a\alpha^n \neq 0$ *for all* n. *Then there exist only finitely many (effectively computable) perfect powers belonging to* (u_n).

The proof of this theorem can be found in Theorem 3.10 in Bugeaud's book (Bugeaud 2018) (we also refer this book to the reader for an introduction to Baker's method together with a large variety of its applications).

We remark that Theorem 1 is applicable to Fibonacci numbers, since, by Binet's formula, one has that $F_n = \alpha^n/\sqrt{5} + O(1)$ (in fact, we remark that the equation $F_n = y^p$ was solved completely in 2003, by Bugeaud (2006, Theorem 1).

Furthermore, for a linear recurrence $(u_n)_n$ to have the form as in (3), it suffices that its characteristic polynomial has a dominant root and at least one of the following conditions is true:

(i) All other roots of $(u_n)_n$ lie inside the unit circle (i.e., α is a *Pisot number*).
(ii) All roots of $(u_n)_n$ are simple.

In fact, by (2), if α_1 is a dominant root and its multiplicity is 1, then the *dominant polynomial* $g_1(n)$ has degree $m_1 - 1 = 0$. So, $g_1(n) = g_1$ is a constant. Since $\max_{j\in[2,\ell]}\{|\alpha_j|\} < |\alpha_1|$, then we have

$$u_n = g_1\alpha^n(1 + o(1)),$$

because $|g_j(n)/\alpha_1^n|$ and $|\alpha_j/\alpha_1|^n$ tend to zero as $n \to \infty$ (which correspond to items (i) and (ii), respectively).

There are some classical works concerning the study of roots of (p, q, k, ℓ)-Fibonacci sequences. For instance, in 1950, Dickinson (1950) proved that all roots are simple for the case $(p, q, 1, 1)$ and, in 1963, Raab (1963) showed the same for the case $(1, q, k, \ell)$.

Thus, the aim of this work is to continue this program by working on the general case (p, q, k, ℓ). For simplicity, we shall denote this polynomial only as $f(x)$ (i.e., we shall suppress the explicit dependence on p, q, k and ℓ in the notation). Our first result is related to the existence of a real dominant root as well as its location:

Theorem 2 *Let p, q, k and ℓ be positive integers with $q > p \geq 1$ and $\gcd(p, q) = 1$. Then the polynomial $f(x) = x^q - kx^{q-p} - \ell$ has a dominant root α. Moreover,*

$$k^{1/p}\left(1 + \frac{1}{q}\log^+\left(\frac{\ell}{k^{q/p}}\right)\right) < \alpha < k^{1/p}\left(1 + \left(\frac{\ell}{p(q-p)k^{q/p}}\right)^{1/2}\right), \tag{4}$$

where $\log^+(x) := \max\{\log x, 0\}$.

Remark 1 In the previous result, the condition $\gcd(p, q) = 1$ is necessary. In fact, if $p = dm$ and $q = dn$ for some $d > 1$, then by the change of variable $y = x^d$, we can write $f(x) = 0$ as $y^n - ky^{n-m} - \ell = 0$. Thus, any solution of the previous equation is a dth power and therefore they have the same absolute value. Thus, there is no a dominant root in this case. For example, if $p = 6, q = 3, k = 2$ and $\ell = 3$, then $f(x) := x^6 - 2x^3 - 3 = (x^3 - 3)(x^3 + 1)$ and so all the roots of $f(x)$ have absolute values equal to $\sqrt[3]{3}$ and 1 (three roots for each one of these two values). In particular, $f(x)$ does not have a dominant root.

Remark 2 Note that many growth properties of the dominant root of $f(x)$ follow from the inequality (4). For instance, one has that

- α tends to 1 as $p \to \infty$ (k and ℓ are held fixed).
- α tends to $k^{1/p}$ as $q \to \infty$ (p, k and ℓ are held fixed).
- α tends to infinity as $k \to \infty$ (ℓ is held fixed).
- α tends to infinity as $\ell \to \infty$ (q is held fixed).

The next result provides a criterion for the simplicity of the roots of $f(x)$. More precisely,

Theorem 3 *Let p, q, k and ℓ be positive integers with $q > p \geq 1$ and $\gcd(p, q) = 1$. Suppose that one of the following conditions is satisfied*:

(i) *The number p is odd.*
(ii) *The number p is even and*

$$\ell^p \neq \frac{(q-p)^{q-p}k^q p^p}{q^q}. \tag{5}$$

Then the polynomial $f(x) = x^q - kx^{q-p} - \ell$ does not have multiple roots. Furthermore, the only possible double root is $-(k(q-p)/q)^{1/p}$.

