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Multi-dimensional Almost Automorphic Type Functions and Applications

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Abstract

In this paper, we introduce and analyze several new classes of multi-dimensional almost automorphic functions which generalize the classical one of Bochner. We develop the basic theory for the introduced classes, investigating the themes like composition principles, convolution invariance and the invariance under the actions of convolution products. We present several illustrative examples and applications to the abstract Volterra integro-differential equations and partial differential equations, providing also a mini appendix about almost automorphic functions on semi-topological groups.

Keywords $(\mathrm{R}, \mathcal{B})$ -multi-almost automorphic type functions · $(\mathrm{R}_X, \mathcal{B})$ -multi-almost automorphic type functions · Abstract Volterra integro-differential equations

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Contents

1 Introduction and Preliminaries

In 1955, S. Bochner discovered the concept of almost automorphy while he was studying problems related to differential geometry (Bochner 1955–1956); after that, it was proved that the almost automorphy is a generalization of the almost periodicity (see Bochner 1961, 1962; Bochner and Von Neumann 1935 and references therein). Starting presumably with the papers of Veech (1965, 1967), many authors have deeply investigated this concept on various classes of (semi-)topological groups.

Suppose that $F : \mathbb{R}^n \to X$ is continuous. Then it is said that $F(\cdot)$ is almost automorphic if and only if for every sequence $(\mathbf{b}_k)$ in $\mathbb{R}^n$ there exist a subsequence $(\mathbf{a}_k)$ of $(\mathbf{b}_k)$ and a map $G : \mathbb{R}^n \to X$ such that

$$\lim_{k\to\infty} F\big(\mathbf{t} + \mathbf{a}_k\big) = G(\mathbf{t}) \text{ and } \lim_{k\to\infty} G\big(\mathbf{t} - a_k\big) = F(\mathbf{t}), \tag{1.1}$$

pointwisely for $\mathbf{t} \in \mathbb{R}^n$. If this is the case, then the range of $F(\cdot)$ is relatively compact in X and the limit function $G(\cdot)$ is bounded on $\mathbb{R}^n$ but not necessarily continuous on $\mathbb{R}^n$. Furthermore, if the convergence of limits appearing in (1.1) is uniform on compact subsets of $\mathbb{R}^n$, resp. the whole space $\mathbb{R}^n$, then it is said that $F(\cdot)$ is compactly almost automorphic, resp. almost periodic. It can be proved that an almost automorphic function $F(\cdot)$ is compactly almost automorphic if and only if it is uniformly continuous (see the doctoral dissertation of Bender (1966) and Sect. 2.1 below). For more details about almost periodic functions in $\mathbb{R}^n$ and their generalizations, the reader may consult Chávez et al. (2020), the forthcoming research monograph by Kostić (2021) and references cited therein; for a fairly complete information about almost periodic functions and almost automorphic functions, the reader may consult the research monographs (Besicovitch 1954; Diagana 2013; Fink 1974; N'Guérékata 2001; Levitan 1953; Levitan and Zhikov 1982; Pankov 1990; Zaidman 1985).

The strong motivational factor for genesis of this paper, which can be viewed as a certain continuation of our previous study Chávez et al. (2020) of multi-dimensional almost periodic functions, presents the fact that almost nothing has been said by now about the space almost automorphic solutions to the (abstract) Volterra integro-differential equations. In support of our investigations of multi-dimensional almost automorphic type functions, we also want to note that we have not been able to find any relevant reference in the existing literature which throws light on some striking peculiarities of almost automorphic functions in $\mathbb{R}^n$ different from those already known for the almost automorphic functions on general topological groups.

The almost automorphic solutions with respect to the time variable for various classes of the (abstract) Volterra integro-differential equations have been intensively sought in numerous research studies (see e.g., Cao et al. 2018; Bugajewski and N'Guérékata 2004; Baroun et al. 2019; Chang and Zheng 2016; Ding et al. 2008 and references quoted therein). Let us recall here that some almost periodic systems do not necessarily carry almost periodic dynamics (see e.g., Ortega and Tarallo 2006; Shen and Yi 1998), while such systems may have bounded oscillating solutions which belong to a broader class of almost automorphic functions (see also the research article of Johnson (1981), who proved the existence of a linear almost periodic system of ordinary differential equations which admits an almost automorphic solution but no almost periodic solution).

Further on, it is well known that the solutions to nonautonomous evolution differential equations satisfy certain integral equations in which the integral kernels are expressed by means of two-parameter evolution families $(U(t,s))_{t\geq s\geq 0}$; see Pazy (1983) for the basic information. In the case of nonautonomous evolution differential equations with almost automorphic dynamics, the notion of bi-almost automorphy of the evolution operator $(U(t,s))_{t\geq s\geq 0}$ is essential in the research studies of the existence and uniqueness of almost automorphic mild solutions. The notion of a (positively) bi-almost automorphic function was introduced by Xiao et al. (2009); in this paper, the authors have obtained some sufficient conditions for the existence of pseudo almost automorphic mild solutions of the following equations in $\mathbb{R}$:

$$x'(t) = A(t)x(t) + f(t, x(t))$$
$$x'(t) = A(t)x(t) + f(t, x(t-h))$$
$$x'(t) = A(t)x(t) + f(t, x(\alpha(t, x(t))) .$$

Three years later, Chen and Lin employed this notion in their investigation of nonautonomous stochastic evolution equations (Chen and Lin 2013); see also Chávez et al. (2014a, b) and Diagana (2013, Appendix A.3), where the authors have analyzed the notion of bi-almost automorphic sequences. More precisely, in Chen and Lin (2013), the authors have introduced the notion of a square-mean bi-almost automorphic function for a stochastic process and analyzed the existence of square-mean almost automorphic solutions of the following non-autonomous linear stochastic evolution equation:

$$dx(t) = A(t)x(t)dt + f(t)dt + \gamma(t)\,dW(t),$$

with f, γ being stochastic processes and W being a two-sided standard one-dimensional Brownian motion. In Chávez et al. (2014a, b), the authors have analyzed the notion of discrete bi-almost automorphy and prove several results concerning the non-autonomous difference equations appearing in the dynamics of the following hybrid system of differential equations:

$$x'(t) = A(t)x(t) + B(t)x([t]) + f(t, x(t), x([t])); \tag{1.2}$$

here, $[t]$ denotes the integer part of a real number t. We also mention that, in Chávez et al. (2021), the authors have used the notion of bi-almost automorphy and the notion of λ-boundedness in their studies of the following nonlinear abstract integral equations of advanced and delayed type:

$$y(t) = f\big(t, y(t), y(a_0(t))\big) + \int_{-\infty}^{t} C_1\big(t, s, y(s), y(a_1(s))\big)\, ds + \int_{t}^{+\infty} C_2\big(t, s, y(s), y(a_2(s))\big)\, ds.$$

Besides the above-mentioned papers, we would like to quote the research studies by Chang and Zheng (2016), Hu and Jin (2013), Qi and Yuan (2020), Xia (2014), Xia and Wang (2018) and Xu et al. (2020). Observing the previous works (and references cited therein), we emphasize that the notion of bi-almost automorphy is crucial in the study of almost automorphic dynamics for various classes of differential, integro-differential and difference equations.

Throughout this paper, we assume that $n \in \mathbb{N}$, $\mathcal{B}$ is a non-empty collection of subsets of X, R is a non-empty collection of sequences in $\mathbb{R}^n$ and R_X is a non-empty collection of sequences in $\mathbb{R}^n \times X$; usually, $\mathcal{B}$ denotes the collection of all bounded subsets of X or all compact subsets of X. Henceforth we will always assume that, for every $x \in X$, there exists $B \in \mathcal{B}$ such that $x \in B$. Since all the norms in $\mathbb{R}^n$ are equivalent, we equiped $\mathbb{R}^n$ with the Euclidean norm denoted by $|\cdot|$. Then, for each $\mathbf{t}_0 \in \mathbb{R}^n$, we denote by $B(\mathbf{t}_0, l)$ the closed ball of $\mathbb{R}^n$ with center $\mathbf{t}_0$ and radius $l > 0$ i.e., $B(\mathbf{t}_0, l) := \{\mathbf{t} \in \mathbb{R}^n : |\mathbf{t} - \mathbf{t}_0| \leq l\}$. The notion of $\mathbb{Z}$-almost automorphy and the notion of bi-almost automorphy, which have been analyzed in the above-mentioned papers, are special cases of the notion (R, $\mathcal{B}$)-multi-almost automorphy, which is a crucial object of our investigations (for example, the notion of bi-almost automorphy is obtained with the collection R of all sequences in $\Delta_2 \equiv \{(w, w) : w \in \mathbb{R}\}$, the diagonal of $\mathbb{R}^2$). Further on, the notion of (R, $\mathcal{B}$)-multi-almost automorphy is a special case of the notion of (R_X, $\mathcal{B}$)-multi-almost automorphy, which has been introduced and analyzed in this paper following the previous investigations of almost automorphic functions on (semi-)topological groups. In this paper, we aim to develop the basic theory of (R_X, $\mathcal{B}$)-multi-almost automorphic type functions as well as to provide some concrete applications to the abstract Volterra integro-differential equations and partial differential equations such as the classical heat equation and the wave equation (we also revisit the theory of integrated semigroups, C-regularized semigroups and their applications here). It is our strong belief that this research study is only the beginning of serious investigations of space almost automorphic solutions of integro-differential equations.

The organization of the present work is as follows. After recalling the basic definitions and results about $(\mathrm{R}_X, \mathcal{B})$-multi-almost periodic type functions and Bohr $(\mathcal{B}, c)$-almost periodic type functions (Sect. 1.1), we introduce the classes of (compactly) $(\mathrm{R}, \mathcal{B})$-multi-almost automorphic functions (Definition 2.1), $(\mathrm{R}, \mathcal{B}, W_{\mathcal{B},\mathrm{R}})$-multi-almost automorphic functions and $(\mathrm{R}, \mathcal{B}, \mathrm{P}_{\mathcal{B},\mathrm{R}})$-multi-almost automorphic functions (Definition 2.2); here, we assume that for each $B \in \mathcal{B}$ and $(\mathbf{b}_k) \in \mathrm{R}$ we have $W_{B,(\mathbf{b}_k)} : B \to P(P(\mathbb{R}^n))$ and $\mathrm{P}_{B,(\mathbf{b}_k)} \in P(P(\mathbb{R}^n \times B))$, where $P(S)$ denotes the power set of S. In Remark 2.3, we single out three most important types of collections R we are working with, the classes of sequences [L1]–[L3]. The introduced class of $(\mathrm{R}, \mathcal{B})$-multi-almost automorphic functions has certain really new features because we provide here, for the first time in the existing literature, an example of an $(\mathrm{R}, \mathcal{B})$-multi-almost automorphic function $F : \mathbb{R}^2 \to X$ (R is the collection of all sequences in Δ_2 and $\mathcal{B}$ denotes the collection of all bounded subsets of X) in which the convergence of limits in Eqs. (2.1)–(2.2) below is uniform not on the whole space (the almost periodic case) and not only on compact subsets of $\mathbb{R}^n$ (the compact almost automorphic case); this example is important for a better understanding of the notion $(\mathrm{R}, \mathcal{B}, W_{\mathcal{B},\mathrm{R}})$-multi-almost automorphy we are working with.

After illustrating this notion with some other examples, we introduce the notions of $(\mathrm{R}_X, \mathcal{B})$-multi-almost automorphy, $(\mathrm{R}_X, \mathcal{B}, W_{\mathcal{B},\mathrm{R}_X})$-multi-almost automorphy and $(\mathrm{R}_X, \mathcal{B}, \mathrm{P}_{\mathcal{B},\mathrm{R}_X})$-multi-almost automorphy in Definition 2.4. In Proposition 2.5, we investigate the relative compactness of range of a two-parameter $(\mathrm{R}_X, \mathcal{B})$-multi-almost automorphic function $F : \mathbb{R}^n \times X \to Y$. After that, we divide the remainder of the second section into several separate subsections. The main aim of Sect. 2.1 is to thoroughly study the compactly $(\mathrm{R}_X, \mathcal{B})$-multi-almost automorphic functions; in Sect. 2.2, we continue our study by clarifying several new structural characterizations of $(\mathrm{R}_X, \mathcal{B})$-multi-almost automorphic type functions. Section 2.3 investigates $\mathbb{D}$-asymptotically $(\mathrm{R}_X, \mathcal{B})$-multi-almost automorphic functions; composition theorems for $(\mathrm{R}, \mathcal{B})$-multi-almost automorphic functions are analyzed in Sect. 2.4, while the invariance of $(\mathrm{R}, \mathcal{B})$-multi-almost automorphic properties under the actions of convolution products are analyzed in Sect. 2.5.

The third section of paper is reserved for applications of our abstract results to the various classes of abstract Volterra integro-differential equations. In Sect. 3.1, we analyze almost automorphic solutions to the abstract semilinear Volterra integral equations. The applications to the heat equation and the wave equation are given in Sect. 3.2; the main aim of Sect. 3.3 is to provide certain applications to the ill-posed abstract Cauchy problems. As mentioned in the abstract, we also provide a small appendix about almost automorphic functions on semi-topological groups at the end of paper. Although a rather long, the paper does not cover many important subjects; for example, we will not consider here the notion of a positively $(\mathrm{R}_X, \mathcal{B})$-multi-almost automorphy and its generalizations (Xiao et al. 2009).

We use the standard notation throughout the work. We assume henceforth that $(X, \|\cdot\|)$, $(Y, \|\cdot\|_Y)$ and $(Z, \|\cdot\|_Z)$ are complex Banach spaces. By $L(X, Y)$ we denote the Banach algebra of all bounded linear operators from X into Y with $L(X, X)$ being denoted $L(X)$. By $(e_1, e_2, \ldots, e_n)$ we denote the standard basis of $\mathbb{R}^n$; $\mathbb{N}_n := \{1, 2, \ldots, n\}$.

Before switching to Sect. 1.1, we would like to motivate our study of the space almost automorphic solutions of evolution equations by the following illustrative example, which is an insignificant modification of the corresponding example given for the space almost periodicity in Chávez et al. (2020, Example 1.1):

Example Let a closed linear operator A generates a strongly continuous semigroup $(T(t))_{t\geq 0}$ on a Banach space X whose elements are certain complex-valued functions defined on $\mathbb{R}^n$, and let $f : \mathbb{R}^+ \longrightarrow X$ be a locally integrable function. Under some reasonable assumptions, the function

$$u(t,x) = \big(T(t)u_0\big)(x) + \int_0^t [T(t-s)f(s)](x)\,ds, \quad t \geq 0,\ x \in \mathbb{R}^n$$

is a unique classical solution of the abstract Cauchy problem

$$u_t(t,x) = Au(t,x) + F(t,x),\ t \geq 0,\ x \in \mathbb{R}^n;\ u(0,x) = u_0(x),$$

where $F(t,x) := [f(t)](x)$, $t \geq 0$, $x \in \mathbb{R}^n$. In some concrete situations (for example, this holds for the Gaussian semigroup on $\mathbb{R}^n$; see Sect. 3.2 and Chávez et al. 2020 for more details), there exists a kernel $(t,y) \mapsto E(t,y)$, $t > 0$, $y \in \mathbb{R}^n$ which is integrable on any set $[0,T] \times \mathbb{R}^n$ $(T > 0)$ and satisfies that

$$[T(t)f(s)](x) = \int_{\mathbb{R}^n} F(s, x-y)E(t,y)\,dy, \quad t > 0,\ s \geq 0,\ x \in \mathbb{R}^n.$$

Let it be the case, and let a positive real number $t_0 > 0$ be fixed. Then the space almost automorphic behaviour of function $x \mapsto u_{t_0}(x) \equiv \int_0^{t_0}[T(t_0-s)f(s)](x)\,ds$, $x \in \mathbb{R}^n$ depends on the space almost automorphic behaviour of function $F(t,x)$. Suppose, for example, that the function $F(t,x)$ is bounded on any region $[0,T] \times \mathbb{R}^n$ $(T > 0)$ as well as that it is R-multi-almost automorphic with respect to the variable $x \in \mathbb{R}^n$, uniformly in the variable t on compact subsets of $[0,\infty)$; that is, for every finite real number $T > 0$ and for every sequence $(\mathbf{b}_k) \in \mathrm{R}$, there exist a subsequence $(\mathbf{b}_{k_l})$ of $(\mathbf{b}_k)$ and a function $F^* : [0,T] \times \mathbb{R}^n \to \mathbb{C}$ such that $\lim_{m\to+\infty} F(t, x+\mathbf{b}_{k_m}) = F^*(t,x)$ and $\lim_{l\to+\infty} F^*(t, x-\mathbf{b}_{k_l}) = F(t,x)$, pointwisely for every $t \in [0,T]$ and $x \in \mathbb{R}^n$. Then the function $u_{t_0}(\cdot)$ is likewise bounded and R-multi-almost automorphic with respect to the variable x, as easily approved using the dominated convergence theorem.

1.1 $(\mathrm{R}_X, \mathcal{B})$-Multi-almost Periodic Type Functions and Bohr $(\mathcal{B}, c)$-Almost Periodic Type Functions

Recall, we assume henceforth that $n \in \mathbb{N}$, $\mathcal{B}$ is a non-empty collection of subsets of X satisfying that, for every $x \in X$, there exists $B \in \mathcal{B}$ such that $x \in B$, as well as that R is a non-empty collection of sequences in $\mathbb{R}^n$ and R_X is a non-empty collection of sequences in $\mathbb{R}^n \times X$.

Suppose that $c \in \mathbb{C}$ and $|c| = 1$. In this subsection, we recall the basic facts about $(\mathrm{R}_X, \mathcal{B})$-multi-almost periodic type functions and Bohr $(\mathcal{B}, c)$-almost periodic type functions; see Chávez et al. (2020) and Kostić (2020) for more details about the subject.

We start by recalling the following definitions:

Definition 1.1 Suppose that $F : \mathbb{R}^n \times X \to Y$ is a continuous function. Then we say that the function $F(\cdot;\cdot)$ is $(\mathrm{R}, \mathcal{B})$-multi-almost periodic if and only if for every $B \in \mathcal{B}$ and for every sequence $(\mathbf{b}_k = (b_k^1, b_k^2, \ldots, b_k^n)) \in \mathrm{R}$ there exist a subsequence $(\mathbf{b}_{k_l} = (b_{k_l}^1, b_{k_l}^2, \ldots, b_{k_l}^n))$ of $(\mathbf{b}_k)$ and a function $F^* : \mathbb{R}^n \times X \to Y$ such that

$$\lim_{l\to+\infty} F\big(\mathbf{t} + (b_{k_l}^1, \ldots, b_{k_l}^n); x\big) = F^*(\mathbf{t}; x)$$

uniformly for all $x \in B$ and $\mathbf{t} \in \mathbb{R}^n$.