A consequence of the previous theorem is:

Corollary 4 *Let p, q, k and ℓ be positive integers with $q > p \geq 1$ and $\gcd(p, q) = 1$. Then all roots of $f(x) = x^q - kx^{q-p} - \ell$ are simple if one of the following conditions is satisfied*:

(i) *If p and q are odd numbers.*
(ii) *If either p does not divide ℓ or q does not divide k.*
(iii) *If ℓ/p and k/q are integers but k/q does not divide ℓ/p.*

In particular, all roots of $f(x)$ are simple if either $k = 1$ or $\ell = 1$.

Remark 3 In the statement of Theorem 3, the technical condition in (5) is necessary. In fact, any polynomial constructed using p, q, k and ℓ which do not satisfy that condition, will have multiple roots. For example, $f(x) = x^5 - 15x^3 - 162$ has a double root at $x = -3$.

We finish our study with a characterization of the location (in $\mathbb{C}$) of roots of $f(x)$ when q tends to infinity, but $q - p$ remains constant, say r (note that, in this case, the dominant root tends to 1 as $q \to \infty$). By using MATHEMATICA software, we observed that the set of roots of $f_q(x) := x^q - kx^r - \ell$ (which we shall denote as $\mathcal{R}_{f_q}$) has an interesting disposal on the complex plane when q increases. In fact, this agrees with the following special case of a result due to Erdös and Turán (see Granville 2007, p. 94–95): *Suppose that $(g_q)_q$ is a sequence of polynomials with fixed coefficients and such that* $\deg(g_q) = q$. *Then, for any* $\epsilon > 0$, *one has*

$$\lim_{q\to\infty} \frac{\#\{z \in \operatorname{ann}(0; 1-\epsilon, 1+\epsilon)) : g_q(z) = 0\}}{\#\{z \in \mathbb{C} : g_q(z) = 0\}} = 1, \tag{6}$$

where the annulus $\operatorname{ann}(z_0; r_1, r_2))$ *is the set* $\{z \in \mathbb{C} : r_1 < |z - z_0| < r_2\}$. In other words, almost all roots of $g_q(x)$ tend to the boundary of the unit circle as $q \to \infty$.

Therefore, the same is valid for the family of polynomials $(f_q)_q$. However, we want to make these quantities explicit by proving that for any $\epsilon > 0$, one has

$$\#\{z \in \operatorname{ann}(0; 1-\epsilon, 1+\epsilon)) : f_q(z) = 0\} = q - r$$

for all q sufficiently large. Since $f_q(x)$ has exactly q roots (by Corollary 4(ii)), then the limit in (6) becomes

$$\lim_{q\to\infty} \frac{\#\{z \in \operatorname{ann}(0; 1-\epsilon, 1+\epsilon)) : f_q(z) = 0\}}{\#\{z \in \mathbb{C} : f_q(z) = 0\}} = \lim_{q\to\infty} \frac{q-r}{q} = 1.$$

Moreover, we find a very interesting geometric pattern for the remaining r roots. We point out that these patterns are illustrated in the next three figures. The first two ones (Figs. 1 and 2) concern the case $k > \ell$:

In these two figures and in their four cases, we have r roots (in red) inside the unit circle, while the other $q - r$ roots (in blue) accumulate on the boundary of that circle. Furthermore, the red points seem to be converging to the vertices of a regular r-sided polygon.

For the case in which $k \le \ell$, all roots seem to be converging to the boundary of the unit circle. See Fig. 3.