Definition 1.2 Suppose that $F : \mathbb{R}^n \times X \to Y$ is a continuous function. Then we say that:

(i) $F(\cdot;\cdot)$ is Bohr $(\mathcal{B}, c)$-almost periodic if and only if for every $B \in \mathcal{B}$ and $\epsilon > 0$ there exists $l > 0$ such that for each $\mathbf{t}_0 \in I$ there exists $\boldsymbol{\tau} \in B(\mathbf{t}_0, l) \cap I$ such that

$$\big\| F(\mathbf{t} + \boldsymbol{\tau}; x) - cF(\mathbf{t}; x)\big\|_Y \le \epsilon, \quad \mathbf{t} \in I,\ x \in B.$$

(ii) $F(\cdot;\cdot)$ is $(\mathcal{B}, c)$-uniformly recurrent if and only if for every $B \in \mathcal{B}$ there exists a sequence $(\boldsymbol{\tau}_k)$ in I such that $\lim_{k\to+\infty} |\boldsymbol{\tau}_k| = +\infty$ and

$$\lim_{k\to+\infty} \sup_{\mathbf{t}\in I; x\in B} \big\| F(\mathbf{t} + \boldsymbol{\tau}_k; x) - cF(\mathbf{t}; x)\big\|_Y = 0. \tag{1.3}$$

If $X \in \mathcal{B}$, then it is also said that $F(\cdot;\cdot)$ is Bohr c-almost periodic (c-uniformly recurrent); if $c = 1$, then we also say that $F(\cdot;\cdot)$ is Bohr $\mathcal{B}$-almost periodic ($\mathcal{B}$-uniformly recurrent) [Bohr almost periodic (uniformly recurrent)].

Assume now that $F : \mathbb{R}^n \times X \to Y$ is continuous, $\mathcal{B}$ is any family of compact subsets of X and R is the collection of all sequences in $\mathbb{R}^n$. Then we know that $F(\cdot;\cdot)$ is Bohr $\mathcal{B}$-almost periodic if and only if $F(\cdot;\cdot)$ is $(\mathrm{R}, \mathcal{B})$-multi-almost periodic.

The notion introduced in Definition 1.1 is a special case of the notion introduced in the following definition:

Definition 1.3 Suppose that $F : \mathbb{R}^n \times X \to Y$ is a continuous function. Then we say that the function $F(\cdot;\cdot)$ is $(\mathrm{R}_X, \mathcal{B})$-multi-almost periodic if and only if for every $B \in \mathcal{B}$ and for every sequence $((\mathbf{b}; \mathbf{x})_k = ((b_k^1, b_k^2, \ldots, b_k^n); x_k)) \in \mathrm{R}_X$ there exist a subsequence $((\mathbf{b}; \mathbf{x})_{k_l} = ((b_{k_l}^1, b_{k_l}^2, \ldots, b_{k_l}^n); x_{k_l}))$ of $((\mathbf{b}; \mathbf{x})_k)$ and a function $F^* : \mathbb{R}^n \times X \to Y$ such that

$$\lim_{l\to+\infty} F\big(\mathbf{t} + (b_{k_l}^1, \ldots, b_{k_l}^n); x + x_{k_l}\big) = F^*(\mathbf{t}; x)$$

uniformly for all $x \in B$ and $\mathbf{t} \in \mathbb{R}^n$.

In this paper, we investigate almost automorphic analogues of $(\mathrm{R}, \mathcal{B})$-multi-almost periodic functions and $(\mathrm{R}_{\mathrm{X}}, \mathcal{B})$-multi-almost periodic functions. For the sequel, we also need the following definition from Chávez et al. (2020):

Definition 1.4 Suppose that $\mathbb{D} \subseteq \mathbb{R}^n$ and the set $\mathbb{D}$ is unbounded. By $C_{0,\mathbb{D},\mathcal{B}}(\mathbb{R}^n \times X : Y)$ we denote the vector space consisting of all continuous functions $Q : \mathbb{R}^n \times X \to Y$ such that, for every $B \in \mathcal{B}$, we have $\lim_{\mathbf{t}\in\mathbb{D}, |\mathbf{t}|\to+\infty} Q(\mathbf{t}; x) = 0$, uniformly for $x \in B$.

2 $(\mathrm{R}_X, \mathcal{B})$-Multi-almost Automorphic Type Functions

In this section, we investigate $(\mathrm{R}_X, \mathcal{B})$-multi-almost automorphic functions and asymptotically $(\mathrm{R}_X, \mathcal{B})$-multi-almost automorphic functions. We start our work with the following definition, which seems to be new even in the one-dimensional setting:

Definition 2.1 Suppose that $F : \mathbb{R}^n \times X \to Y$ is a continuous function. Then we say that the function $F(\cdot; \cdot)$ is $(\mathrm{R}, \mathcal{B})$-multi-almost automorphic if and only if for every $B \in \mathcal{B}$ and for every sequence $(\mathbf{b}_k = (b_k^1, b_k^2, \ldots, b_k^n)) \in \mathrm{R}$ there exist a subsequence $(\mathbf{b}_{k_l} = (b_{k_l}^1, b_{k_l}^2, \ldots, b_{k_l}^n))$ of $(\mathbf{b}_k)$ and a function $F^* : \mathbb{R}^n \times X \to Y$ such that

$$\lim_{l\to+\infty} F\big(\mathbf{t} + (b_{k_l}^1, \ldots, b_{k_l}^n); x\big) = F^*(\mathbf{t}; x) \tag{2.1}$$

and

$$\lim_{l\to+\infty} F^*\big(\mathbf{t} - (b_{k_l}^1, \ldots, b_{k_l}^n); x\big) = F(\mathbf{t}; x), \tag{2.2}$$

pointwisely for all $x \in B$ and $\mathbf{t} \in \mathbb{R}^n$. If for each $x \in B$ the above limits converge uniformly on compact subsets of $\mathbb{R}^n$, then we say that $F(\cdot; \cdot)$ is compactly $(\mathrm{R}, \mathcal{B})$-multi-almost automorphic. By $AA_{(\mathrm{R},\mathcal{B})}(\mathbb{R}^n \times X : Y)$ and $AA_{(\mathrm{R},\mathcal{B},\mathbf{c})}(\mathbb{R}^n \times X : Y)$ we denote the spaces consisting of all $(\mathrm{R}, \mathcal{B})$-multi-almost automorphic functions and compactly $(\mathrm{R}, \mathcal{B})$-multi-almost automorphic functions, respectively.

In the case that $X = \{0\}$ and $\mathcal{B} = \{X\}$, i.e., if we consider the function $F : \mathbb{R}^n \to Y$, then we also say that $F(\cdot)$ is (compactly) R-multi-almost automorphic and denote the corresponding spaces by $AA_{\mathrm{R}}(\mathbb{R}^n : Y)$ and $AA_{\mathrm{R},\mathbf{c}}(\mathbb{R}^n : Y)$ [in the remainder of paper, we will tacitly omit the term "$\mathcal{B}$" from the notation in such situations].

The following definition seems to be new in the one-dimensional setting, as well:

Definition 2.2 Suppose that $F : \mathbb{R}^n \times X \to Y$ is a continuous function as well as that for each $B \in \mathcal{B}$ and $(\mathbf{b}_k = (b_k^1, b_k^2, \ldots, b_k^n)) \in \mathrm{R}$ we have that $W_{B,(\mathbf{b}_k)} : B \to P(P(\mathbb{R}^n))$ and $\mathrm{P}_{B,(\mathbf{b}_k)} \in P(P(\mathbb{R}^n \times B))$. Then we say that $F(\cdot; \cdot)$ is:

(i) $(\mathrm{R}, \mathcal{B}, W_{\mathcal{B},\mathrm{R}})$-multi-almost automorphic if and only if for every $B \in \mathcal{B}$ and for every sequence $(\mathbf{b}_k = (b_k^1, b_k^2, \ldots, b_k^n)) \in \mathrm{R}$ there exist a subsequence $(\mathbf{b}_{k_l} = (b_{k_l}^1, b_{k_l}^2, \ldots, b_{k_l}^n))$ of $(\mathbf{b}_k)$ and a function $F^* : \mathbb{R}^n \times X \to Y$ such that (2.1)–(2.2) hold pointwisely for all $x \in B$ and $\mathbf{t} \in \mathbb{R}^n$ as well as that for each $x \in B$ the convergence in $\mathbf{t}$ is uniform for any element of the collection $W_{B,(\mathbf{b}_k)}(x)$;

(ii) $(\mathrm{R}, \mathcal{B}, \mathrm{P}_{\mathcal{B},\mathrm{R}})$-multi-almost automorphic if and only if for every $B \in \mathcal{B}$ and for every sequence $(\mathbf{b}_k = (b_k^1, b_k^2, \ldots, b_k^n)) \in \mathrm{R}$ there exist a subsequence $(\mathbf{b}_{k_l} = (b_{k_l}^1, b_{k_l}^2, \ldots, b_{k_l}^n))$ of $(\mathbf{b}_k)$ and a function $F^* : \mathbb{R}^n \times X \to Y$ such that (2.1)–(2.2) hold pointwisely for all $x \in B$ and $\mathbf{t} \in \mathbb{R}^n$ as well as that the convergence in (2.1)–(2.2) is uniform in $(\mathbf{t}; x)$ for any set of the collection $\mathrm{P}_{B,(\mathbf{b}_k)}$.

Before we go any further, we would like to present the following illustrative example of the notion introduced above:

Example (Terras 1972; Milnes 1977) Let us write the set $\mathbb{R}$ as the disjoint union of intervals $\bigcup_{k=1}^{\infty} V_k$, where $V_k := \bigcup_{m\in\mathbb{Z}}([0,1)+s_k+2^k m)$ and $s_k := ((-2)^{k-1}-1)/3$ for all $k \in \mathbb{N}$. After that, we define a continuous function $f : \mathbb{R} \to \mathbb{R}$ through $f(t) := \sin(2^k \pi t)$ if $t \in V_k$ for some $k \in \mathbb{N}$. We know that the function $f(\cdot)$ is almost automorphic as well as that the sequence of translations $(f(\cdot + s_k))_{k\in\mathbb{N}}$ does not converge uniformly on the set $[0, 1]$, so that $f(\cdot)$ is not uniformly continuous and not compactly almost automorphic. If $f_1(\cdot), \ldots, f_{n-1}(\cdot)$ are almost automorphic complex-valued functions, then we set $F(t_1, \ldots, t_{n-1}, t_n) := f_1(t_1)\ldots f_{n-1}(t_{n-1})f(t_n)$, $\mathbf{t} = (t_1, \ldots, t_{n-1}, t_n) \in \mathbb{R}^n$. It can be easily shown that $F(\cdot)$ is an almost automorphic function which is not compactly almost automorphic, as well as that $F(\cdot)$ cannot be $(\mathrm{R}, W_{\mathrm{R}})$-multi-almost automorphic for any collection of sequences in $\mathbb{R}^n$ which contains the sequence $(\mathbf{b}_k = (0, \ldots, 0; s_k))_{k\in\mathbb{N}}$ and for any collection $W_{\mathbf{b}_k}$ of subsets of $\mathbb{R}^n$ which contains the set $S \times [0, 1]$, where $S = (t_1^0, \ldots, t_n^0) \in \mathbb{R}^{n-1}$ and $f_1(t_1^0)\ldots f_{n-1}(t_{n-1}^0) \neq 0$.

Remark 2.3 The following special cases are very important (see also Chávez et al. 2020):

L1. Let $\mathrm{R} := \{b : \mathbb{N} \to \mathbb{R}^n ;\ \text{ for all } j \in \mathbb{N} \text{ we have } b_j \in \Delta_n \equiv \{(a, a, a, \ldots, a) \in \mathbb{R}^n : a \in \mathbb{R}\}\}$. In the case when $n = 2$ and $\mathcal{B}$ is the collection of all bounded subsets of X, we say that the function $F(\cdot;\cdot)$ is bi-almost automorphic. Concerning this notion, let us also mention that, Baroun et al. (2019) have proved the existence and uniqueness of μ-pseudo almost automorphic solutions to a class of nonautonomous evolution equations with inhomogeneous boundary conditions, using the notion of bi-almost automorphic Green functions; in addition, the authors have established sufficient weak conditions on the initial data of the equation insuring the bi-almost automorphy of the associated Green function. Clearly, with this choice of collection R, we have that the function $F(\cdot;\cdot)$ is $(\mathrm{R}, \mathcal{B})$-multi-almost automorphic if and only if, for every $B \in \mathcal{B}$ and for every real sequence (b_k), there exist a subsequence (a_k) of (b_k) and a function $F^* : \mathbb{R}^n \times X \to Y$ such that

$$\lim_{k\to+\infty} \Big\| F\big(\mathbf{t} + (a_k, \ldots, a_k); x\big) - F^*(\mathbf{t}; x) \Big\|_Y = 0$$

and

$$\lim_{k\to+\infty} \Big\| F^*\big(\mathbf{t} - (a_k, \ldots, a_k); x\big) - F(\mathbf{t}; x) \Big\|_Y = 0,$$

pointwisely for all $x \in B$ and $\mathbf{t} \in \mathbb{R}^n$.

L2. $\mathrm{R} = \{b : \mathbb{N} \to \mathbb{R}^n ;$ for all $j \in \mathbb{N}$ we have $b_j \in \{(a, 0, 0, \ldots, 0) \in \mathbb{R}^n : a \in \mathbb{R}\}\}$. Without going into full details, we want only to consider here the case in which $X \in \mathcal{B}$ (the choice in which $\mathcal{B}$ is a collection of all bounded or compact subsets of X is a bit complicated but the obtained conclusions are similar; the difficulty actually lies in the fact that a bounded (compact) set in the space $\mathbb{R}^{n-1} \times X$ is not necessarily a direct product of a bounded (compact) set in $\mathbb{R}^{n-1}$ and a bounded (compact) set in X). It can be simply approved that the study of (compact) $(\mathrm{R}, \mathcal{B})$-multi-almost automorphy of function $F(\cdot; \cdot)$ cannot be reduced to the study of the corresponding notion for the function $\mathbb{F} : \mathbb{R} \times \mathcal{X} \to Y$, given by $\mathcal{F}(t; \S) := F((t, \mathbf{t}'); x)$, $t \in \mathbb{R}$, $\S = (\mathbf{t}'; x) \in \mathcal{X} = \mathbb{R}^{n-1} \times X$. Therefore, the notion introduced above cannot be viewed as some special case of the notion of almost automorphy of function from $\mathbb{R} \times X$ into Y.

L3. R is a collection of all sequences $b(\cdot)$ in $\mathbb{R}^n$. This is the limit case in our analysis because this assumption clearly implies that any $(\mathrm{R}, \mathcal{B})$-multi-almost automorphic function is automatically $(\mathrm{R}_1, \mathcal{B})$-multi-almost automorphic for any other collection R_1 of sequences $b(\cdot)$ in $\mathbb{R}^n$.

The notion in which R is not the collection of all sequences in $\mathbb{R}^n$ is far from being comparable with the usual almost automorphy (see e.g., Proposition 4.3 below). In several important research studies of spatially almost periodic solutions of (abstract) Volterra integro-differential equations, the Bochner criterion is essentially employed with the collection R of all sequences in $\mathbb{R}^n$; here we would like to emphasize, without going into full details, that some established results concerning this problematic can be further extended by allowing that R is an arbitrary collection of sequences (in $\mathbb{R}^n$) in their formulations:

Example It is well known that the Euler equations in $\mathbb{R}^n$, where $n \geq 2$, describe the motion of perfect incompressible fluids. It is problem to find the unknown functions $u = u(x, t) = (u^1(x, t), \ldots, u^n(x, t))$ and $p = p(x, t)$ denoting the velocity field and the pressure of the fluid, respectively, such that

$$
\begin{aligned}
&\frac{\partial u}{\partial t} + (u \cdot \nabla)u + \nabla p = 0 \quad \text{in } \mathbb{R}^n \times (0, T), \\
&\operatorname{div} u = 0 \text{ in } \mathbb{R}^n \times (0, T), \\
&u(x, 0) = u_0(x) \text{ in } \mathbb{R}^n,
\end{aligned}
\tag{2.3}
$$

where $u_0 = u_0(x) = (u_0^1(x), ..., u_0^n(x))$ denotes the given initial velocity field. There are many results concerning the well-posedness of (2.3) in the case that the initial velocity field $u_0(x)$ belongs to some direct product of (fractional) Sobolev spaces. For our observation, it is crucial to remind the readers of the research article by Pak and Park (2004), who investigated the well-posedness of (2.3) in the case that the initial velocity field $u_0(x)$ belongs to the space $B^1_{\infty,1}(\mathbb{R}^n)^n$, where $B^1_{\infty,1}(\mathbb{R}^n)$ denotes the usual Besov space (see e.g., (Sawada and Takada 2011, Definition 2.1)). The authors have proved that for any function $u_0 \in B^1_{\infty,1}(\mathbb{R}^n)^n$ such that $\operatorname{div} u_0 = 0$ there exists a finite real number $T > 0$ such that there exists a solution $u \in C([0, T] : B^1_{\infty,1}(\mathbb{R}^n)^n)$

of (2.3). Using some known results proved by H. C. Pak, Y. J. Park and the fact that a function $f \in B^0_{\infty,1}(\mathbb{R}^n)$ is almost periodic in $\mathbb{R}^n$ if and only if the set of all its translations is relatively compact in $B^0_{\infty,1}(\mathbb{R}^n)$ (see (Sawada and Takada 2011, Lemma 4.2)), Sawada and Takada have proved in (2011, Theorem 1.5), that the assumption that $u_0(x)$ is almost periodic in $\mathbb{R}^n$ implies that the solution $u(\cdot, t)$ of (2.3) is almost periodic in $\mathbb{R}^n$ for all $t \in [0, T]$. Let R denote an arbitrary collection of sequences in $\mathbb{R}^n$, and let $u_0(\cdot)$ has the property that for each sequence $(\mathbf{b}_k)$ in R there exists a subsequence $(\mathbf{b}_{k_l})$ of $(\mathbf{b}_k)$ such that the sequence of translations $(u_0(\cdot + \mathbf{b}_{k_l}))$ is convergent in the space $B^0_{\infty,1}(\mathbb{R}^n)^n$. Then for each sequence $(\mathbf{b}_k)$ in R there exists a subsequence $(\mathbf{b}_{k_l})$ of $(\mathbf{b}_k)$ such that, for every $t \in [0, T]$, the sequence of translations $(u(\cdot + \mathbf{b}_{k_l}, t))$ is convergent in the space $B^0_{\infty,1}(\mathbb{R}^n)^n$; let us only note that the assumptions on function $u_0(\cdot)$ used here can serve one to introduce a new notion of multi-dimensional R-almost automorphy which is not so simply connected, in general case, with the notion introduced in Definitions 2.1 and 2.2 (more details will appear somewhere else). See also the research studies by Giga et al. (2007), Li (2018), Taniuchi et al. (2010), and the references quoted in Kostić (2021) for further information concerning spatially almost periodic solutions of (abstract) Volterra integro-differential equations.