These interpretations are confirmed in the following result:

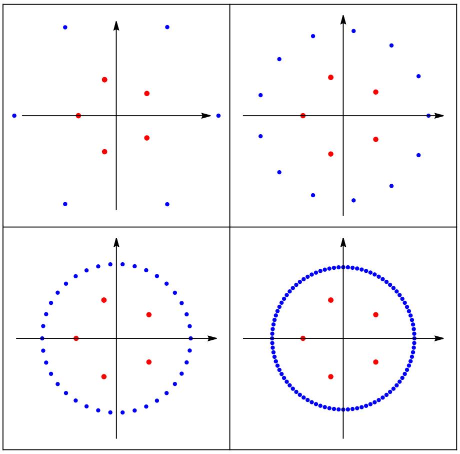

Fig. 1 Roots of $x^q - 15x^5 - 1$, for $q = 11$, $q = 18$, $q = 43$ and $q = 101$, respectively

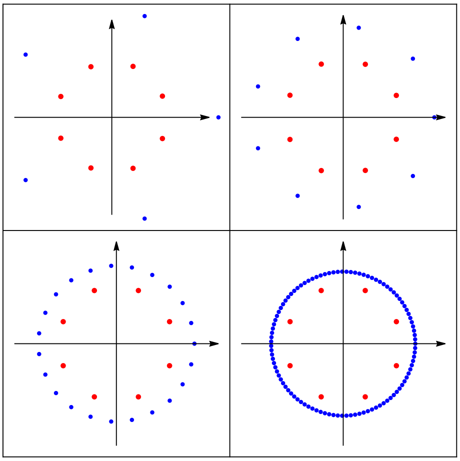

Fig. 2 Roots of $x^q - 10x^8 - 2$, for $q = 13$, $q = 17$, $q = 31$ and $q = 111$, respectively

Theorem 5 *Let r, k and ℓ be positive integers. For an integer $q > \max\{k, r\}$ with $\gcd(q, r) = 1$, set $f_q(x) := x^q - kx^r - \ell$. We have*

(i) *If $k > \ell$, then*

$$\mathcal{R}_{f_q} = \left\{\alpha_1^{(q)}, \ldots, \alpha_{q-r}^{(q)}\right\} \cup \left\{\beta_1^{(q)}, \ldots, \beta_r^{(q)}\right\}$$

has cardinality q and $|\alpha_j^{(q)}|$ tends to 1, while each $\beta_j^{(q)}$ tends to an r-th root of $-\ell/k$ as $q \to \infty$.

(ii) *If $\ell \geq k$, then*

$$\mathcal{R}_{f_q} = \left\{\alpha_1^{(q)}, \ldots, \alpha_q^{(q)}\right\}$$

has cardinality q and $|\alpha_j^{(q)}|$ tends to 1 as $q \to \infty$.

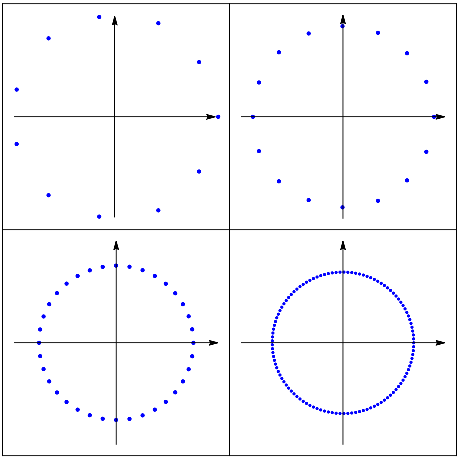

Fig. 3 Roots of $x^q - 2x^5 - 100$, for $q = 11$, $q = 16$, $q = 36$ and $q = 113$, respectively

Remark 4 In particular, the previous theorem (item (i)) implies that the set $\{\beta_1^{(q)}, \ldots, \beta_q^{(q)}\}$ "tends to" the set $\{(\ell/k)^{1/r} \exp((2j+1)\pi/r) : j \in [0, r-1]\}$ of the rth complex roots of $-\ell/k$ as $q \to \infty$. In particular, the numbers $\beta_1^{(q)}, \ldots, \beta_q^{(q)}$ approximate to the vertices of the regular r-sided polygon inscribed in the circle centered at origin with radius $(\ell/k)^{1/r}$.

2 Auxiliary Results

In this section, we shall present some results which will be essential ingredients in the proof of our results. For clarity, we record some notations. As usual, $[a, b]$ denotes the set $\{a, a+1, \ldots, b\}$, for integers $a < b$. Also, $B(z_0, r) = \{z \in \mathbb{C} : |z - z_0| < r\}$ denotes the open ball of radius r with center at z_0, $\partial\Omega$ is the boundary of the set Ω and $B[z_0, r] = B(z_0, r) \cup \partial B(z_0, r)$. To finish, $\mathcal{R}_g$ denotes the set of all complex zeros of the polynomial $g(x)$.

The first tool is the famous *Descartes' sign rule* which gives an upper bound on the number of positive or negative real roots of a polynomial with real coefficients. For the sake of completeness, we shall state it as a lemma.

Lemma 6 (Descartes' sign rule) *Let $f(x) = a_{n_1}x^{n_1} + \cdots + a_{n_k}x^{n_k}$ be a polynomial with nonzero real coefficients and such that $n_1 > \cdots > n_k \geq 0$. Set*

$$\nu := \#\{i \in [1, k-1] : a_{n_i}a_{n_{i+1}} < 0\}.$$

Then, there exists a non-negative integer r such that $\#\mathcal{R}_f = \nu - 2r$ (multiple roots of the same value are counted separately).

As a corollary, we have that for obtaining information on the number of negative real roots, we must apply the previous rule for $f(-x)$.

Another useful and essential result is due to Eneström (1920) and Kakeya (1912) which provides information on the size of the roots of a polynomial depending on the ordering of its coefficients:

Lemma 7 (Eneström–Kakeya theorem) *Let* $f(x) = a_0 + a_1x + \cdots + a_nx^n$ *be an* n*-degree polynomial with real coefficients. If* $0 \le a_0 \le a_1 \le \cdots \le a_n$, *then all zeros of* $f(x)$ *lie in* $B[0, 1]$.

The following two lemmas are from complex analysis. In what follows, $\mathcal{Z}(f : \Omega)$ denotes the set of zeros of $f(z)$ belonging to $\Omega \subseteq \mathbb{C}$. The first lemma is known *Rouché's theorem* which is very useful to simplify the problem of locating zeros of holomorphic functions (see Conway 1973, Theorem 3.8). More precisely:

Lemma 8 (Rouché's theorem) *Let* $g, h : \Omega \to \mathbb{C}$ *be analytic functions in a region* Ω *(open and connected subset of* $\mathbb{C}$*). If* $|h(z)| < |g(z)|$ *on the boundary of the closed ball* $B[a, r] \subseteq \Omega$, *then*

$$\#\mathcal{Z}(g : B(a, r)) = \#\mathcal{Z}(g + h : B(a, r)),$$

where the zeroes are counted according to multiplicity.

It is particularly known that the *Mean Value Theorem* does not hold for complex-valued functions. However, the situation is favorable for polynomial functions, as can be seen in Çakmak and Tiryaki (2012, Theorem D):

Lemma 9 *Let* f *be a polynomial of degree at most* n. *Furthermore, let* z_1 *and* z_2 *be any pair of distinct points in the complex plane. Then, there exists* z_3 *with* $|z_3 - (z_1 + z_2)/2| \le |z_2 - z_1| \cot(\pi/n)/2$ *and such that*

$$f(z_2) - f(z_1) = f'(z_3)(z_2 - z_1).$$

Our last lemma is a theoretical result concerning *Möbius transformation*:

Lemma 10 (Lemma 4 in Trojovský 2021) *Let* $f : \mathbb{C} \to \mathbb{C}$ *be the Möbius transformation*

$$f(z) = \frac{az + b}{cz + d},$$

where a, b, c, d *are real numbers with* $ad - bc \neq 0$. *Then* $f^{-1}(\mathbb{R}) \subseteq \mathbb{R}$, *that is, if* $f(z)$ *is a real number, then so is* z.

Now, we are ready to deal with the proof of the theorems.

3 Proof of Theorem 2

3.1 Proof of the Existence of a Dominant Root

By the Descartes' sign rule, we have the existence of only one positive real root α of $f(x)$ (in fact, $\alpha > k^{1/p}$, by the intermediate value theorem). Note that we also have