In what follows, we will provide several elaborate examples illustrating the concepts introduced in Definitions 2.1 and 2.2:

Example Let $\varphi : \mathbb{R} \to \mathbb{C}$ be a (compactly) almost automorphic function, and let $(T(t))_{t\in\mathbb{R}} \subseteq L(X, Y)$ be a strongly continuous operator family. Suppose first that R is the collection of all sequences in Δ_2 as well as that $X \in \mathcal{B}$. Define a function $G : \mathbb{R}^2 \times X \to Y$ by

$$F(t, s; x) := e^{\int_s^t \varphi(\tau)\, d\tau} T(t - s)x, \quad (t, s) \in \mathbb{R}^2,\ x \in X. \tag{2.4}$$

The function $F(\cdot, \cdot; \cdot)$ is (compactly) bi-almost automorphic, which can be simply shown (see also (Chen and Lin 2013, Example 7.1) and (Xiao et al. 2009, Example 4.1)).

Suppose now that $\varphi : \mathbb{R} \to \mathbb{C}$ is almost periodic as well as that R is the collection of all sequences in Δ_2 and $\mathcal{B}$ denotes the collection of all bounded subsets of X. Let for each bounded subset B of X and for each sequence $(\mathbf{b}_k = (b_k, b_k))$ in R the colection $P_{B,(\mathbf{b}_k)}$ be constituted of all sets of form $\{(t, s) \in \mathbb{R}^2 : |t - s| \leq L\} \times B$, where $L > 0$. Then the function $F(\cdot, \cdot; \cdot)$ is $(\mathrm{R}, \mathcal{B}, \mathrm{P}_{\mathcal{B},\mathrm{R}})$-multi-almost automorphic, which can be deduced as follows. Let a real number $L > 0$ and a bounded subset B of X be fixed, and let $(t, s) \in \mathbb{R}^2$ satisfy $|t - s| \leq L$. By Bochner's criterion, there exist a subsequence (b_{k_l}, b_{k_l}) of (b_k, b_k) and a function $\varphi^* : \mathbb{R} \to \mathbb{C}$ such that $\lim_{l\to+\infty} \varphi(r + b_{k_l}) = \varphi^*(r)$, uniformly in $r \in \mathbb{R}$. Set

$$F^*(t, s; x) := e^{\int_s^t \varphi^*(\tau)\, d\tau} T(t - s)x, \quad (t, s) \in \mathbb{R}^2,\ x \in X.$$

Then the function $\varphi^*(\cdot)$ is bounded and there exists a finite real constant $c_{L,B} > 0$ such that, for every integer $l \geq l_0(\epsilon)$,

$$\begin{aligned}
&\left\| e^{\int_{s+b_{k_l}}^{t+b_{k_l}} \varphi(\tau)\,d\tau} T(t-s)x - e^{\int_s^t \varphi^*(\tau)\,d\tau} T(t-s)x \right\|_Y \\
&\leq c_{L,B} \left| e^{\int_{s+b_{k_l}}^{t+b_{k_l}} \varphi(\tau)\,d\tau} - e^{\int_s^t \varphi^*(\tau)\,d\tau} \right| \leq c_{L,B} \left| e^{\int_s^t \varphi(\tau+b_{k_l})\,d\tau} - e^{\int_s^t \varphi^*(\tau)\,d\tau} \right| \\
&\leq c_{L,B} e^{L\|\varphi^*\|_\infty} \left| e^{\int_s^t [\varphi(\tau+b_{k_l})-\varphi^*(\tau)]\,d\tau} - 1 \right| \\
&\leq c_{L,B} \left| \int_s^t \big[\varphi(\tau+b_{k_l}) - \varphi^*(\tau)\big]\,d\tau \right| e^{\left|\int_s^t [\varphi(\tau+b_{k_l})-\varphi^*(\tau)]\,d\tau\right|} \leq c_{L,B} L\epsilon e^{L\epsilon},
\end{aligned}$$

which simply implies the required. A large class of relatively simple examples shows that the function $F(\cdot, \cdot; \cdot)$ is not (R, $\mathcal{B}$)-multi-almost periodic in general (let us only note here that the obtained conclusions can be simply applied to some partial differential equations in the distributional spaces as well as that it would be very difficult to aggregate all such applications; put e.g. $\varphi \equiv 0$ in (2.4)).

We can simply construct the corresponding analogue of this example in the higher dimensions $n > 2$; for example, if $\varphi_j : \mathbb{R} \to \mathbb{R}$ is (compactly) almost automorphic or almost periodic and $(T(t))_{t\in\mathbb{R}} \subseteq L(X, Y)$ is a strongly continuous operator family $(1 \leq j \leq n-1)$, resp., if $\varphi_j : \mathbb{R} \to \mathbb{R}$ is (compactly) almost automorphic or almost periodic and $(T(t))_{t\in\mathbb{R}} \subseteq L(X, Y)$ is a strongly continuous operator family $(1 \leq j \leq n)$, then the similar conclusions hold for the function $F : \mathbb{R}^n \times X \to X$ defined through $((t_1, t_2, \ldots, t_n) \in \mathbb{R}^n,\ x \in X)$:

$$F\big(t_1, t_2, \ldots, t_n; x\big) := \sum_{j=1}^{n-1} T_j\big(t_{j+1} - t_j\big) e^{\int_{t_j}^{t_{j+1}} \varphi_j(\xi)\,d\xi} x,$$

with $\mathrm{R} := \{b : \mathbb{N} \to \mathbb{R}^n;\ \text{ for all } j \in \mathbb{N} \text{ we have } b_j \in \{(a, a, a, \ldots, a) \in \mathbb{R}^n : a \in \mathbb{R}\}\}$, resp., for the function

$$F\big(t_1, t_2, \ldots, t_{2n}; x\big) := \sum_{j=1}^{n} T_j\big(t_{2j} - t_{2j-1}\big) e^{\int_{t_{2j-1}}^{t_{2j}} \varphi_j(\xi)\,d\xi} x,$$

with $\mathrm{R} := \{b : \mathbb{N} \to \mathbb{R}^n;\ \text{ for all } j \in \mathbb{N} \text{ we have } b_j \in \{(a_1, a_1, a_2, a_2, \ldots, a_n, a_n) \in \mathbb{R}^{2n} : a_i \in \mathbb{R}\}\}$.

Example Let $f_j : \mathbb{R} \to \mathbb{R}$ be a (compactly) almost automorphic function $(1 \leq j \leq n)$. The function $F : \mathbb{R}^{2n} \to \mathbb{R}$, defined by

$$\begin{aligned}
\big(s_1, s_2, \ldots, s_n, t_1, t_2, \ldots, t_n\big) &\mapsto F\big(s_1, s_2, \ldots, s_n, t_1, t_2, \ldots, t_n\big) \\
&:= \prod_{j=1}^{n} \int_{s_j}^{t_j} f_j(\xi)\,d\xi\,,
\end{aligned}$$

is (compactly) R-multi-almost automorphic, where $\mathrm{R} := \{b : \mathbb{N} \to \mathbb{R}^n \times \mathbb{R}^n ;$ for all $j \in \mathbb{N}$ we have $b_j \in \{(a_1, a_2, \ldots, a_n, a_1, a_2, \ldots, a_n) \in \mathbb{R}^n \times \mathbb{R}^n : a_i \in \mathbb{R}$ for $1 \leq i \leq n\}\}$.

In Chávez et al. (2020, Example 2.15(i)), we have analyzed case in which the functions $t \mapsto \int_0^t f_j(s)\,ds,\ t \in \mathbb{R}$ are almost periodic ($1 \leq j \leq n$); if we assume that the functions $t \mapsto f_j(t),\ t \in \mathbb{R}$ are almost periodic ($1 \leq j \leq n$), then we can simply prove that the function $F(\cdot)$ will be $(\mathrm{R}, \mathrm{P}_{\mathrm{R}})$-multi-almost automorphic, where for each sequence $b \in \mathrm{R}$ the collection P_{R} consists of all sets of the form $\{(s_1, s_2, \ldots, s_n, t_1, t_2, \ldots, t_n) \in \mathbb{R}^{2n} : |s_i - t_i| \leq L_i$ for all $i \in \mathbb{N}_n\}$ with $L_i > 0$ for all $i \in \mathbb{N}_n$.

Example This example substantially generalizes the previous one. Let R be any collection of sequences in $\mathbb{R}^n$ such that each subsequence of a sequence $(\mathbf{b}_k) \in \mathrm{R}$ also belongs to R, and let R′ be any collection of sequences in $\mathbb{R}^m$ such that each subsequence of a sequence $(\mathbf{b}_k') \in \mathrm{R}'$ also belongs to R′. Let $f_i : \mathbb{R}^n \to \mathbb{R}$ be a bounded, (compactly) R-almost automorphic function ($1 \leq i \leq p$), and let $g_j : \mathbb{R}^m \to \mathbb{R}$ be a bounded, (compactly) R′-almost automorphic function ($1 \leq j \leq q$). Define the functions $F : \mathbb{R}^n \to \mathbb{R}^q$ by $F(\mathbf{t}) := \sum_{i=1}^p f_i(\mathbf{t})e_i$ and $G : \mathbb{R}^m \to \mathbb{R}^q$ by $G(\mathbf{s}) := \sum_{j=1}^q g_j(\mathbf{s})e_j$. Now, we define the function $F \bigotimes G : \mathbb{R}^n \times \mathbb{R}^m \to M_{p\times q}(\mathbb{R})$ by ($\mathbf{t} \in \mathbb{R}^n,\ \mathbf{s} \in \mathbb{R}^m$)

$$F \bigotimes G\,(\mathbf{t}, \mathbf{s}) := \begin{pmatrix} f_1(\mathbf{t})g_1(\mathbf{s}) & f_1(\mathbf{t})g_2(\mathbf{s}) & \cdots & f_1(\mathbf{t})g_q(\mathbf{s}) \\ f_2(\mathbf{t})g_1(\mathbf{s}) & f_2(\mathbf{t})g_2(\mathbf{s}) & \cdots & f_2(\mathbf{t})g_q(\mathbf{s}) \\ \vdots & \vdots & \ddots & \vdots \\ f_p(\mathbf{t})g_1(\mathbf{s}) & f_p(\mathbf{t})g_2(\mathbf{s}) & \cdots & f_p(\mathbf{t})g_q(\mathbf{s}) \end{pmatrix},$$

where $M_{p\times q}(\mathbb{R})$ denotes the set of all real matrices of format $p \times q$. Suppose that the sequences $\mathbf{b} \in \mathrm{R}$ and $\mathbf{b}' \in \mathrm{R}'$ are given. Due to our assumption, we get the existence of a subsequence $\mathbf{b}_0 \in \mathrm{R}$ of $\mathbf{b}$, a subsequence $\mathbf{b}_0'$ of $\mathbf{b}'$ and the corresponding limit functions $f_j^*(\cdot),\ g_k^*(\cdot)$ ($1 \leq j \leq p,\ 1 \leq k \leq q$) from the definition of R-multi-almost automorphy (R′-multi-almost automorphy) of functions $f_j(\cdot),\ g_k(\cdot)$ ($1 \leq j \leq p$, $1 \leq k \leq q$). Set, for every $\mathbf{t} \in \mathbb{R}^n$ and $\mathbf{s} \in \mathbb{R}^m$,

$$\left[F \bigotimes G\right]^*(\mathbf{t}, \mathbf{s}) := \begin{pmatrix} f_1^*(\mathbf{t}) \cdot g_1^*(\mathbf{s}) & f_1^*(\mathbf{t}) \cdot g_2^*(\mathbf{s}) & \cdots & f_1^*(\mathbf{t}) \cdot g_q^*(\mathbf{s}) \\ f_2^*(\mathbf{t}) \cdot g_1^*(\mathbf{s}) & f_2^*(\mathbf{t}) \cdot g_2^*(\mathbf{s}) & \cdots & f_2^*(\mathbf{t}) \cdot g_q^*(\mathbf{s}) \\ \vdots & \vdots & \ddots & \vdots \\ f_p^*(\mathbf{t}) \cdot g_1^*(\mathbf{s}) & f_p^*(\mathbf{t}) \cdot g_2^*(\mathbf{s}) & \cdots & f_p^*(\mathbf{t}) \cdot g_q^*(\mathbf{s}) \end{pmatrix}.$$

Using this limit function, it is not difficult to prove that $F \bigotimes G$ is (compactly) $(\mathrm{R}\times\mathrm{R}')$-almost automorphic, where $\mathrm{R} \times \mathrm{R}' := \{(\mathbf{b}, \mathbf{b}') : \mathbf{b} \in \mathrm{R},\ \mathbf{b}' \in \mathrm{R}'\}$. Furthermore, if for each $i \in \mathbb{N}_p$ we have that $f_i : \mathbb{R}^n \to \mathbb{R}$ is a bounded $(\mathrm{R}, \mathrm{P}_{\mathrm{R}})$-almost automorphic function as well as that for each $j \in \mathbb{N}_q$ we have that $g_j : \mathbb{R}^n \to \mathbb{R}$ is a bounded

$(\mathrm{R}', \mathrm{P}'_{\mathrm{R}'})$-almost automorphic function, then the function $F \bigotimes G$ is $(\mathrm{R} \times \mathrm{R}', \mathrm{P}''_{\mathrm{R}\times\mathrm{R}'})$-almost automorphic function, provided that for each sequence $\mathbf{b}$ from R ($\mathbf{c}$ from R') each set of the collection $\mathrm{P}_{\mathbf{b}}$ ($\mathrm{P}_{\mathbf{c}}$) belongs to the collection $\mathrm{P}_{\mathbf{b}'}$ ($\mathrm{P}_{\mathbf{c}'}$) for any subsequence $\mathbf{b}'$ of $\mathbf{b}$ ($\mathbf{c}'$ of $\mathbf{c}$) and for each sequence $(\mathbf{b};\mathbf{c})$ belonging to $\mathrm{R} \times \mathrm{R}'$ the collection $\mathrm{P}''_{(\mathbf{b};\mathbf{c})}$ consists of all direct products of sets from the collections $\mathrm{P}_{\mathbf{b}}$ and $\mathrm{P}'_{\mathbf{c}}$.

From the point of view of the theory of differential equations with piecewise constant argument (see e.g., the references quoted in Chávez et al. (2014a, b)), the continuity of function $F(\cdot;\cdot)$ in Definition 2.1 is a slightly redundant condition; we will not go into further details concerning this question here (see e.g., (Chávez et al. 2014b, Definition 2.3)). Further on, the notion introduced in Definition 2.1 is a special case of the notion introduced in the following definition (with $\mathrm{R}_{\mathrm{X}} := \{b : \mathbb{N} \to \mathbb{R}^n \times X\,;\, (\exists a \in \mathrm{R})\, b(l) = (a(l); 0) \text{ for all } l \in \mathbb{N}\}$); this is an extremely important notion because, in case that $X \in \mathcal{B}$ and R_X denotes the collection of all sequences in $\mathbb{R}^n \times X$, the notion of $(\mathrm{R}_{\mathrm{X}}, \mathcal{B})$-multi-almost automorphy is equivalent with the usual notion of almost automorphy on the topological group $\mathbb{R}^n \times X$ (see appendix for more details):

Definition 2.4 Suppose that $F : \mathbb{R}^n \times X \to Y$ is a continuous function. Then we say that the function $F(\cdot;\cdot)$ is $(\mathrm{R}_{\mathrm{X}}, \mathcal{B})$-multi-almost automorphic if and only if for every $B \in \mathcal{B}$ and for every sequence $((\mathbf{b};\mathbf{x})_k = ((b_k^1, b_k^2, \ldots, b_k^n); x_k)) \in \mathrm{R}_{\mathrm{X}}$ there exist a subsequence $((\mathbf{b};\mathbf{x})_{k_l} = ((b_{k_l}^1, b_{k_l}^2, \ldots, b_{k_l}^n); x_{k_l}))$ of $((\mathbf{b};\mathbf{x})_k)$ and a function $F^* : \mathbb{R}^n \times X \to Y$ such that

$$\lim_{m\to+\infty} F\big(\mathbf{t} + (b_{k_m}^1, \ldots, b_{k_m}^n); x + x_{k_m}\big) = F^*(\mathbf{t}; x) \tag{2.5}$$

and

$$\lim_{l\to+\infty} F^*\big(\mathbf{t} - (b_{k_l}^1, \ldots, b_{k_l}^n); x - x_{k_l}\big) = F(\mathbf{t}; x), \tag{2.6}$$

pointwisely for all $x \in B$ and $\mathbf{t} \in \mathbb{R}^n$. We say that the function $F(\cdot;\cdot)$ is compactly $(\mathrm{R}_{\mathrm{X}}, \mathcal{B})$-multi-almost automorphic if and only if the convergence of limits in (2.5)–(2.6) is uniform on any compact subset K of $\mathbb{R}^n \times X$ which is a subset of $\mathbb{R}^n \times B$. By $AA_{(\mathrm{R}_{\mathrm{X}},\mathcal{B})}(\mathbb{R}^n \times X : Y)$ and $AA_{(\mathrm{R}_{\mathrm{X}},\mathcal{B},\mathbf{c})}(\mathbb{R}^n \times X : Y)$ we denote the spaces consisting of all $(\mathrm{R}_{\mathrm{X}}, \mathcal{B})$-multi-almost automorphic functions and compactly $(\mathrm{R}_{\mathrm{X}}, \mathcal{B})$-multi-almost automorphic functions, respectively.