$f(x) > 0$, for all $x > \alpha$. Let β be a complex root of $f(x)$ with $|\beta| \geq \alpha$. To prove that α is the dominant root, we must show that $\beta = \alpha$. For that, since $f(|\beta|) \geq 0$, we have $|\beta^q| \geq |k\beta^{q-p}| + \ell$. On the other hand, by the triangle inequality, one has $|\beta^q| \leq |k\beta^{q-p}| + \ell$ and so $|\beta^q| = |k\beta^{q-p} + \ell|$. Thus, $\ell, k\beta^{q-p}$ and β^q belong to the same ray. This implies that there exists $t_0 \in \mathbb{R}$ such that $\beta^q = \ell + t_0(k\beta^{q-p} - \ell)$. Since $\beta^q = k\beta^{q-p} + \ell$, we deduce that $g(\beta) = k\beta^{q-p}/(k\beta^{q-p} - \ell) = t_0$ is a real number (where $g(z) := kz^{q-p}/(kz^{q-p} - \ell)$) and so is β^{q-p}, by Lemma 10. It follows that β^q is also a real number and so is $\beta^p = \beta^q/\beta^{q-p}$. Now, since $\gcd(p, q) = 1$, then $mp + qn = 1$, for some integers m and n. Thus $\beta = (\beta^p)^m(\beta^q)^n$ is a real number. Now, we shall split the proof according to the parities of p and q.

3.1.1 $p \equiv q \pmod 2$

Since $\gcd(p, q) = 1$, then p and q are odd numbers. Now, observe that $f(-x) = -x^q - kx^{q-p} - \ell$ and so, by Descartes' sign rule, $f(x)$ does not have negative roots. Therefore, $\beta = \alpha$ as desired.

3.1.2 $p \not\equiv q \pmod 2$

Now, let us define $\psi(x) := f(\alpha x)$. Then, we have (by using $\alpha^q = k\alpha^{q-p} + \ell$) the explicit form

$$\psi(x) = \alpha^{q-1}(x^{q-1} + \cdots + x^{q-p}) + (\alpha^{q-1} - k\alpha^{q-p-1})(x^{q-p-1} + \cdots + x + 1).$$

Since α is a positive real number and clearly $\alpha^{q-1} > \alpha^{q-1} - k\alpha^{q-p-1} > 0$ (since $\alpha^p > k$), we can apply Lemma 7 to deduce that $\mathcal{R}_\psi \subseteq B[0, 1]$. However, $z \in \mathcal{R}_f$ if and only if $z/\alpha \in \mathcal{R}_\psi$. Note that β/α is a root of $\psi(x)$ (since $\psi(\beta/\alpha) = f(\beta) = 0$) and so $|\beta/\alpha| \leq 1$. On the other hand, by hypothesis, we have $|\beta| \geq \alpha$, which yields that $|\beta| = \alpha$. Remember that we want to prove that $\beta = \alpha$. Aiming for a contradiction, suppose that $\beta \neq \alpha$. Since $|\beta| = \alpha$ and $\beta \in \mathbb{R}$, then $\beta = -\alpha$. This implies that $f(\alpha) = f(-\alpha) = 0$. Thus,

$$2k\alpha^{q-p} = f(-\alpha) - f(\alpha) = 0,$$

when p is odd and q is even. Also,

$$-2\ell = f(-\alpha) + f(\alpha) = 0,$$

when p is even and q is odd. In both cases, we arrive at contradictions, such as that either $k\alpha^{q-p}$ or ℓ is equal to zero. Hence $\beta = \alpha$ and the proof is complete. □

3.2 Proof of the Estimate (4)

To simplify the notation, we set

$$\theta := k^{1/p}\left(1+\frac{1}{q}\log\left(\frac{\ell}{k^{q/p}}\right)\right) \text{ and } \gamma := k^{1/p}\left(1+\left(\frac{\ell}{p(q-p)k^{q/p}}\right)^{1/2}\right).$$

Since α is the unique positive root of $f(x)$, in order to prove that $\theta < \alpha < \gamma$, it suffices to show (by the *Intermediate Value Theorem*) that $f(\theta) < 0$ and $f(\gamma) > 0$.

Indeed, since $f(x) = x^{q-p}(x^p - k) - \ell$, then $f(k^{1/p}) = -\ell < 0$. Furthermore,

$$f(\theta) = k^{(q-p)/p}\left(1+\frac{1}{q}\log\left(\frac{\ell}{k^{q/p}}\right)\right)^{q-p}\left(k\left(1+\frac{1}{q}\log\left(\frac{\ell}{k^{q/p}}\right)\right)^{p} - k\right) - \ell.$$

We use that $(1+x)^n < e^{nx}$, for all real numbers x and $n > 0$, to deduce that

$$\begin{aligned} f(\theta) &< k^{q/p}e^{((q-p)/q)\log(\ell/k^{q/p})} \cdot e^{(p/q)\log(\ell/k^{q/p})} - \ell \\ &= k^{q/p}e^{\log(\ell/k^{q/p})} - \ell \\ &= 0. \end{aligned}$$

On the other hand,

$$\begin{aligned} f(\gamma) = k^{(q-p)/p}&\left(1+\left(\frac{\ell}{p(q-p)k^{q/p}}\right)^{1/2}\right)^{q-p} \\ &\times\left(k\left(1+\left(\frac{\ell}{p(q-p)k^{q/p}}\right)^{1/2}\right)^{p} - k\right) - \ell. \end{aligned}$$

By using the *Bernoulli inequality* $(1+x)^n \geq 1+nx$, for all $n \geq 1$ and $x \in \mathbb{R}_{>-1}$, together with a straightforward calculation, we obtain

$$\begin{aligned} f(\gamma) &\geq k^{(q-p)/p}\left(1+(q-p)\left(\frac{\ell}{p(q-p)k^{q/p}}\right)^{1/2}\right)\times k\left(\frac{p\ell}{(q-p)k^{q/p}}\right)^{1/2} - \ell \\ &> k^{q/p}\left(\frac{(q-p)\ell}{pk^{q/p}}\right)^{1/2}\left(\frac{p\ell}{(q-p)k^{q/p}}\right)^{1/2} - \ell \\ &= 0. \end{aligned}$$

Thus, inequality (4) holds, which finishes the proof. □

4 Proof of Theorem 3

4.1 The Proof

We must prove that $f(x)$ and $f'(x)$ do not have common zeroes. Since $f'(x) = qx^{q-1} - k(q-p)x^{q-p-1}$ and we can suppose that $x \neq 0$ (since $f(0) = 1$). Then $f'(\gamma) = 0$ implies that $\gamma^p = k(q-p)/q$. Thus $\gamma = (k(q-p)/q)^{1/p}\exp(2t\pi i/p)$, with $t \in [0, p-1]$. Now, it suffices to prove that $f(\gamma) \neq 0$. Indeed, on the contrary, we would have

$$0 = f(\gamma) = \left(k\left(1 - \frac{1}{r}\right)\right)^r \exp(2tr\pi i) - k\left(k\left(1 - \frac{1}{r}\right)\right)^{r-1} \exp(2t(r-1)\pi i) - \ell,$$

where $r := q/p > 1$. Since $\exp(2tr\pi i) = \exp(2tr\pi i - 2t\pi i) = \exp(2t(r-1)\pi i)$, we get

$$\ell = \left(k\left(1 - \frac{1}{r}\right)\right)^r \exp(2rt\pi i) - k\left(k\left(1 - \frac{1}{r}\right)\right)^{r-1} \exp(2rt\pi i)$$

and so

$$\ell = -\frac{k^r}{r}\left(1 - \frac{1}{r}\right)^{r-1} \exp(2rt\pi i).$$

Thus $\ell = -C_{k,r} \exp(2tr\pi i)$, where

$$C_{k,r} := \frac{k^r}{r}\left(1 - \frac{1}{r}\right)^{r-1}.$$

First, note that $r > 0$ (otherwise $\ell = -C_{k,r} < 0$). Now, by comparing the real and imaginary parts of $\ell = -C_{k,r} \exp(2tr\pi i)$, we obtain

$$\begin{cases} \ell = -C_{k,r} \cos(2tr\pi); \\ 0 = \sin(2tr\pi). \end{cases}$$

From $\sin(2tr\pi) = 0$, we deduce that $2tr\pi = s\pi$, for some integer s, which yields that p divides $2tq$. Since $\gcd(p, q) = 1$ and $0 < t < p-1$, then $p \mid 2t$ and $p \nmid t$ and so p is even (therefore item (i) is proved). Note that q is odd (since p and q are coprime) and hence

$$\ell = -C_{k,r} \cos(2tr\pi) = -C_{k,r} \cos\left(\frac{2tq}{p}\pi\right) = -C_{k,r}(-1)^{2tq/p} = -C_{k,r}(-1)^{2t/p},$$

where we used that $2t/p$ is an integer and $q \equiv 1 \pmod 2$. Furthermore, since $p \mid 2t$, then either $p = 2t$ or $p \leq (2t)/2 = t \in [1, p-1]$. Since the last fact can not happen, then $p = 2t$ yielding $C_{k,r} = \ell$ (and $\gamma = (k(q-p)/q)^{1/p} \exp(\pi i) = -(k(q-p)/q)^{1/p}$). Note that $\ell = C_{k,r}$ may be rewritten as

$$\ell^p = \frac{(q-p)^{q-p} k^q p^p}{q^q}.$$

However, this equality is exactly the case which can not happen by item (ii). So, $f(\gamma) \neq 0$. In conclusion, $f(x)$ and $f'(x)$ do not have common zeroes. This completes the proof. □