Further on, let for each $B \in \mathcal{B}$ and $(\mathbf{b};\mathbf{x}) = ((\mathbf{b};\mathbf{x})_k = ((b_k^1, b_k^2, \ldots, b_k^n); x_k)_k) \in \mathrm{R}_{\mathrm{X}}$ we have $W_{B,(\mathbf{b};\mathbf{x})} : B \to P(P(\mathbb{R}^n))$ and $\mathrm{P}_{B,(\mathbf{b};\mathbf{x})} \in P(P(\mathbb{R}^n \times B))$. Then the following notion generalizes the corresponding notion from Definition 2.2; we say that $F(\cdot;\cdot)$ is:

(i) $(\mathrm{R}_X, \mathcal{B}, W_{\mathcal{B},\mathrm{R}_X})$-multi-almost automorphic if and only if for every $B \in \mathcal{B}$ and for every sequence $((\mathbf{b};\mathbf{x})_k = ((b_k^1, b_k^2, \ldots, b_k^n); x_k)_k) \in \mathrm{R}_{\mathrm{X}}$ there exist a subsequence $((\mathbf{b};\mathbf{x})_{k_l})$ of $((\mathbf{b};\mathbf{x})_k)$ and a function $F^* : \mathbb{R}^n \times X \to Y$ such that (2.5)–(2.6) hold pointwisely for all $x \in B$ and $\mathbf{t} \in \mathbb{R}^n$ as well as that for each $x \in B$ the convergence in (2.5)–(2.6) is uniform in $\mathbf{t}$ for any set of the collection $W_{B,(\mathbf{b};\mathbf{x})}(x)$;

(ii) $(\mathrm{R}_X, \mathcal{B}, \mathrm{P}_{\mathcal{B},\mathrm{R}_X})$-multi-almost automorphic if and only if for every $B \in \mathcal{B}$ and for every sequence $((\mathbf{b}; \mathbf{x})_k = ((b_k^1, b_k^2, \ldots, b_k^n); x_k)) \in \mathrm{R}_{\mathrm{X}}$ there exist a subsequence $((\mathbf{b}; \mathbf{x})_{k_l})$ of $((\mathbf{b}; \mathbf{x})_k)$ of $((\mathbf{b}; \mathbf{x})_k)$ and a function $F^* : \mathbb{R}^n \times X \to Y$ such that (2.5)–(2.6) hold pointwisely for all $x \in B$ and $\mathbf{t} \in \mathbb{R}^n$ as well as that the convergence in (2.5)–(2.6) is uniform in $(\mathbf{t}; x)$ for any set of the collection $\mathrm{P}_{B,(\mathbf{b};\mathbf{x})}$.

It is clear that the assumption $X \in \mathcal{B}$ implies that a continuous function $F : \mathbb{R}^n \times X \to Y$ is (compactly) $(\mathrm{R}_{\mathrm{X}}, \mathcal{B})$-multi-almost automorphic if and only if the above requirements hold for any sequence $((\mathbf{b}; \mathbf{x})_k) \in \mathrm{R}_{\mathrm{X}}$ and the set $B = X$.

The following result holds true:

Proposition 2.5 (i) *Suppose that $F : \mathbb{R}^n \times X \to Y$ is an $(\mathrm{R}, \mathcal{B})$-multi-almost automorphic function, where R denotes the collection of all sequences in $\mathbb{R}^n$ and $\mathcal{B}$ denotes any collection of compact subsets of X. If for every $B \in \mathcal{B}$ there exists a finite real constant $L_B > 0$ such that, for every $x,\ y \in B$ and $\mathbf{t} \in \mathbb{R}^n$, we have*

$$\left\| F(\mathbf{t}; x) - F(\mathbf{t}; y) \right\|_Y \leq L_B \|x - y\|, \tag{2.7}$$

then, for every set $B \in \mathcal{B}$, we have that the set $\{F(\mathbf{t}, x) : \mathbf{t} \in \mathbb{R}^n,\ x \in B\}$ is relatively compact in Y.

(ii) *Suppose that $F : \mathbb{R}^n \times X \to Y$ is an $(\mathrm{R}_X, \mathcal{B})$-multi-almost automorphic function, where R_X denotes the collection of all sequences in $\mathbb{R}^n \times X$ and $\mathcal{B}$ denotes any collection of compact subsets of X. Then, for every set $B \in \mathcal{B}$, we have that the set $\{F(\mathbf{t}, x) : \mathbf{t} \in \mathbb{R}^n,\ x \in B\}$ is relatively compact in Y.*

Proof To prove (i), it suffices to show that, for every sequence $((\mathbf{t}_k; x_k))_{k\in\mathbb{N}}$ in $\mathbb{R}^n \times B$, there exists a subsequence $((\mathbf{t}_{k_l}; x_{k_l}))_{l\in\mathbb{N}}$ which converges for topology of Y. Since B is compact, we may assume without loss of generality that $x_k \to x$, $k \to +\infty$ for some element $x \in B$. Applying the definition of $(\mathrm{R}, \mathcal{B})$-multi-almost automorphy, we can find a subsequence $((\mathbf{t}_{k_l}; x_{k_l}))_{l\in\mathbb{N}}$ of $((\mathbf{t}_k; x_k))_{k\in\mathbb{N}}$ such that $F(0 + \mathbf{t}_{k_l}; x) = F(\mathbf{t}_{k_l}; x)$ converges to some element $y \in Y$ as $l \to +\infty$. Then the final conclusion follows from (2.7) and the decomposition

$$\begin{aligned}\left\| F\big(\mathbf{t}_{k_l}; x_{k_l}\big) - y \right\|_Y &\leq \left\| F\big(\mathbf{t}_{k_l}; x_{k_l}\big) - F(\mathbf{t}_{k_l}, x) \right\|_Y + \left\| F(\mathbf{t}_{k_l}; x) - y \right\|_Y \\ &\leq L_B \left\| x_{k_l} - x \right\| + \left\| F(\mathbf{t}_{k_l}; x) - y \right\|_Y.\end{aligned}$$

The proof of (ii) is similar but, in this part, we do not need any Lipschitz type condition because there exists a subsequence of sequence $((\mathbf{t}_{k_l}; x_{k_l}))_{l\in\mathbb{N}} \in \mathrm{R}_X$ of $((\mathbf{t}_k; x_k))_{n\in\mathbb{N}}$ obeying the properties in the definition of $(\mathrm{R}_X, \mathcal{B})$-multi-almost automorphy. □

Before we move ourselves to Sect. 2.1, we would like to note that it is very simple to show that the assumption $X \in \mathcal{B}$ implies that a continuous function $F : \mathbb{R}^n \times X \to Y$ is $(\mathrm{R}_{\mathrm{X}}, \mathcal{B})$-multi-almost automorphic if and only if for every sequence $((\mathbf{b}; \mathbf{x})_k) \in \mathrm{R}_{\mathrm{X}}$ there exists a subsequence $((\mathbf{b}; \mathbf{x})_{k_l})$ of $((\mathbf{b}; \mathbf{x})_k)$ such that

$$\lim_{l\to+\infty} \lim_{m\to+\infty} F\big(\mathbf{t} - \mathbf{b}_{k_l} + \mathbf{b}_{k_m}; x - x_{k_l} + x_{k_m}\big) = F(\mathbf{t}; x),$$

pointwisely for all $x \in X$ and $\mathbf{t} \in \mathbb{R}^n$; in general case ($X \in \mathcal{B}$ or $X \notin \mathcal{B}$), the $(\mathrm{R}, \mathcal{B})$-multi-almost automorphy of a continuous function $F : \mathbb{R}^n \times X \to Y$ is equivalent to saying that for every $B \in \mathcal{B}$ and for every sequence $(\mathbf{b}_k) \in \mathrm{R}$ there exists a subsequence $(\mathbf{b}_{k_l})$ of $(\mathbf{b}_k)$ such that

$$\lim_{l\to+\infty} \lim_{m\to+\infty} F\big(\mathbf{t} - \mathbf{b}_{k_l} + \mathbf{b}_{k_m}; x\big) = F(\mathbf{t}; x), \tag{2.8}$$

pointwisely for all $x \in B$ and $\mathbf{t} \in \mathbb{R}^n$.

2.1 Compactly $(\mathrm{R}_X, \mathcal{B})$-Multi-almost Automorphic Functions

In this subsection, we analyze compactly $(\mathrm{R}_X, \mathcal{B})$-multi-almost automorphic functions. The following result is crucial:

Theorem 2.6 *Suppose that $F : \mathbb{R}^n \times X \to Y$ is an $(\mathrm{R_X}, \mathcal{B})$-multi-almost automorphic function as well as that, for every $B \in \mathcal{B}$ and for every sequence $((\mathbf{b}; \mathbf{x})_k) \in \mathrm{R_X}$, there exist a subsequence $((\mathbf{b}; \mathbf{x})_{k_l})$ of $((\mathbf{b}; \mathbf{x})_k)$ and a function $F^* : \mathbb{R}^n \times X \to Y$ such that (2.5)–(2.6) hold pointwisely for all $x \in B$ and $\mathbf{t} \in \mathbb{R}^n$. Let for each $B \in \mathcal{B}$ and $(\mathbf{b}; \mathbf{x}) \in \mathrm{R_X}$ we have $\mathrm{P}_{B,(\mathbf{b};\mathbf{x})} \in P(P(\mathbb{R}^n \times B))$. Suppose also that the following conditions hold:*

(a) *if $(\mathbf{b}; \mathbf{x}) \in \mathrm{R_X}$, then every subsequence of $(\mathbf{b}; \mathbf{x})$ also belongs to R_X;*
(b) *if $B \in \mathcal{B}$, $((\mathbf{b}; \mathbf{x})_k = ((b_k^1, b_k^2, \ldots, b_k^n); x_k)) \in \mathrm{R_X}$ and $D \in P_{B,((\mathbf{b};\mathbf{x})_k)}$, then $D \in P_{B,((\mathbf{b};\mathbf{x})_{k_l})}$ for every subsequence $((\mathbf{b}; \mathbf{x})_{k_l})$ of $((\mathbf{b}; \mathbf{x})_k)$.*

Then the following holds:

(i) *If $F(\cdot; \cdot)$ is $(\mathrm{R_X}, \mathcal{B}, P_{\mathcal{B},\mathrm{R}_X})$-multi-almost automorphic, then the following statements are equivalent:*

  (c) *for every $B \in \mathcal{B}$ and $((\mathbf{b}; \mathbf{x})_k = ((b_k^1, b_k^2, \ldots, b_k^n); x_k)) \in \mathrm{R_X}$, the limit function $F^*(\cdot; \cdot)$ is uniformly continuous on any set D of the collection $P_{B,((\mathbf{b};\mathbf{x})_k)}$;*
  (d) *for every $\epsilon > 0$, $B \in \mathcal{B}$, $((\mathbf{b}; \mathbf{x})_k = ((b_k^1, b_k^2, \ldots, b_k^n); x_k)) \in \mathrm{R_X}$ and $D \in P_{B,((\mathbf{b};\mathbf{x})_k)}$, there exist a subsequence $((\mathbf{b}; \mathbf{x})_{k_l})$ of $((\mathbf{b}; \mathbf{x})_k)$, an integer $l_0 \in \mathbb{N}$ and a finite real number $\delta > 0$ such that, for every $(\mathbf{t}; x)$, $(\mathbf{t}'; x') \in D$ with $|\mathbf{t} - \mathbf{t}'| + \|x - x'\| \le \delta$ and for every integer $l \ge l_0$, we have*

$$\Big\| F\big(\mathbf{t} + b_{k_l}; x + x_{k_l}\big) - F\big(\mathbf{t}' + b_{k_l}; x' + x_{k_l}\big) \Big\|_Y \le \epsilon. \tag{2.9}$$

  Moreover, (c) *and* (d) *hold provided that condition* (Q) *holds, where:*

  (Q) *For every $B \in \mathcal{B}$ and $(\mathbf{b}; \mathbf{x}) \in \mathrm{R}_X$, we have that every set D of the collection $P_{B,((\mathbf{b};\mathbf{x})_k)}$ is compact in $\mathbb{R}^n \times X$.*

(ii) *If* (Q) *holds, then the validity of condition* (d) *and*

$(d)_s$: for every $\epsilon > 0$, $B \in \mathcal{B}$, $((\mathbf{b}; \mathbf{x})_k = ((b_k^1, b_k^2, \ldots, b_k^n); x_k)) \in \mathrm{R_X}$ and $D \in P_{B,((\mathbf{b};\mathbf{x})_k)}$, there exist a subsequence $((\mathbf{b}; \mathbf{x})_{k_l})$ of $((\mathbf{b}; \mathbf{x})_k)$, integers l_0, $m_0 \in$

$\mathbb{N}$, *and a finite real number* $\delta > 0$ *such that, for every* $(\mathbf{t}; x),\ (\mathbf{t}'; x') \in D$ *with* $|\mathbf{t} - \mathbf{t}'| + \|x - x'\| \le \delta$ *and for every integers* $l \ge l_0$ *and* $m \ge m_0$*, we have* $x - x_{k_l} \in B$ *and*

$$\left\| F\big(\mathbf{t} - b_{k_l} + b_{k_m}; x - x_{k_l} + x_{k_m}\big) - F\big(\mathbf{t}' - b_{k_l} + b_{k_m}; x' - x_{k_l} + x_{k_m}\big) \right\|_Y \le \epsilon,$$

implies that the function $F(\cdot;\cdot)$ *is* $(\mathrm{R_X}, \mathcal{B}, P_{\mathcal{B},\mathrm{R}_X})$*-multi-almost automorphic.*

Proof We will firstly prove that (d) implies (c). Let $\epsilon > 0$, $B \in \mathcal{B}$, $((\mathbf{b}; \mathbf{x})_k = ((b_k^1, b_k^2, \ldots, b_k^n); x_k)) \in \mathrm{R_X}$ and $D \in P_{B,((\mathbf{b};\mathbf{x})_k)}$. Further on, let a subsequence $((\mathbf{b}; \mathbf{x})_{k_l} = ((b_{k_l}^1, b_{k_l}^2, \ldots, b_{k_l}^n); x_{k_l}))$ of $((\mathbf{b}; \mathbf{x})_k)$ and a function $F^* : \mathbb{R}^n \times X \to Y$ be such that (2.5)–(2.6) hold pointwisely for all $x \in B$ and $\mathbf{t} \in \mathbb{R}^n$. Then $((\mathbf{b}; \mathbf{x})_{k_l})$ is a sequence which belongs to the collection R_X and $D \in P_{B,((\mathbf{b};\mathbf{x})_{k_l})}$ due to conditions (a) and (b). Since (d) holds, we may assume without loss of generality that there exist an integer $l_0 \in \mathbb{N}$ and a finite real number $\delta > 0$ such that, for every $(\mathbf{t}; x),\ (\mathbf{t}'; x') \in D$ with $|\mathbf{t}-\mathbf{t}'|+\|x-x'\| \le \delta$ and for every integer $l \ge l_0$, we have (2.9) with the number ϵ replaced therein with the number $\epsilon/3$. Since $F(\cdot;\cdot)$ is $(\mathrm{R_X}, \mathcal{B}, P_{\mathcal{B},\mathrm{R}_X})$-multi-almost automorphic, (c) simply follows from the decomposition

$$\begin{aligned}
&\left\| F^*(\mathbf{t}; x) - F^*(\mathbf{t}'; x') \right\|_Y \\
&\le \left\| F^*(\mathbf{t}; x) - F\big(\mathbf{t} + b_{k_l}; x + x_{k_l}\big) \right\|_Y \\
&\quad + \left\| F\big(\mathbf{t} + b_{k_l}; x + x_{k_l}\big) - F\big(\mathbf{t}' + b_{k_l}; x' + x_{k_l}\big) \right\|_Y \\
&\quad + \left\| F\big(\mathbf{t}' + b_{k_l}; x' + x_{k_l}\big) - F^*(\mathbf{t}'; x') \right\|_Y \\
&\le 2\epsilon/3 + \left\| F\big(\mathbf{t} + b_{k_l}; x + x_{k_l}\big) - F\big(\mathbf{t}' + b_{k_l}; x' + x_{k_l}\big) \right\|_Y \le \epsilon, \quad l \ge l_0.
\end{aligned}$$

The proof of implication (c) $\Rightarrow$ (d) is similar and follows from the decomposition:

$$\begin{aligned}
&\left\| F\big(\mathbf{t} + b_{k_l}; x + x_{k_l}\big) - F\big(\mathbf{t}' + b_{k_l}; x' + x_{k_l}\big) \right\|_Y \\
&\le \left\| F^*(\mathbf{t}; x) - F\big(\mathbf{t} + b_{k_l}; x + x_{k_l}\big) \right\|_Y + \left\| F^*(\mathbf{t}; x) - F^*(\mathbf{t}'; x') \right\|_Y \\
&\quad + \left\| F^*(\mathbf{t}'; x') - F\big(\mathbf{t}' + b_{k_l}; x' + x_{k_l}\big) \right\|_Y.
\end{aligned}$$

Assume now that (Q) holds and $\epsilon > 0$. Then, for every fixed set $B \in \mathcal{B}$ and for every sequence $(\mathbf{b}; \mathbf{x}) \in \mathrm{R}_X$, we have that every set D of the collection $P_{B,((\mathbf{b};\mathbf{x})_k)}$ is compact. Furthermore, the above argumentation yields that there exists an integer $l_0 \in \mathbb{N}$ such that, for every $(\mathbf{t}; x),\ (\mathbf{t}'; x') \in D$, we have

$$\left\| F^*(\mathbf{t}; x) - F^*(\mathbf{t}'; x') \right\|_Y \le 2\epsilon/3 + \left\| F\big(\mathbf{t} + b_{k_{l_0}}; x + x_{k_{l_0}}\big) - F\big(\mathbf{t}' + b_{k_{l_0}}; x' + x_{k_{l_0}}\big) \right\|_Y.$$

Since the function $F(\cdot;\cdot)$ is uniformly continuous on the compact set $D+(\mathbf{b}_{k_{l_0}}; x_{k_{l_0}})$, the above estimate simply implies (c). In order to show (ii), suppose again that condition (Q) holds. Let (d) hold, and let $\epsilon > 0$ be fixed. We need to prove that the function

$F(\cdot;\cdot)$ is $(\mathrm{R_X}, \mathcal{B}, P_{\mathcal{B},\mathrm{R}_X})$-multi-almost automorphic. If the set D from the the collection $P_{B,((\mathbf{b};\mathbf{x})_k)}$ is fixed, then (d) implies the existence of a subsequence $((\mathbf{b};\mathbf{x})_{k_l})$ of $((\mathbf{b};\mathbf{x})_k)$, an integer $l_0 \in \mathbb{N}$ and a finite real number $\delta_1 > 0$ such that, for every $(\mathbf{t};x)$, $(\mathbf{t}';x') \in D$ with $|\mathbf{t}-\mathbf{t}'| + \|x-x'\| \leq \delta_1$ and for every integer $l \geq l_0$, we have (2.9) with the number ϵ replaced therein with the number $\epsilon/3$. Since (c) holds, there exists a number $\delta \in (0,\delta_1]$ such that, for every $(\mathbf{t};x)$, $(\mathbf{t}';x') \in D$ with $|\mathbf{t}-\mathbf{t}'| + \|x-x'\| \leq \delta$, we have

$$\left\| F^*(\mathbf{t};x) - F^*(\mathbf{t}';x') \right\|_Y \leq \epsilon/3.$$

Moreover, since D is compact and $F(\cdot;\cdot)$ is uniformly continuous on D, there exists a finite net $\{(\mathbf{t}_i;x_i)\}_{1\leq i\leq n}$ in D such that, for every $(\mathbf{t};x) \in D$, we have the existence of a number $i \in \mathbb{N}_n$ such that $|\mathbf{t}-\mathbf{t}_i| + \|x-x_i\| \leq \delta$ and

$$\left\| F\big(\mathbf{t}_i;x_i\big) - F(\mathbf{t};x) \right\|_Y \leq \epsilon/3.$$

Then there exists an integer $l_0 \in \mathbb{N}$ such that, for every integer $l \geq l_0$ and for every tuple $(\mathbf{t};x) \in D$, we have:

$$\begin{aligned}
&\left\| F\big(\mathbf{t}+b_{k_l};x+x_{k_l}\big) - F^*(\mathbf{t};x) \right\|_Y \\
&\quad\leq \left\| F\big(\mathbf{t}+b_{k_l};x+x_{k_l}\big) - F\big(\mathbf{t}_i+b_{k_l};x_i+x_{k_l}\big) \right\|_Y \\
&\qquad + \left\| F\big(\mathbf{t}_i+b_{k_l};x_i+x_{k_l}\big) - F^*\big(\mathbf{t}_i;x_i\big) \right\|_Y + \left\| F^*\big(\mathbf{t}_i;x_i\big) - F^*(\mathbf{t};x) \right\|_Y \\
&\quad\leq 2\epsilon/3 + \left\| F\big(\mathbf{t}_i+b_{k_l};x_i+x_{k_l}\big) - F^*\big(\mathbf{t}_i;x_i\big) \right\|_Y \leq 2\epsilon/3 + \epsilon/3 = \epsilon,
\end{aligned}$$

due to condition (d). Moreover, we have:

$$\begin{aligned}
&\left\| F^*\big(\mathbf{t}-b_{k_l};x-x_{k_l}\big) - F(\mathbf{t};x) \right\|_Y \\
&\quad\leq \left\| F^*\big(\mathbf{t}-b_{k_l};x-x_{k_l}\big) - F^*\big(\mathbf{t}_i-b_{k_l};x_i-x_{k_l}\big) \right\|_Y \\
&\qquad + \left\| F^*\big(\mathbf{t}_i-b_{k_l};x_i-x_{k_l}\big) - F\big(\mathbf{t}_i;x_i\big) \right\|_Y + \left\| F\big(\mathbf{t}_i;x_i\big) - F(\mathbf{t};x) \right\|_Y \\
&\quad\leq \left\| F^*\big(\mathbf{t}-b_{k_l};x-x_{k_l}\big) - F^*\big(\mathbf{t}_i-b_{k_l};x_i-x_{k_l}\big) \right\|_Y \\
&\qquad + \left\| F^*\big(\mathbf{t}_i-b_{k_l};x_i-x_{k_l}\big) - F\big(\mathbf{t}_i;x_i\big) \right\|_Y + \epsilon/3 \\
&\quad\leq \left\| F^*\big(\mathbf{t}-b_{k_l};x-x_{k_l}\big) - F^*\big(\mathbf{t}_i-b_{k_l};x_i-x_{k_l}\big) \right\|_Y + 2\epsilon/3 \quad \big(l \geq l_0\big) \\
&\quad= \Big\| \lim_{m\to+\infty} \Big[F\big(\mathbf{t}-b_{k_l}+b_{k_m};x-x_{k_l}+x_{k_m}\big) \\
&\qquad - F\big(\mathbf{t}_i-b_{k_l}+b_{k_m};x_i-x_{k_l}+x_{k_m}\big)\Big]\Big\|_Y \\
&\qquad + 2\epsilon/3 \leq \epsilon, \quad l \geq l_0,\ m \geq m_0,
\end{aligned}$$

where we have applied $(d)_s$ in the last estimate. □

Now we would like to state the following important corollary of Theorem 2.6:

Corollary 2.7 *Suppose that* $F : \mathbb{R}^n \times X \to Y$ *is an* $(\mathrm{R}_{\mathrm{X}}, \mathcal{B})$*-multi-almost automorphic function,* $X \in \mathcal{B}$ *and* R_{X} *denotes the collection of all sequences in* $\mathbb{R}^n \times X$*. Then* $F(\cdot;\cdot)$ *is compactly* $(\mathrm{R}_{\mathrm{X}}, \mathcal{B})$*-multi-almost automorphic if and only if* $F(\cdot;\cdot)$ *is uniformly continuous.*

Proof Without loss of generality, we may assume that $\mathcal{B} = \{X\}$ as well as that, for every sequence $(\mathbf{b}; \mathbf{x})$ in $\mathbb{R}^n \times X$, we have that $P_{B,(\mathbf{b};\mathbf{x})}$ is the collection of all compact sets in $\mathbb{R}^n \times X$. Let $F(\cdot;\cdot)$ be uniformly continuous. Then conditions (d) and $(d)_s$ hold, so that the conclusion simply follows from Theorem 2.6. Assume that $F(\cdot;\cdot)$ is compactly $(\mathrm{R}_{\mathrm{X}}, \mathcal{B})$-multi-almost automorphic and not uniformly continuous. Then there exist $\epsilon > 0$ and two sequences $(\mathbf{b}_k; x_k)$ and $(\mathbf{b}_k'; x_k')$ in $\mathbb{R}^n \times X$ such that, for every $k \in \mathbb{N}$, we have $|\mathbf{b}_k - \mathbf{b}_k'| + \|x_k - x_k'\| \leq 1/k$ and $\|F(\mathbf{b}_k; x_k) - F(\mathbf{b}_k'; x_k')\| \geq \epsilon$. The set $D := \{(0; 0)\} \cup \{(\mathbf{b}_k' - \mathbf{b}_k; x_k' - x_k) : k \in \mathbb{N}\}$ is compact in $\mathbb{R}^n \times X$ and this violets condition (d) from Theorem 2.6 with the number $\epsilon > 0$, $B = X$, and the sequence $(\mathbf{b}_k; x_k)$. □

Similarly we can prove the following result (see also (Bender 1966, Lemma 5.1, Theorem 5.1), (Fink 1969, Lemma 1) and (N'Guérékata 2005, Theorem 2.6) for some particular cases of Theorem 2.6 and Corollary 2.7–2.8, as well as (Bender 1966, Definition 5.2, Definition 5.3) where the notion of compact almost automorphy has been defined for the first time):

Corollary 2.8 *Suppose that* $F : \mathbb{R}^n \times X \to Y$ *is an* $(\mathrm{R}, \mathcal{B})$*-multi-almost automorphic function, where* R *denotes the collection of all sequences in* $\mathbb{R}^n$ *and* $X \in \mathcal{B}$*. Then* $F(\cdot;\cdot)$ *is compactly* $(\mathrm{R}, \mathcal{B})$*-multi-almost automorphic if and only if for every fixed element* $x \in X$ *we have that the function* $F(\cdot; x)$ *is uniformly continuous on* $\mathbb{R}^n$*.*

Before proceeding further, we would like to note that the notion of a compact almost automorphic function $F : \mathbb{R} \times X \to X$ has been introduced by Ait Dads et al. (Ait Dads et al. 2017, Definition 5) in a slightly artificial way, following the results obtained in the previous two corollaries. The approach of these authors can be also used for the introduction of several new types of compactly $(\mathrm{R}_X, \mathcal{B})$-multi-almost automorphic functions which will not be considered here. For compactly almost automorphic solutions of evolution equations, we may refer also to Es-Sebbar (2016) and Es-sebbar et al. (2020).

We close the subsection with the following example (see also (Chávez et al. 2020, Example 2.22)):

Example Suppose that $f : \mathbb{R}^n \to X$ and $g : \mathbb{R}^n \to \mathbb{R}^n$ are (compactly) almost automorphic functions. Define the function

$$F(\mathbf{t}) := f(\mathbf{t} - g(\mathbf{t})), \quad \mathbf{t} \in \mathbb{R}^n.$$

Then the function $F(\cdot)$ is (compactly) almost automorphic, as well. This can be shown as in Ait Dads et al. (Ait Dads et al. 2017, Lemma 7), where the corresponding statement has been analyzed in the one-dimensional setting (see also Abbas et al. 2021).

2.2 Further Properties of $(\mathrm{R}_X, \mathcal{B})$-Multi-almost Automorphic Functions

In this subsection, we further explore the class of $(\mathrm{R}_X, \mathcal{B})$-multi-almost automorphic functions. First of all, it is clear that we have the following: Suppose that $F : \mathbb{R}^n \times X \to Y$ is a continuous function. If $\mathcal{B}'$ is a certain collection of subsets of X which contains $\mathcal{B}$, R'_X is a certain collection of sequences in $\mathbb{R}^n \times X$ which contains R_X and $F(\cdot;\cdot)$ is (compactly) $(\mathrm{R}'_X, \mathcal{B}')$-multi-almost automorphic, then $F(\cdot;\cdot)$ is (compactly) $(\mathrm{R}_X, \mathcal{B})$-multi-almost automorphic. This also holds for any other class of functions introduced so far.

It is very simple to deduce the following result, which can be also reformulated for $(\mathrm{R}_X, \mathcal{B})$-multi-almost automorphy by assuming additionally that $X \in \mathcal{B}$; see also (2.8) and Levitan and Zhikov (Levitan and Zhikov 1982, Property 4, p. 3):

Proposition 2.9 *Suppose that $F : \mathbb{R}^n \times X \to Y$ is $(\mathrm{R}, \mathcal{B})$-multi-almost automorphic, resp. $(\mathrm{R}, \mathcal{B}, W_{\mathcal{B},\mathrm{R}})$-multi-almost automorphic [$(\mathrm{R}, \mathcal{B}, P_{\mathcal{B},\mathrm{R}})$-multi-almost automorphic] and $\phi : Y \to Z$ is continuous, resp. $\phi : Y \to Z$ is continuous and satisfies that, for every $B \in \mathcal{B}$ as well as for every element $x \in B$, for every sequence $(\mathbf{b}_k) \in \mathrm{R}$ and every its subsequence $(\mathbf{b}_{k_l})$, there exists an integer $s \in \mathbb{N}$ such that the function $\phi(\cdot)$ is uniformly continuous on the closure of the set $\{F(\mathbf{t} + b_{k_m}; x) : m \geq s,\ \mathbf{t} \in W_{B,(\mathbf{b}_k)}(x)\} \cup \{F(\mathbf{t} - b_{k_l} + b_{k_m}; x) : m,\ l \geq s,\ \ \mathbf{t} \in W_{B,(\mathbf{b}_k)}(x)\}$ [$\phi : Y \to Z$ is continuous and satisfies that, for every $B \in \mathcal{B}$ as well as for every sequence $(\mathbf{b}_k) \in \mathrm{R}$ and every its subsequence $(\mathbf{b}_{k_l})$, there exists an integer $s \in \mathbb{N}$ such that the function $\phi(\cdot)$ is uniformly continuous on the closure of the set $\{F(\mathbf{t} + b_{k_m}; x) : m \geq s,\ (\mathbf{t}; x) \in P_{B,(\mathbf{b}_k)}\} \cup \{F(\mathbf{t} - b_{k_l} + b_{k_m}; x) : m,\ l \geq s,\ \ (\mathbf{t}; x) \in P_{B,(\mathbf{b}_k)}\}$]. Then $\phi \circ F : \mathbb{R}^n \times X \to Z$ is $(\mathrm{R}, \mathcal{B})$-multi-almost automorphic, resp. $(\mathrm{R}, \mathcal{B}, W_{\mathcal{B},\mathrm{R}})$-multi-almost automorphic [$(\mathrm{R}, \mathcal{B}, P_{\mathcal{B},\mathrm{R}})$-multi-almost automorphic].*

In Kostić (2019, Lemma 3.9.9), we have clarified the supremum formula for the one-dimensional almost automorphic functions. This formula can be extended in our framework as follows:

Proposition 2.10 (The supremum formula) *Let $F : \mathbb{R}^n \times X \to Y$ be $(\mathrm{R}, \mathcal{B})$-multi-almost automorphic. Suppose that there exists a sequence $b(\cdot)$ in R whose any subsequence is unbounded. Then for any $a \geq 0$ we have*

$$\sup_{\mathbf{t}\in\mathbb{R}^n, x\in X} \left\| F(\mathbf{t}; x) \right\|_Y = \sup_{\mathbf{t}\in\mathbb{R}^n, |t|\geq a, x\in X} \left\| F(\mathbf{t}; x) \right\|_Y. \tag{2.10}$$

Proof We will include all relevent details of the proof for the sake of completeness. Let $\epsilon > 0$, $a \geq 0$ and $x \in X$ be given. Then (2.10) holds if we prove that

$$\left\| F(\mathbf{t}; x) \right\|_Y \leq \epsilon + \sup_{\mathbf{t}\in\mathbb{R}^n, |t|\geq a} \left\| F(\mathbf{t}; x) \right\|_Y. \tag{2.11}$$

By assumption, there exists $B \in \mathcal{B}$ with $x \in B$. Let $b(\cdot)$ be a sequence in R whose any subsequence is unbounded. Then we have (2.8), and consequently, there exist two integers $l_0 \in \mathbb{N}$ and $m_0 \in \mathbb{N}$ such that

$$\left\| F(\mathbf{t};x) \right\|_Y \le \epsilon + \left\| F\Big(\mathbf{t} - \big(b^1_{k_l},\ldots,b^n_{k_l}\big) + \big(b^1_{k_m},\ldots,b^n_{k_m}\big);x\Big) \right\|_Y, \quad l \ge l_0,\ m \ge m_0.$$

In particular,

$$\left\| F(\mathbf{t};x) \right\|_Y \le \epsilon + \left\| F\Big(\mathbf{t} - \big(b^1_{k_{l_0}},\ldots,b^n_{k_{l_0}}\big) + \big(b^1_{k_m},\ldots,b^n_{k_m}\big);x\Big) \right\|_Y, \quad m \ge m_0.$$

Since the sequence $(b^1_{k_m},\ldots,b^n_{k_m})_{m\ge m_0}$ is unbounded, (2.11) follows immediately. □

Arguing similarly as in the proofs of Chávez et al. (2020, Proposition 2.7, Proposition 2.8) (cf. also the proof of N'Guérékata (2001, Theorem 2.1.10)), we may deduce the following:

Proposition 2.11 (i) *Suppose that for each integer $j \in \mathbb{N}$ the function $F_j(\cdot;\cdot)$ is $(\mathrm{R_X},\mathcal{B})$-multi-almost automorphic and, for every sequence which belongs to $\mathrm{R_X}$, any its subsequence also belongs to $\mathrm{R_X}$. If the sequence $(F_j(\cdot;\cdot))$ converges uniformly to a function $F(\cdot;\cdot)$ on X, then the function $F(\cdot;\cdot)$ is $(\mathrm{R_X},\mathcal{B})$-multi-almost automorphic. If, additionally, for each $B \in \mathcal{B}$ and $(\mathbf{b};\mathbf{x}) \in \mathrm{R_X}$ we have $W_{B,(\mathbf{b};\mathbf{x})} : B \to P(P(\mathbb{R}^n))$, $\mathrm{P}_{B,(\mathbf{b};\mathbf{x})} \in P(P(\mathbb{R}^n \times B))$, $W_{B,(\mathbf{b};\mathbf{x})}(x) \subseteq W_{B,(\mathbf{b};\mathbf{x})'}(x)$ and $\mathrm{P}_{B,(\mathbf{b};\mathbf{x})} \subseteq \mathrm{P}_{B,(\mathbf{b};\mathbf{x})'}$ for any $x \in B$ and any subsequence $(\mathbf{b};\mathbf{x})'$ of $(\mathbf{b};\mathbf{x})$, and the function $F_j(\cdot;\cdot)$ is $(\mathrm{R}_X,\mathcal{B},W_{\mathcal{B},\mathrm{R}_X})$-multi-almost automorphic, resp. $(\mathrm{R}_X,\mathcal{B},\mathrm{P}_{\mathcal{B},\mathrm{R}_X})$-multi-almost automorphic, then the function $F(\cdot;\cdot)$ is likewise $(\mathrm{R}_X,\mathcal{B},W_{\mathcal{B},\mathrm{R}_X})$-multi-almost automorphic, resp. $(\mathrm{R}_X,\mathcal{B},\mathrm{P}_{\mathcal{B},\mathrm{R}_X})$-multi-almost automorphic.*

(ii) *Suppose that for each integer $j \in \mathbb{N}$ the function $F_j(\cdot;\cdot)$ is $(\mathrm{R},\mathcal{B})$-multi-almost automorphic and, for every sequence which belongs to R, any its subsequence also belongs to R. If for each $B \in \mathcal{B}$ there exists $\epsilon_B > 0$ such that the sequence $(F_j(\cdot;\cdot))$ converges uniformly to a function $F(\cdot;\cdot)$ on the set $B^\circ \cup \bigcup_{x\in\partial B} B(x,\epsilon_B)$, then the function $F(\cdot;\cdot)$ is $(\mathrm{R},\mathcal{B})$-multi-almost automorphic. If, additionally, for each $B \in \mathcal{B}$ and $(\mathbf{b}_k) \in \mathrm{R}$ we have $W_{B,(\mathbf{b}_k)} : B \to P(P(\mathbb{R}^n))$, $\mathrm{P}_{B,(\mathbf{b}_k)} \in P(P(\mathbb{R}^n \times B))$, $W_{B,(\mathbf{b})}(x) \subseteq W_{B,(\mathbf{b})'}(x)$ and $\mathrm{P}_{B,(\mathbf{b})} \subseteq \mathrm{P}_{B,(\mathbf{b})'}$ for any $x \in B$ and any subsequence $(\mathbf{b})'$ of $(\mathbf{b})$, and $F_j(\cdot;\cdot)$ is $(\mathrm{R},\mathcal{B},W_{\mathcal{B},\mathrm{R}})$-multi-almost automorphic, resp. $(\mathrm{R},\mathcal{B},\mathrm{P}_{\mathcal{B},\mathrm{R}})$-multi-almost automorphic, then the function $F(\cdot;\cdot)$ is likewise $(\mathrm{R},\mathcal{B},W_{\mathcal{B},\mathrm{R}})$-multi-almost automorphic, resp. $(\mathrm{R},\mathcal{B},\mathrm{P}_{\mathcal{B},\mathrm{R}})$-multi-almost automorphic.*