4.2 The Proof of Corollary 4

By Theorem 5, a possible double root of $f(x)$ must be a real negative number. However, if $p \equiv q \equiv 1 \pmod 2$, then by Lemma 6, we have $f(-x) = (-1)^q x^q - (-1)^{q-p} k x^{q-p} - \ell = x^q - kx^{q-p} - \ell$ which does not have negative roots. So, item (i) is proved.

For the other items, suppose that (5) is false. This would imply that

$$\ell^p q^q = (q-p)^{q-p} k^q p^p. \tag{7}$$

Thus, q^q divides $(q-p)^{q-p} k^q p^p$ and since $\gcd(q^q, p^p) = \gcd(q^q, (q-p)^{q-p}) = 1$, we obtain that q^q divides k^q. In particular, q divides k. Similarly, p divides ℓ, since p^p divides $\ell^p q^q$, but it is coprime with q^q. This proves item (ii). Moreover, even when k/q and ℓ/p, we can rewrite (7) as

$$\left(\frac{\ell}{p}\right)^p = (q-p)^{q-p} \left(\frac{k}{q}\right)^q.$$

Therefore, since $p < q$, one has that $(k/q)^p \mid (k/q)^q \mid (\ell/p)^p$ yielding that k/q divides ℓ/p. This proves item (iii) and the finishing the proof of the corollary. □

5 Proof of Theorem 5

5.1 Proof of Item (i)

Since $q > k$, by Corollary 4(i), one has that all the roots of $f_q(x)$ are simple and so $\#\mathcal{R}_{f_q} = q$. Choose a real number ϵ such that $0 < \epsilon < 1 - (\ell/k)^{1/r}$ (this is possible, because $k > \ell$). Now, we wish to apply Lemma 8 for $g(x) := -kx^r - \ell$, $h(x) := x^q$, $a = 0$ and $r = 1 - \epsilon$. Indeed, if $|x| = 1 - \epsilon$, then

$$|h(x)| = |x^q| = |x|^q = (1-\epsilon)^q < |kx^r + \ell| = |g(x)|,$$

for all q sufficiently large (since $(1-\epsilon)^q \to 0$ as $q \to \infty$ and $|kx^r + \ell| > 0$ does not depend on q). Thus, by Lemma 8 and the fact that $g(x) + h(x) = f_q(x)$, we infer that

$$\#\mathcal{Z}(-kx^r - \ell : B(0, 1-\epsilon)) = \#\mathcal{Z}(f_q(x) : B(0, 1-\epsilon)).$$

However, $1 - \epsilon > (\ell/k)^{1/r}$ which yields $\#\mathcal{Z}(-kx^r - \ell : B(0, 1-\epsilon)) = r$. Thus, $f_q(x)$ has exactly r roots belonging to $B(0, 1-\epsilon)$. Additionally, $|x| < 1 - \epsilon < 1$ implies that $f_q(x)$ tends to $-kx^r - \ell$ uniformly as $q \to \infty$.

We claim that the roots of $f_q(x)$ in $B(0, 1-\epsilon)$ tend to the roots of $-kx^r - \ell$, which are the rth roots of $-\ell/k$. To prove this, we first note that

$$|f_q(x) - g(x)| < (1-\epsilon)^q, \quad \forall x \in B(0, 1-\epsilon). \tag{8}$$

Let $\omega \in \mathcal{R}_g$. Since $f_q'(x) = qx^{q-1} - krx^{r-1} = x^{r-1}(qx^{q-r} - kr)$, then $\mathcal{R}_{f_q'} \backslash \{0\} \subseteq \partial B(0, (kr/q)^{1/(q-r)})$. We have that $(kr/q)^{1/(q-r)}$ tends to 1 as $q \to \infty$ and so $(kr/q)^{1/(q-r)} > 1 - \epsilon$, for all $q \geq q_0$. Therefore, by setting $\gamma^{(q)}$ the root of $f_q(x)$ nearest to ω, then there exists $\delta > 0$ such that

$$\gamma^{(q)} \in B(\omega, \delta) \subseteq B(0, 1 - \epsilon) \text{ and } M := \inf_{x \in B(\omega,\delta)} |f_q'(x)| > 0.$$