Concerning the convolution invariance of space consisting of all $(\mathrm{R}_X,\mathcal{B})$-multi-almost automorphic functions, we would like to state the following result:

Proposition 2.12 *Suppose that $h \in L^1(\mathbb{R}^n)$ and $F : \mathbb{R}^n \times X \to Y$ is an $(\mathrm{R}_X,\mathcal{B})$-multi-almost automorphic function satisfying that for each $B \in \mathcal{B}$ there exists a finite real number $\epsilon_B > 0$ such that $\sup_{\mathbf{t}\in\mathbb{R}^n, x\in B^\cdot} \|F(\mathbf{t},x)\|_Y < +\infty$, where $B^\cdot \equiv B^\circ \cup \bigcup_{x\in\partial B} B(x,\epsilon_B)$. Let condition (CI) holds, where:*

(CI) $\mathrm{R_X} = \mathrm{R}$, *which corresponds to the case $F : \mathbb{R}^n \to Y$, or $X \in \mathcal{B}$ and R_X is general.*

Then the function

$$(h * F)(\mathbf{t}; x) := \int_{\mathbb{R}^n} h(\sigma) F(\mathbf{t} - \sigma; x)\, d\sigma, \quad \mathbf{t} \in \mathbb{R}^n, \ x \in X$$

is well defined, $(\mathrm{R}_X, \mathcal{B})$*-multi-almost automorphic, and for each* $B \in \mathcal{B}$ *we have* $\sup_{\mathbf{t} \in \mathbb{R}^n, x \in B^{\cdot}} \|(h * F)(\mathbf{t}; x)\|_Y < +\infty.$

Proof It is clear that the function $(h * F)(\cdot; \cdot)$ is well defined as well as that $\sup_{\mathbf{t} \in \mathbb{R}^n, x \in B^{\cdot}} \|(h * F)(\mathbf{t}; x)\|_Y < +\infty$ for all $B \in \mathcal{B}$. The continuity of function $(h * F)(\cdot; \cdot)$ at the fixed point $(\mathbf{t}_0; x_0) \in \mathbb{R}^n \times X$ follows from the continuity of function $F(\cdot; \cdot)$ at this point, the existence of a set $B \in \mathcal{B}$ such that $x_0 \in B$, the assumption $\sup_{\mathbf{t} \in \mathbb{R}^n, x \in B^{\cdot}} \|F(\mathbf{t}; x)\|_Y < +\infty$ and the dominated convergence theorem. We will prove the remainder provided that the second part of condition (CI) holds. Let $((\mathbf{b}; \mathbf{x})_k) \in \mathrm{R}_{\mathrm{X}}$ be fixed. Then we know that there exist a subsequence $((\mathbf{b}; \mathbf{x})_{k_l} = ((b_{k_l}^1, b_{k_l}^2, \ldots, b_{k_l}^n); x_{k_l}))$ of $((\mathbf{b}; \mathbf{x})_k)$ and a function $F^* : \mathbb{R}^n \times X \to Y$ such that (2.5)–(2.6) hold pointwisely for all $x \in X$ and $\mathbf{t} \in \mathbb{R}^n$. It is not difficult to prove that the function $F^*(\cdot; x)$ is measurable for every fixed element $x \in X$. Clearly, the function

$$\big(h * F\big)^*(\mathbf{t}; x) := \int_{\mathbb{R}^n} h(\sigma) F^*(\mathbf{t} - \sigma; x)\, d\sigma, \quad \mathbf{t} \in \mathbb{R}^n, \ x \in B$$

is well defined. Using the dominated convergence theorem, it can be simply shown that we have

$$\lim_{m \to +\infty} (h * F)\big(\mathbf{t} + (b_{k_m}^1, \ldots, b_{k_m}^n); x + x_{k_m}\big) = \big(h * F\big)^*(\mathbf{t}; x)$$

and

$$\lim_{l \to +\infty} \big(h * F\big)^*\big(\mathbf{t} - (b_{k_l}^1, \ldots, b_{k_l}^n); x - x_{k_l}\big) = (h * F)(\mathbf{t}; x),$$

pointwisely for all $x \in X$ and $\mathbf{t} \in \mathbb{R}^n$. This completes the proof. □

2.3 $\mathbb{D}$-Asymptotically $(\mathrm{R}_X, \mathcal{B})$-Multi-almost Automorphic Functions

This subsection investigates $\mathbb{D}$-asymptotically $(\mathrm{R}_X, \mathcal{B})$-multi-almost automorphic functions. We start by introducing the following notion:

Definition 2.13 Suppose that the set $\mathbb{D} \subseteq \mathbb{R}^n$ is unbounded, $i = 1, 2$ and $F : \mathbb{R}^n \times X \to Y$ is a continuous function. Then we say that $F(\cdot; \cdot)$ is $\mathbb{D}$-asymptotically (compactly) $(\mathrm{R}_X, \mathcal{B})$-multi-almost automorphic if and only if there exist a function $G(\cdot; \cdot)$ which is (compactly) $(\mathrm{R}_X, \mathcal{B})$-multi-almost automorphic and a function $Q \in C_{0,\mathbb{D},\mathcal{B}}(\mathbb{R}^n \times X : Y)$ such that $F(\mathbf{t}; x) = G(\mathbf{t}; x) + Q(\mathbf{t}; x)$ for all $\mathbf{t} \in \mathbb{R}^n$ and $x \in X$.

It is said that $F(\cdot;\cdot)$ is asymptotically (compactly) $(\mathrm{R}_X, \mathcal{B})$-multi-almost automorphic if and only if $F(\cdot;\cdot)$ is $\mathbb{R}^n$-asymptotically (compactly) $(\mathrm{R}_X, \mathcal{B})$-multi-almost automorphic.

We similarly introduce the classes of $\mathbb{D}$-asymptotically (compactly) $(\mathrm{R}, \mathcal{B})$-multi-almost automorphic functions and asymptotically (compactly) $(\mathrm{R}, \mathcal{B})$-multi-almost automorphic functions, as well as the corresponding classes of functions in which the notion of $(\mathrm{R}, \mathcal{B})$-multi-almost automorphy ($(\mathrm{R}_X, \mathcal{B})$-multi-almost automorphy) is replaced with some of the notions introduced in Definition 2.2 or Definition 2.4. We will not consider here the notion in which the space $C_{0,\mathbb{D},\mathcal{B}}(\mathbb{R}^n \times X : Y)$ is replaced with some space of weighted ergodic components in $\mathbb{R}^n$, analyzed recently in [47].

The proof of the following proposition can be given as for the usually considered almost automorphic functions (N'Guérékata 2001); all clarifications also hold if the notion of $(\mathrm{R}, \mathcal{B})$-multi-almost automorphy ($(\mathrm{R}_X, \mathcal{B})$-multi-almost automorphy) is replaced with some of the notions introduced in Definition 2.2 or Definition 2.4:

Proposition 2.14 (i) *Suppose that* $\tau \in \mathbb{R}^n$, $x_0 \in X$ *and* $F(\cdot;\cdot)$ *is (compactly)* $(\mathrm{R}_X, \mathcal{B})$*-multi-almost automorphic. Then* $F(\cdot + \tau; \cdot + x_0)$ *is (compactly)* $(\mathrm{R}_X, \mathcal{B}_{x_0})$*-multi-almost automorphic, where* $\mathcal{B}_{x_0} \equiv \{-x_0 + B : B \in \mathcal{B}\}$. *Furthermore, if* $F(\cdot;\cdot)$ *is* $\mathbb{D}$*-asymptotically (compactly)* $(\mathrm{R}_X, \mathcal{B})$*-multi-almost automorphic, then* $F(\cdot + \tau; \cdot + x_0)$ *is* $(\mathbb{D} - \tau)$*-asymptotically (compactly)* $(\mathrm{R}_X, \mathcal{B}_{x_0})$*-multi-almost automorphic.*

(ii) *Suppose that* $c_1 \in \mathbb{C} \setminus \{0\}$, $c_2 \in \mathbb{C} \setminus \{0\}$, *and* $F(\cdot;\cdot)$ *is (compactly)* $(\mathrm{R}_X, \mathcal{B})$*-multi-almost automorphic. Then* $F(c_1\cdot; c_2\cdot)$ *is (compactly)* $((\mathrm{R}_{c_1})_X, \mathcal{B}_{c_2})$*-multi-almost automorphic, where* $(\mathrm{R}_{c_1})_X \equiv \{c_1^{-1}\mathbf{b}(\cdot) := (c_1^{-1}b^1, \cdots, c_1^{-1}b^n) : \mathbf{b} \in \mathrm{R}_X\}$ *and* $\mathcal{B}_{c_2} \equiv \{c_2^{-1}B : B \in \mathcal{B}\}$. *Furthermore, if* $F(\cdot;\cdot)$ *is* $\mathbb{D}$*-asymptotically (compactly)* $(\mathrm{R}_X, \mathcal{B})$*-multi-almost automorphic, then* $F(c_1\cdot; c_2\cdot)$ *is* $\mathbb{D}/c_1$*-asymptotically (compactly)* $((\mathrm{R}_{c_1})_X, \mathcal{B}_{c_2})$*-multi-almost automorphic.*

(iii) *Suppose that* α, $\beta \in \mathbb{C}$ *and, for every sequence which belongs to* R_X, *we have that any its subsequence belongs to* R_X. *If* $F(\cdot;\cdot)$ *and* $G(\cdot;\cdot)$ *are (compactly)* $(\mathrm{R}_X, \mathcal{B})$*-multi-almost automorphic, then* $\alpha F(\cdot;\cdot) + \beta G(\cdot;\cdot)$ *is also (compactly)* $(\mathrm{R}_X, \mathcal{B})$*-multi-almost automorphic. The same holds for* $\mathbb{D}$*-asymptotically (compactly)* $(\mathrm{R}_X, \mathcal{B})$*-multi-almost automorphic functions.*

(iv) *If* $X \in \mathcal{B}$ *and* $F(\cdot;\cdot)$ *is asymptotically* $(\mathrm{R}, \mathcal{B})$*-multi-almost automorphic, then* $F(\cdot;\cdot)$ *is bounded in case* [L3]*; furthermore, if* $F(\cdot;\cdot)$ *is asymptotically* $(\mathrm{R}_X, \mathcal{B})$*-multi-almost automorphic, then* $F(\cdot;\cdot)$ *is bounded in case that* R_X *denotes the collection of all sequences in* $\mathbb{R}^n \times X$.

Using Proposition 2.14(iv) and the supremum formula clarified in Proposition 2.10 (see also the estimate (2.11)), we can simply deduce that the decomposition in Definition 2.13 is unique (the same holds for the class of $\mathbb{D}$-asymptotically $(\mathrm{R}, \mathcal{B})$-multi-almost automorphic functions, where $\mathbb{D}$ contains the complement of a ball centered at the origin):

Proposition 2.15 *Suppose that there exist a function* $G_i(\cdot;\cdot)$ *which is* $(\mathrm{R}, \mathcal{B})$*-multi-almost automorphic and a function* $Q_i \in C_{0,\mathbb{R}^n,\mathcal{B}}(\mathbb{R}^n \times X : Y)$ *such that* $F(\mathbf{t}; x) =$

$G_i(\mathbf{t}; x) + Q_i(\mathbf{t}; x)$ for all $\mathbf{t} \in \mathbb{R}^n$ and $x \in X$ ($i = 1, 2$). Then we have $G_1 \equiv G_2$ and $Q_1 \equiv Q_2$, provided that the collection R *satisfies the following two conditions:*

- D1. *There exists a sequence in* R *whose any subsequence is unbounded.*
- D2. *For every sequence which belongs to* R, *we have that any its subsequence belongs to* R.

Furthermore, arguing as in the proof of Diagana (2013, Lemma 4.28), we may deduce the following:

Lemma 2.16 *Suppose that there exist an* $(\mathrm{R}, \mathcal{B})$*-multi-almost automorphic function* $G(\cdot; \cdot)$ *and a function* $Q \in C_{0,\mathbb{R}^n,\mathcal{B}}(\mathbb{R}^n \times X : Y)$ *such that* $F(\mathbf{t}; x) = G(\mathbf{t}; x) + Q(\mathbf{t}; x)$ *for all* $\mathbf{t} \in \mathbb{R}^n$ *and* $x \in X$*. Then we have*

$$\overline{\{G(\mathbf{t}; x) : \mathbf{t} \in \mathbb{R}^n,\ x \in X\}} \subseteq \overline{\{F(\mathbf{t}; x) : \mathbf{t} \in \mathbb{R}^n,\ x \in X\}},$$

provided that condition [D1] *holds.*

Proposition 2.17 *Suppose that conditions* [D1]–[D2] *hold and for each integer* $j \in \mathbb{N}$ *the function* $F_j(\cdot; \cdot)$ *is asymptotically (compactly)* $(\mathrm{R}, \mathcal{B})$*-multi-almost automorphic. If the sequence* $(F_j(\cdot; \cdot))$ *converges uniformly to a function* $F(\cdot; \cdot)$*, then the function* $F(\cdot; \cdot)$ *is asymptotically (compactly)* $(\mathrm{R}, \mathcal{B})$*-multi-almost automorphic.*

Proof Due to Proposition 2.15, we know that there exist a uniquely determined function $G(\cdot; \cdot)$ which is $(\mathrm{R}, \mathcal{B})$-multi-almost automorphic and a uniquely determined function $Q \in C_{0,\mathbb{R}^n,\mathcal{B}}(\mathbb{R}^n \times X : Y)$ such that $F(\mathbf{t}; x) = G(\mathbf{t}; x) + Q(\mathbf{t}; x)$ for all $\mathbf{t} \in \mathbb{R}^n$ and $x \in X$. Furthermore, we have

$$F_j(\mathbf{t}; x) - F_m(\mathbf{t}; x) = \big[G_j(\mathbf{t}; x) - G_m(\mathbf{t}; x)\big] + \big[Q_j(\mathbf{t}; x) - Q_m(\mathbf{t}; x)\big],$$

for all $\mathbf{t} \in \mathbb{R}^n$, $x \in X$ and $j,\ m \in \mathbb{N}$. Due to Proposition 2.14(iv), we have that the function $F_j(\cdot; \cdot) - F_m(\cdot; \cdot)$ is asymptotically $(\mathrm{R}, \mathcal{B})$-multi-almost automorphic as well as that the function $G_j(\cdot; \cdot) - G_m(\cdot; \cdot)$ is $(\mathrm{R}, \mathcal{B})$-multi-almost automorphic ($j,\ m \in \mathbb{N}$). Keeping in mind this fact, Lemma 2.16 and the argumentation used in the proof of Diagana (2013, Theorem 4.29), we get that

$$\begin{aligned} 3 \sup_{\mathbf{t} \in \mathbb{R}^n, x \in X} \Big\| F_j(\mathbf{t}; x) - F_m(\mathbf{t}; x) \Big\|_Y & \\ \geq \sup_{\mathbf{t} \in \mathbb{R}^n, x \in X} \Big\| G_j(\mathbf{t}; x) - G_m(\mathbf{t}; x) \Big\|_Y & \\ + \sup_{\mathbf{t} \in \mathbb{R}^n, x \in X} \Big\| Q_j(\mathbf{t}; x) - Q_m(\mathbf{t}; x) \Big\|_Y &, \end{aligned}$$

for any $j,\ m \in \mathbb{N}$. This implies that the sequences $(G_j(\cdot; \cdot))$ and $(Q_j(\cdot; \cdot))$ converge uniformly to the functions $G(\cdot; \cdot)$ and $Q(\cdot; \cdot)$, respectively. Due to Proposition 2.11, we get that the function $G(\cdot; \cdot)$ is $(\mathrm{R}, \mathcal{B})$-multi-almost automorphic. The final conclusion follows from the obvious equality $F = G + Q$ and the fact that $C_{0,\mathbb{R}^n,\mathcal{B}}(\mathbb{R}^n \times X : Y)$ is a Banach space. □

Remark 2.18 The previous proposition is also true in the one-dimensional case, with $\mathbb{D} = [0, \infty)$ and R being any collection of sequences in $[0, \infty)$ satisfying conditions [D1]–[D2].

Concerning the partial derivatives of (asymptotically) $(\mathrm{R}_X, \mathcal{B})$-multi-almost automorphic functions, we will state and prove only one partial result (cf. (Chávez et al. 2020, Sect. 2.3) for more details given in the almost periodic case):

Proposition 2.19 (i) *Suppose that the function $F(\cdot;\cdot)$ is (compactly) $(\mathrm{R}, \mathcal{B})$-multi-almost automorphic,* [D2] *holds, the partial derivative*

$$\frac{\partial F(\mathbf{t}; x)}{\partial t_i} := \lim_{h \to 0} \frac{F(\mathbf{t} + he_i; x) - F(\mathbf{t}; x)}{h}, \quad \mathbf{t} \in \mathbb{R}^n, \ x \in X$$

exists and it is uniformly continuous on $\mathcal{B}$, i.e.,

$$(\forall B \in \mathcal{B}) \ (\forall \epsilon > 0) \ (\exists \delta > 0) \ (\forall \mathbf{t}', \ \mathbf{t}'' \in \mathbb{R}^n) \ (\forall x \in B)$$
$$\left(\left| \mathbf{t}' - \mathbf{t}'' \right| < \delta \Rightarrow \left\| \frac{\partial F(\mathbf{t}'; x)}{\partial t_i} - \frac{\partial F(\mathbf{t}''; x)}{\partial t_i} \right\|_Y < \epsilon \right).$$

Then the function $\partial F(\cdot;\cdot)/\partial t_i$ is (compactly) $(\mathrm{R}, \mathcal{B})$-multi-almost automorphic.

(ii) *Suppose that the function $F(\cdot;\cdot)$ is asymptotically (compactly) $(\mathrm{R}, \mathcal{B})$-multi-almost automorphic,* [D1]–[D2] *hold, the partial derivative $\partial F(\mathbf{t}; x)/\partial t_i$ exists for all $\mathbf{t} \in \mathbb{R}^n$, $x \in X$ and it is uniformly continuous on $\mathcal{B}$. Then the function $\partial F(\cdot;\cdot)/\partial t_i$ is asymptotically (compactly) $(\mathrm{R}, \mathcal{B})$-multi-almost automorphic.*

Proof We will prove only (i) because (ii) follows similarly, by appealing to Proposition 2.17 instead of Proposition 2.11. The proof immediately follows from the fact that the sequence $(F_j(\cdot;\cdot) \equiv j[F(\cdot + j^{-1}e_i;\cdot) - F(\cdot;\cdot)])$ of (compactly) $(\mathrm{R}, \mathcal{B})$-multi-almost automorphic functions converges uniformly to the function $\partial F(\cdot;\cdot)/\partial t_i$ as $j \to +\infty$. This can be shown as in one-dimensional case, by observing that

$$F_j(\cdot;\cdot) - \frac{\partial F(\cdot;\cdot)}{\partial t_i} = j \int_0^{1/j} \left[\frac{\partial F(\cdot + se_i;\cdot)}{\partial t_i} - \frac{\partial F(\cdot;\cdot)}{\partial t_i} \right] ds.$$

□

2.4 Composition Theorems for $(\mathrm{R}, \mathcal{B})$-Multi-almost Automorphic Functions

Suppose that $F : \mathbb{R}^n \times X \to Y$ and $G : \mathbb{R}^m \times Y \to Z$ are given functions, where $m \in \mathbb{N}$. The main aim of this subsection is to analyze the $(\mathrm{R}, \mathcal{B})$-multi-almost automorphic properties of the following multi-dimensional Nemytskii operator $W : \mathbb{R}^n \times X \to Z$, given by

$$W(\mathbf{t}; x) := G\big(\mathbf{t}; F(\mathbf{t}; x)\big), \quad \mathbf{t} \in \mathbb{R}^n, \ x \in X.$$

We will first state the following generalization of Diagana (2013, Theorem 4.16); the proof is similar to the proof of the above-mentioned theorem but we will present all details for the sake of completeness:

Theorem 2.20 *Suppose that* $F : \mathbb{R}^n \times X \to Y$ *is* $(\mathrm{R}, \mathcal{B})$*-multi-almost automorphic and* $G : \mathbb{R}^n \times X \to Y$ *is* $(\mathrm{R}', \mathcal{B}')$*-multi-almost automorphic, where* R' *is a collection of all sequences* $b : \mathbb{N} \to \mathbb{R}^n$ *from* R *and all their subsequences, as well as*

$$\mathcal{B}' := \Bigl\{ B' \equiv \overline{\bigcup_{\mathbf{t}\in\mathbb{R}^n} F(\mathbf{t}; B)} : B \in \mathcal{B} \Bigr\}. \tag{2.12}$$

If there exists a finite constant $L > 0$ *such that*

$$\bigl\| G(\mathbf{t}; x) - G(\mathbf{t}; y) \bigr\|_Z \le L \|x - y\|_Y, \quad \mathbf{t} \in \mathbb{R}^n, \ x, \ y \in Y, \tag{2.13}$$

then the Nemytskii function $W(\cdot; \cdot)$ *is* $(\mathrm{R}, \mathcal{B})$*-multi-almost automorphic. Furthermore, let* $W_{B,(\mathbf{b}_k)} : B \to P(P(\mathbb{R}^n))$ *and* $\mathrm{P}_{B,(\mathbf{b}_k)} \in P(P(\mathbb{R}^n \times B))$*. Then we have the following:*

(i) *Suppose that* $F(\cdot; \cdot)$ *is* $(\mathrm{R}, \mathcal{B}, W_{\mathcal{B},\mathrm{R}})$*-multi-almost automorphic, for every* $B \in \mathcal{B}$*,* $x \in B$ *and* $(\mathbf{b}_k) \in \mathrm{R}$*, we have that any set of collection* $W_{B,(\mathbf{b}_k)}(x)$ *is an element of the collection* $W_{B,(\mathbf{b}_{k_l})}(x)$ *for any subsequence* $(\mathbf{b}_{k_l})$ *of* $(\mathbf{b}_k)$*. If the following condition*

(DB) *For every* $B \in \mathcal{B}$*,* $(\mathbf{b}_k) \in \mathrm{R}$*,* $x \in B$*,* $D \in W_{B,(\mathbf{b}_k)}(x)$ *as well as for every subsequence* $(\mathbf{b}_{k_l})$ *of* $(\mathbf{b}_k)$*, we can find a subsequence* $(\mathbf{b}_{k_{l_m}})$ *of* $(\mathbf{b}_{k_l})$ *and a function* $G^* : \mathbb{R}^n \times Y \to Z$ *such that*

$$\lim_{m\to+\infty} \Bigl\| G\bigl(\mathbf{t} + (b^1_{k_{l_m}}, \cdots, b^n_{k_{l_m}}); y\bigr) - G^*(\mathbf{t}; y) \Bigr\|_Z = 0, \tag{2.14}$$

holds uniformly for $(\mathbf{t}, y) \in D \times \overline{F([D \times \{x\}] + \{(\mathbf{b}_{k_{l_m}}; 0) : m \in \mathbb{N}\})}$ *and*

$$\lim_{m\to+\infty} \Bigl\| G^*\bigl(\mathbf{t} - (b^1_{k_{l_m}}, \cdots, b^n_{k_{l_m}}); y\bigr) - G(\mathbf{t}; y) \Bigr\|_Z = 0, \tag{2.15}$$

holds uniformly for $(\mathbf{t}; y) \in D \times F(D \times \{x\})$*,*

holds, then the function $W(\cdot; \cdot)$ *is* $(\mathrm{R}, \mathcal{B}, W_{\mathcal{B},\mathrm{R}})$*-multi-almost automorphic.*

(ii) Suppose that $F(\cdot; \cdot)$ *is* $(\mathrm{R}, \mathcal{B}, \mathrm{P}_{\mathcal{B},\mathrm{R}})$*-multi-almost automorphic and, for every* $B \in \mathcal{B}$ *and* $(\mathbf{b}_k) \in \mathrm{R}$*, we have that any set of collection* $\mathrm{P}_{B,(\mathbf{b}_k)}$ *is an element of the collection* $\mathrm{P}_{B,(\mathbf{b}_{k_l})}$ *for any subsequence* $(\mathbf{b}_{k_l})$ *of* $(\mathbf{b}_k)$*. If the following condition*

(DB1) *For every* $B \in \mathcal{B}$*,* $(\mathbf{b}_k) \in \mathrm{R}$*,* $D \in \mathrm{P}_{B,(\mathbf{b}_k)}$ *as well as for every subsequence* $(\mathbf{b}_{k_l})$ *of* $(\mathbf{b}_k)$*, we can find a subsequence* $(\mathbf{b}_{k_{l_m}})$ *of* $(\mathbf{b}_{k_l})$ *and a function* $G^* : \mathbb{R}^n \times Y \to Z$ *such that* (2.14) *holds uniformly for* $(\mathbf{t}, y) \in D \times \overline{F(D + \{(\mathbf{b}_{k_{l_m}}; 0) : m \in \mathbb{N}\})}$ *and* (2.15) *holds uniformly for* $(\mathbf{t}; y) \in D \times F(D \times \{x\})$

holds, then the function $W(\cdot;\cdot)$ *is* $(\mathrm{R}, \mathcal{B}, \mathrm{P}_{\mathcal{B},\mathrm{R}})$*-multi-almost automorphic.*

Proof Let the set $B \in \mathcal{B}$ and the sequence $(\mathbf{b}_k = (b_k^1, b_k^2, \ldots, b_k^n)) \in \mathrm{R}$ be given. By definition, there exist a subsequence $(\mathbf{b}_{k_l} = (b_{k_l}^1, b_{k_l}^2, \ldots, b_{k_l}^n))$ of $(\mathbf{b}_k)$ and a function $F^* : \mathbb{R}^n \times X \to Y$ such that (2.1)–(2.2) hold true. Then there exist a subsequence $(\mathbf{b}_{k_{l_m}} = (b_{k_{l_m}}^1, b_{k_{l_m}}^2, \ldots, b_{k_{l_m}}^n))$ of $(\mathbf{b}_{k_l})$ and a function $G^* : \mathbb{R}^n \times X \to Y$ such that (2.14)–(2.15) hold pointwisely for all $y \in B'$ and $\mathbf{t} \in \mathbb{R}^n$. Using (2.13) and (2.14), we get that

$$\left\| G^*(\mathbf{t}; x) - G^*(\mathbf{t}; y) \right\|_Z \leq L\|x - y\|_Y, \quad \mathbf{t} \in \mathbb{R}^n,\ x,\ y \in B'. \tag{2.16}$$

In order to see that the function $W(\cdot;\cdot)$ is $(\mathrm{R}, \mathcal{B})$-multi-almost automorphic, it suffices to show that

$$\left\| G\big(\mathbf{t} + (b_{k_{l_m}}^1, \ldots, b_{k_{l_m}}^n); F\big(\mathbf{t} + (b_{k_{l_m}}^1, \ldots, b_{k_{l_m}}^n); x\big)\big) - G^*(\mathbf{t}; F^*(\mathbf{t}; x)) \right\|_Z \to 0, \tag{2.17}$$

as $m \to +\infty$, and

$$\left\| G^*\big(\mathbf{t} - (b_{k_{l_m}}^1, \ldots, b_{k_{l_m}}^n); F^*\big(\mathbf{t} - (b_{k_{l_m}}^1, \ldots, b_{k_{l_m}}^n); x\big)\big) - G(\mathbf{t}; F(\mathbf{t}; x)) \right\|_Z \to 0, \tag{2.18}$$

as $m \to +\infty$, pointwisely for $t \in \mathbb{R}^n$ and $x \in B$. The proof of (2.17) goes as follows. For simplicity, denote $\boldsymbol{\o}_{\mathbf{m}} := (b_{k_{l_m}}^1, \ldots, b_{k_{l_m}}^n)$ for all $m \in \mathbb{N}$. We have ($\mathbf{t} \in \mathbb{R}^n$, $x \in B$, $m \in \mathbb{N}$):

$$\begin{aligned}
&\left\| G\big(\mathbf{t} + \boldsymbol{\o}_{\mathbf{m}}; F\big(\mathbf{t} + \boldsymbol{\o}_{\mathbf{m}}; x\big)\big) - G^*(\mathbf{t}; F^*(\mathbf{t}; x)) \right\|_Z \\
&\leq \left\| G\big(\mathbf{t} + \boldsymbol{\o}_{\mathbf{m}}; F\big(\mathbf{t} + \boldsymbol{\o}_{\mathbf{m}}; x\big)\big) - G(\mathbf{t} + \boldsymbol{\o}_{\mathbf{m}}; F^*(\mathbf{t}; x)) \right\|_Z \\
&\quad + \left\| G(\mathbf{t} + \boldsymbol{\o}_{\mathbf{m}}; F^*(\mathbf{t}; x)) - G^*(\mathbf{t}; F^*(\mathbf{t}; x)) \right\|_Z \\
&\leq L \left\| F\big(\mathbf{t} + \boldsymbol{\o}_{\mathbf{m}}; x\big) - F^*(\mathbf{t}; x) \right\|_Y + \left\| G(\mathbf{t} + \boldsymbol{\o}_{\mathbf{m}}; F^*(\mathbf{t}; x)) - G^*(\mathbf{t}; F^*(\mathbf{t}; x)) \right\|_Z.
\end{aligned}$$

Since $x \in B$ and $F^*(\mathbf{t}; x) \in B'$ for all $\mathbf{t} \in \mathbb{R}^n$, (2.17) follows by applying (2.1) and (2.14). Keeping in mind the estimate (2.16) and the estimate

$$\begin{aligned}
&\left\| G^*\big(\mathbf{t} - \tau_l; F^*\big(\mathbf{t} - \tau_l; x\big)\big) - G(\mathbf{t}; F(\mathbf{t}; x)) \right\|_Z \\
&\leq \left\| G^*\big(\mathbf{t} - \tau_l; F^*\big(\mathbf{t} - \tau_l; x\big)\big) - G^*(\mathbf{t} - \tau_l; F(\mathbf{t}; x)) \right\|_Z \\
&\quad + \left\| G^*(\mathbf{t} - \tau_l; F(\mathbf{t}; x)) - G(\mathbf{t}; F(\mathbf{t}; x)) \right\|_Z,
\end{aligned}$$

the proof of (2.18) is quite analogous, which completes the proof of the first part of theorem. The proofs of (i)–(ii) follows from the already shown part and an elementary argumentation involving the corresponding definitions and the prescribed conditions. □

In the one-dimensional case, some composition principles for compactly almost automorphic functions are stated in Diagana (2013, Lemma 4.36, Lemma 4.37, Lemma 4.38) and Ait Dads et al. (2017). We will clarify only one, almost immediate, corollary of Theorem 2.20 for compactly $(\mathrm{R}, \mathcal{B})$-multi-almost automorphic type functions:

Corollary 2.21 *Suppose that* $F : \mathbb{R}^n \times X \rightarrow Y$ *is compactly* $(\mathrm{R}, \mathcal{B})$*-multi-almost automorphic and* $G : \mathbb{R}^n \times X \rightarrow Y$ *is* $(\mathrm{R}, \mathcal{B}, \mathrm{P}_{\mathrm{R},\mathcal{B}})$*-multi-almost automorphic, where* R *is a collection of all sequences* $b : \mathbb{N} \rightarrow \mathbb{R}^n$*,* $\mathcal{B}$ *is the collection of all compact subsets of* X*, as well as for every* $B \in \mathcal{B}$ *we have that* $\mathrm{P}_{\mathrm{R},\mathcal{B}}(B)$ *is the collection of all compact subsets of* $\mathbb{R}^n \times X$*, and there exists a finite constant* $L > 0$ *such that* (2.13) *holds. Then the function* $W(\cdot;\cdot)$ *is compactly* $(\mathrm{R}, \mathcal{B})$*-multi-almost automorphic.*

A slight modification of the proof of Theorem 2.20 (cf. also the proof of Diagana (2013, Theorem 4.17)) shows that the following result holds true:

Theorem 2.22 *Suppose that* $F : \mathbb{R}^n \times X \rightarrow Y$ *is* $(\mathrm{R}, \mathcal{B})$*-multi-almost automorphic and* $G : \mathbb{R}^n \times X \rightarrow Y$ *is* $(\mathrm{R}', \mathcal{B}')$*-multi-almost automorphic, where* R' *is a collection of all sequences* $b : \mathbb{N} \rightarrow \mathbb{R}^n$ *from* R *and all their subsequences, as well as* $\mathcal{B}'$ *be given by* (2.12)*. If*

$$(\forall B \in \mathcal{B})\ (\forall \epsilon > 0)\ (\exists \delta > 0)$$
$$\Big(x,\ y \in B' \text{ and } \|x - y\|_Y < \delta \Rightarrow \|G(\mathbf{t}; x) - G(\mathbf{t}; y)\|_Z < \epsilon,\ \mathbf{t} \in \mathbb{R}^n\Big),$$

then the function $W(\cdot;\cdot)$ *is* $(\mathrm{R}, \mathcal{B})$*-multi-almost automorphic. Furthermore, let* $W_{B,(\mathbf{b}_k)} : B \rightarrow P(P(\mathbb{R}^n))$ *and* $\mathrm{P}_{B,(\mathbf{b}_k)} \in P(P(\mathbb{R}^n \times B))$*. Then we have the following:*

(i) *The requirements in* (i) *of* Theorem 2.20 *imply that the function* $W(\cdot;\cdot)$ *is* $(\mathrm{R}, \mathcal{B}, W_{\mathcal{B},\mathrm{R}})$*-multi-almost automorphic.*
(ii) *The requirements in* (ii) *of* Theorem 2.20 *imply that the function* $W(\cdot;\cdot)$ *is* $(\mathrm{R}, \mathcal{B}, \mathrm{P}_{\mathcal{B},\mathrm{R}})$*-multi-almost automorphic.*

Now we proceed with the analysis of composition theorems for asymptotically $(\mathrm{R}, \mathcal{B})$-multi-almost automorphic functions. Our first result corresponds to Theorem 2.20 and Diagana (2013, Theorem 4.34):

Theorem 2.23 *Suppose that* $F_0 : \mathbb{R}^n \times X \rightarrow Y$ *is* $(\mathrm{R}, \mathcal{B})$*-multi-almost automorphic,* $Q_0 \in C_{0,\mathbb{R}^n,\mathcal{B}}(\mathbb{R}^n \times X : Y)$ *and* $F(\mathbf{t}; x) = F_0(\mathbf{t}; x) + Q_0(\mathbf{t}; x)$ *for all* $\mathbf{t} \in \mathbb{R}^n$ *and* $x \in X$*. Suppose further that* $G_1 : \mathbb{R}^n \times X \rightarrow Y$ *is* $(\mathrm{R}', \mathcal{B}')$*-multi-almost automorphic, where* R' *is a collection of all sequences* $b : \mathbb{N} \rightarrow \mathbb{R}^n$ *from* R *and all their subsequences as well as* $\mathcal{B}'$ *is defined by* (2.12) *with the function* $F(\cdot;\cdot)$ *replaced therein by the function* $F_0(\cdot;\cdot)$*,* $Q_1 \in C_{0,\mathbb{R}^n,\mathcal{B}_1}(\mathbb{R}^n \times Y : Z)$*, where*

$$\mathcal{B}_1 := \Big\{ \bigcup_{\mathbf{t} \in \mathbb{R}^n} F(\mathbf{t}; B) : B \in \mathcal{B} \Big\}, \tag{2.19}$$

and $G(\mathbf{t}; x) = G_1(\mathbf{t}; x) + Q_1(\mathbf{t}; x)$ *for all* $\mathbf{t} \in \mathbb{R}^n$ *and* $x \in X$. *If there exists a finite constant* $L > 0$ *such that the estimate* (2.13) *holds with the function* $G(\cdot; \cdot)$ *replaced therein by the function* $G_1(\cdot; \cdot)$, *then the function* $W(\cdot; \cdot)$ *is asymptotically* $(\mathrm{R}, \mathcal{B})$*-multi-almost automorphic.*