By (8) (for $x = \omega$), we have that $|f_q(\omega)| < (1 - \epsilon)^q$ and by Lemma 9, there exists $\theta \in \{z \in \mathbb{C} : |z - (\omega + \gamma^{(q)})/2| \leq |\omega - \gamma^{(q)}| \cot(\pi/q)/2\}$ such that

$$f_q(\omega) - f_q(\gamma^{(q)}) = f_q'(\theta)(\omega - \gamma^{(q)}).$$

By taking the absolute values of both sides of the previous relation together with the previous inequalities, we deduce that

$$|\omega - \gamma^{(q)}| < \frac{(1 - \epsilon)^q}{M},$$

for all $q \geq q_0$. In particular, $\gamma^{(q)}$ tends to ω as $q \to \infty$. This completes the proof of the first part.

In the second part (the largest roots), for any $\epsilon > 0$, we set $q_\epsilon := r + \lceil \log(k + \ell)/ \log(1 + \epsilon) \rceil$. Thus, if $|x| > 1 + \epsilon$, then

$$|f_q(x)| = |x^q - kx^r - \ell| \geq (1 + \epsilon)^r((1 + \epsilon)^{q-r} - k) - \ell > (1 + \epsilon)^{q-r} - k - \ell > 0,$$

for all $q \geq q_\epsilon$. Hence, in particular, the set of roots of $f_q(x)$ is contained in $B[0, 1+\epsilon]$. However, $f_q(x)$ has exactly r roots inside $B(0, 1-\epsilon)$ which forces the existence of $q-r$ roots (which we named as $\alpha_1^{(q)}, \ldots, \alpha_{q-r}^{(q)}$) in the closure of annulus $\text{ann}(0; 1-\epsilon, 1+\epsilon)$. Therefore, $1 - \epsilon \leq |\alpha_i^{(q)}| \leq 1 + \epsilon$ (for all $i \in [1, q - r]$) and since ϵ can be take arbitrarily small, we have that each $\alpha_i^{(q)}$ will tend to $\partial B(0, 1)$ as desired. This completes the proof. □

5.2 Proof of Item (ii)

This item has a simpler proof. In fact, for any $\epsilon > 0$, one has that if $|x| = 1 - \epsilon$, then

$$|x^q| = |x|^q = (1 - \epsilon)^q < |kx^r + \ell|,$$

for all q sufficiently large, say $q \geq q_0$. Thus, by Lemma 8, the polynomials $f_q(x)$ and $-kx^r - \ell$ have the same number of zeros in the ball $B(0, 1 - \epsilon)$, i.e.,

$$\#\mathcal{Z}(-kx^r - \ell : B(0, 1 - \epsilon)) = \#\mathcal{Z}(f_q(x) : B(0, 1 - \epsilon)).$$

However, each root of $-kx^r - \ell$ has absolute value equals to $(\ell/k)^{1/r} \geq 1 > 1-\epsilon$ yielding $\#\mathcal{Z}(-kx^r - \ell : B(0, 1-\epsilon)) = 0$. In particular, by the previous relation, $\mathcal{R}_{f_q} \cap B(0, 1-\epsilon) = \emptyset$. On the other hand, for any $\epsilon > 0$, as before, one has that

$$|f_q(x)| = |x^q - kx^r - \ell| \geq (1+\epsilon)^r((1+\epsilon)^{q-r} - k) - \ell > (1+\epsilon)^{q-r} - k - \ell > 0,$$

holds for all q large enough, say $q \geq q_1$. Therefore, all roots of $f_q(x)$ lie in the closure of $B(0, 1+\epsilon)$. We then conclude that $\mathcal{R}_{f_q}$ is a subset of the closure of annulus $\mathrm{ann}(0 : 1-\epsilon, 1+\epsilon)$ (for all $q \geq \max\{q_0, q_1\}$) which gives our desired result, since ϵ can be taken arbitrarily small. This finishes the proof. □

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