Proof Using the above assumptions and Theorem 2.20, we have that the function $(\mathbf{t}; x) \mapsto G_1(\mathbf{t}; F_0(\mathbf{t}; x))$, $\mathbf{t} \in \mathbb{R}^n$, $x \in X$ is $(\mathrm{R}, \mathcal{B})$-multi-almost automorphic. Furthermore, we have the following decomposition

$$W(\mathbf{t}; x) = G_1(\mathbf{t}; F_0(\mathbf{t}; x)) + \Big[G_1(\mathbf{t}; F(\mathbf{t}; x)) - G_1(\mathbf{t}; F_0(\mathbf{t}; x))\Big] + Q_1(\mathbf{t}; F(\mathbf{t}; x)),$$

for any $\mathbf{t} \in \mathbb{R}^n$ and $x \in X$. Since

$$\Big\|G_1(\mathbf{t}; F(\mathbf{t}; x)) - G_1(\mathbf{t}; F_0(\mathbf{t}; x))\Big\|_Z \leq L\big\|Q_0(\mathbf{t}; x)\big\|_Y, \quad \mathbf{t} \in \mathbb{R}^n, \; x \in X,$$

we have that the function $(\mathbf{t}; x) \mapsto G_1(\mathbf{t}; F(\mathbf{t}; x)) - G_1(\mathbf{t}; F_0(\mathbf{t}; x))$, $\mathbf{t} \in \mathbb{R}^n$, $x \in X$ belongs to the space $C_{0,\mathbb{R}^n,\mathcal{B}}(\mathbb{R}^n \times X : Z)$. The same holds for the function $(\mathbf{t}; x) \mapsto Q_1(\mathbf{t}; F(\mathbf{t}; x))$, $\mathbf{t} \in \mathbb{R}^n$, $x \in X$ because of our choice of the collection $\mathcal{B}_1$ in (2.19). The proof of the theorem is thereby complete. □

Similarly we can prove the following result which corresponds to Theorem 2.22 and Diagana (2013, Theorem 4.35):

Theorem 2.24 *Suppose that* $F_0 : \mathbb{R}^n \times X \to Y$ *is* $(\mathrm{R}, \mathcal{B})$*-multi-almost automorphic,* $Q_0 \in C_{0,\mathbb{R}^n,\mathcal{B}}(\mathbb{R}^n \times X : Y)$ *and* $F(\mathbf{t}; x) = F_0(\mathbf{t}; x) + Q_0(\mathbf{t}; x)$ *for all* $\mathbf{t} \in \mathbb{R}^n$ *and* $x \in X$. *Suppose further that* $G_1 : \mathbb{R}^n \times X \to Y$ *is* $(\mathrm{R}', \mathcal{B}')$*-multi-almost automorphic, where* R' *is a collection of all sequences* $b : \mathbb{N} \to \mathbb{R}^n$ *from* R *and all their subsequences as well as* $\mathcal{B}'$ *is defined by* (2.12) *with the function* $F(\cdot; \cdot)$ *replaced therein by the function* $F_0(\cdot; \cdot)$, $Q_1 \in C_{0,\mathbb{R}^n,\mathcal{B}_1}(\mathbb{R}^n \times Y : Z)$, *where* $\mathcal{B}_1$ *is given through* (2.19), *and* $G(\mathbf{t}; x) = G_1(\mathbf{t}; x) + Q_1(\mathbf{t}; x)$ *for all* $\mathbf{t} \in \mathbb{R}^n$ *and* $x \in X$. *For every* $B \in \mathcal{B}$, *we set* $B' := \overline{\bigcup_{\mathbf{t}\in\mathbb{R}^n} F_0(\mathbf{t}; B)}$. *If*

$$(\forall B \in \mathcal{B})\; (\forall \epsilon > 0)\; (\exists \delta > 0)$$
$$\Big(x,\; y \in B' \text{ and } \big\|x - y\big\|_Y < \delta \Rightarrow \big\|G_1(\mathbf{t}; x) - G_1(\mathbf{t}; y)\big\|_Z < \epsilon,\; \mathbf{t} \in \mathbb{R}^n\Big),$$

then the function $W(\cdot; \cdot)$ *is asymptotically* $(\mathrm{R}, \mathcal{B})$*-multi-almost automorphic.*

The statements of Theorem 2.23 and Theorem 2.24 can be reformulated for the asymptotical $(\mathrm{R}, \mathcal{B}, W_{B,(\mathbf{b}_k)})$-multi-almost automorphy and the asymptotical $(\mathrm{R}, \mathcal{B}, \mathrm{P}_{B,(\mathbf{b}_k)})$-multi-almost automorphy by taking into consideration conditions (i) and (ii) from the formulation of Theorem 2.20.

2.5 Invariance of (R, $\mathcal{B}$)-Multi-almost Automorpic Properties Under Actions of Convolution Products

If $\mathbf{t} = (t_1, t_2, \ldots, t_n)$, then we use the notation $\mathcal{I}_{\mathbf{t}} = (-\infty, t_1] \times (-\infty, t_2] \times \cdots \times (-\infty, t_n]$. We impose the following condition:

(E1) $(K(\mathbf{t}))_{\mathbf{t}\in(0,\infty)^n} \subseteq L(X,Y)$ is a strongly continuous operator family and $\int_{(0,\infty)^n} \|K(\mathbf{t})\|_{L(X,Y)}\, d\mathbf{t} < +\infty$.

The main results of this subsection, Theorem 2.25 and Theorem 2.27, are new even in the one-dimensional setting. This enables one to provide numerous applications in the analysis of the existence and uniqueness of time almost automorphic solutions of the abstract (degenerate) Volterra integro-differential equations (see Kostić 2019 and the last example of this section):

Theorem 2.25 *Let $f : \mathbb{R}^n \to X$ be a bounded* R*-multi-almost automorphic function and* (E1) *holds. Define*

$$F(\mathbf{t}) := \int_{\mathcal{I}_{\mathbf{t}}} K(\mathbf{t}-\eta) f(\eta)\, d\eta, \quad \mathbf{t} \in \mathbb{R}^n.$$

Then $F(\cdot)$ is a bounded R*-multi-almost automorphic function. Furthermore, if $f : \mathbb{R}^n \to X$ is a bounded $(\mathrm{R}, W_{\mathrm{R}})$-multi-almost automorphic function, then $F(\cdot)$ is likewise a bounded $(\mathrm{R}, W_{\mathrm{R}})$-multi-almost automorphic function provided that, for every set $D \in W_{\mathrm{R}}$ and for every compact set $C \subseteq [0,\infty)^n$, we have that $D - C \subseteq D'$ for some set $D' \in W_{\mathrm{R}}$.*

Proof First of all, observe that the Lebesgue dominated convergence theorem implies in view of condition (E1) that $F(\cdot)$ is a continuous function on $\mathbb{R}^n$; it is also clear that (E1) implies that the function $F(\cdot)$ is bounded on $\mathbb{R}^n$. On the other hand, since $f(\cdot)$ is R-multi-multi-almost automorphic, given a sequence $(b_n) \in \mathrm{R}$, there exist a subsequence (c_n) of (b_n) and a function $\tilde{f}(\cdot)$ such that $\lim_{n\to\infty} f(\mathbf{t}+c_n) = \tilde{f}(\mathbf{t})$ and $\lim_{n\to\infty} \tilde{f}(\mathbf{t}-c_n) = f(\mathbf{t})$ pointwisely for all $\mathbf{t} \in \mathbb{R}^n$. It is clear that the function $\tilde{f}(\cdot)$ is measurable and bounded. Now, let us define

$$F^*(\mathbf{t}) := \int_{\mathcal{I}_{\mathbf{t}}} K(\mathbf{t}-\eta) \tilde{f}(\eta)\, d\eta, \quad \mathbf{t} \in \mathbb{R}^n.$$

Then we have

$$\begin{aligned} &\left\| F(\mathbf{t}+c_n) - F^*(t) \right\|_Y \\ &= \left\| \int_{\mathcal{I}_{\mathbf{t}+c_n}} K(\mathbf{t}+c_n-\eta) f(\eta)\, d\eta - \int_{\mathcal{I}_{\mathbf{t}}} K(\mathbf{t}-\eta)\tilde{f}(\eta)\, d\eta \right\| \\ &\leq \int_{\mathcal{I}_{\mathbf{t}}} \|K(\mathbf{t}-\eta)\|_{L(X,Y)} \cdot \left\| f(\eta+c_n) - \tilde{f}(\eta) \right\| d\eta, \quad \mathbf{t} \in \mathbb{R}^n. \end{aligned}$$

Using condition (E1), the above estimate and the Lebesgue dominated convergence theorem, we get

$$\lim_{n\to\infty} F(\mathbf{t}+c_n) = F^*(\mathbf{t}), \quad \mathbf{t} \in \mathbb{R}^n.$$

Similarly we get

$$\lim_{n\to\infty} F^*(\mathbf{t}-c_n) = F(\mathbf{t}), \quad \mathbf{t} \in \mathbb{R}^n,$$

which completes the proof of the first part of theorem. Suppose now that $f : \mathbb{R}^n \to X$ is a bounded (R, W_{R})-multi-almost automorphic function, $\epsilon > 0$ and $D \in W_{\mathrm{R}}$. Then there exists $L > 0$ such that

$$\|f\|_\infty \int_{\eta\in[0,\infty)^n; |\boldsymbol{\eta}|\geq L} \|K(\eta)\|_{L(X,Y)}\, d\eta < \epsilon/4.$$

Due to our assumption, we have the existence of a set $D' \in W_{\mathrm{R}}$ such that $D - \{\mathbf{t} \in [0,\infty)^n : |\mathbf{t}| \leq L\} \subseteq D'$. Choose after that a natural number $n_0 \in \mathbb{N}$ such that

$$\left\| f(\mathbf{t}+c_n-\eta) - \tilde{f}(\mathbf{t}-\eta)\right\| < \frac{\epsilon}{2\Big(1+\int_{\eta\in[0,\infty)^n; |\boldsymbol{\eta}|\leq L} \|K(\eta)\|_{L(X,Y)}\, d\eta\Big)}.$$

Arguing as above, we get

$$\begin{aligned}
&\left\|F(\mathbf{t}+c_n) - F^*(t)\right\|_Y \leq \int_{(0,\infty)^n} \|K(\eta)\|_{L(X,Y)} \left\| f(\mathbf{t}+c_n-\eta) - \tilde{f}(\mathbf{t}-\eta)\right\| d\eta \\
&\leq 2\|f\|_\infty \int_{\eta\in[0,\infty)^n; |\boldsymbol{\eta}|\geq L} \|K(\eta)\|_{L(X,Y)}\, d\eta \\
&\quad + \int_{\eta\in(0,\infty)^n; |\eta|\leq L} \|K(\eta)\|_{L(X,Y)} \left\| f(\mathbf{t}+c_n-\eta) - \tilde{f}(\mathbf{t}-\eta)\right\| d\eta \\
&\leq (\epsilon/2) + (\epsilon/2) = \epsilon, \quad \mathbf{t} \in D.
\end{aligned}$$

We can similarly prove that $\lim_{n\to\infty} F^*(\mathbf{t}-c_n) = F(\mathbf{t})$, uniformly in $\mathbf{t} \in D$. □

Remark 2.26 It is clear that the above requirements hold if W_{R} denotes the collection of all compact subsets of $\mathbb{R}^n$, so that Theorem 2.25 transfers the well known result of Henríquez and Lizama (2009, Lemma 3.1) to the multi-dimensional setting. On the other hand, W_{R} need not be consisted of compact sets; for example, in our previous analyses, we have analyzed case in which W_{R} is a collection of sets of the form $\{(x,y) \in \mathbb{R}^2 : |x-y| \leq L\}$, when $L > 0$ ($n = 2$). Then the requirements of Theorems 2.25 and 2.27 are also satisfied.

Let $\mathbb{D}$ be an unbounded subset of $\mathbb{R}^n$; then we set $\mathbb{D}_t := \mathcal{I}_{\mathbf{t}} \cap \mathbb{D}$ for all $\mathbf{t} \in \mathbb{R}^n$. For the invariance of $\mathbb{D}$-asymptotical R-multi-almost automorphy under the actions of multi-dimensional finite convolution products, we impose the following conditions:

(E2) $$\lim_{|\mathbf{t}|\to+\infty, \mathbf{t}\in\mathbb{D}} \int_{\mathcal{I}_{\mathbf{t}}\cap\mathbb{D}^c} \left\|K(\mathbf{t}-)\right\|_{L(X,Y)} d\eta = 0\,.$$

(E3) there exists $r_0 > 0$ such that, for every $r > r_0$, we have

$$\lim_{|\mathbf{t}|\to+\infty,\mathbf{t}\in\mathbb{D}} \int_{\mathcal{I}_{\mathbf{t}}\cap\mathbb{D}\cap B(0,r)} \|K(\mathbf{t}-\,)\|_{L(X,Y)}\, d\eta = 0.$$

Theorem 2.27 *Let conditions* (E1)–(E3) *be fulfilled, let for every set $D \in W_{\mathrm{R}}$ and for every compact set $C \subseteq [0,\infty)^n$, we have that $D - C \subseteq D'$ for some set $D' \in W_{\mathrm{R}}$, and let $f = f_a + f_0$, where $f_a(\cdot)$ is a bounded* R*-multi-almost automorphic function (bounded* (R, W_{R})*-multi-almost automorphic function) and $f_0(\cdot) \in C_{0,\mathbb{D}}(\mathbb{R}^n : X) \cap L^\infty(\mathbb{R}^n : X)$. Define*

$$\Gamma f(\mathbf{t}) := \int_{\mathbb{D}_{\mathbf{t}}} K(\mathbf{t}-\eta) f(\eta)\, d\eta, \quad \mathbf{t} \in \mathbb{R}^n.$$

Then $\Gamma f(\cdot)$ can be written as a sum of a bounded R*-multi-almost automorphic function (bounded* (R, W_{R})*-multi-almost automorphic function) and a bounded function belonging to the space $C_{0,\mathbb{D}}(\mathbb{R}^n : Y)$.*

Proof We will only prove the result provided that $f_a(\cdot)$ is a bounded R-multi-almost automorphic function. We have

$$\begin{aligned}\int_{\mathbb{D}_{\mathbf{t}}} K(\mathbf{t}-\eta) f(\eta)\, d\eta &= \int_{\mathbb{D}_{\mathbf{t}}} K(\mathbf{t}-\eta)\left(f_a(\eta)+f_0(\eta)\right) d\eta \\ &= \int_{\mathcal{I}_{\mathbf{t}}} K(\mathbf{t}-\eta) f_a(\eta)\, d\eta - \int_{\mathcal{I}_{\mathbf{t}}\cap\mathbb{D}^c} K(\mathbf{t}-\eta) f_a(\eta)\, d\eta \\ &\quad + \int_{\mathbb{D}_{\mathbf{t}}} K(\mathbf{t}-\eta) f_0(\eta)\, d\eta \\ &= \Gamma_1 f_a(\mathbf{t}) + \Gamma_2 f(\mathbf{t}), \quad \mathbf{t} \in \mathbb{R}^n,\end{aligned}$$

where

$$\Gamma_1 f_a(\mathbf{t}) := \int_{\mathcal{I}_{\mathbf{t}}} K(\mathbf{t}-\eta) f_a(\eta)\, d\eta, \quad \mathbf{t} \in \mathbb{R}^n,$$

and

$$\Gamma_2 f(\mathbf{t}) := \int_{\mathbb{D}_{\mathbf{t}}} K(\mathbf{t}-\eta) f_0(\eta)\, d\eta - \int_{\mathcal{I}_{\mathbf{t}}\cap\mathbb{D}^c} K(\mathbf{t}-\eta) f_a(\eta)\, d\eta, \quad \mathbf{t} \in \mathbb{R}^n.$$

It can be simply shown that these functions are bounded. By Theorem 2.25, we have that $\Gamma_1 f_a(\cdot)$ is bounded, R-multi-almost automorphic. Now we will prove the following decay at infinity across $\mathbb{D}$:

$$\lim_{\mathbf{t}\in\mathbb{D};|\mathbf{t}|\to+\infty} \Gamma_2 f(\mathbf{t}) = 0.$$

Towards this end, observe that the estimate $||f_a||_\infty < \infty$ and condition (E2) together imply:

$$\lim_{\mathbf{t}\in\mathbb{D};|\mathbf{t}|\to+\infty} \int_{\mathcal{I}_{\mathbf{t}}\cap\mathbb{D}^c} K(\mathbf{t}-\eta) f_a(\eta)\, d\eta = 0.$$

Therefore, the second integral in the representation of $\Gamma_2 f(\cdot)$ vanishes at infinity. To estimate the first integral, fix a real number $\epsilon > 0$. Since $f_0 \in C_{0,\mathbb{D}}(\mathbb{R}^n : X)$, there exists $r > 0$ such that for all $\mathbf{t} \in \mathbb{D}$ with $|\mathbf{t}| > r$ we have $\| f_0(\mathbf{t})\| < \epsilon$; moreover, due to condition (E3), we have (for this r and for $|\mathbf{t}|$ a large enough):

$$\int_{B(0,r)\cap\mathbb{D}_{\mathbf{t}}} \|K(\mathbf{t}-\eta)\|_{L(X,Y)}\, d\eta < \epsilon, \quad \mathbf{t} \in \mathbb{R}^n.$$

Since

$$\int_{\mathbb{D}_{\mathbf{t}}} K(\mathbf{t}-\eta) f_0(\eta)\, d\eta = \int_{B(0,r)\cap\mathbb{D}_{\mathbf{t}}} K(\mathbf{t}-\eta) f_0(\eta) d\eta + \int_{B(0,r)^c\cap\mathbb{D}_{\mathbf{t}}} K(\mathbf{t}-\eta) f_0(\eta)\, d\eta,$$

the above implies

$$\left\| \int_{\mathbb{D}_{\mathbf{t}}} K(\mathbf{t}-\eta) f_0(\eta)\, d\eta \right\|_Y < \left(||f_0||_\infty + \int_{B(0,r)^c\cap\mathbb{D}_{\mathbf{t}}} \|K(\mathbf{t}-\eta)\|_{L(X,Y)}\, d\eta \right) \epsilon.$$

The proof of the theorem is thereby complete. □

Remark 2.28 In $\mathbb{R}^2$, let us consider the set $\mathbb{D}$ formed by the union of lines containing a fixed point $p \in \mathbb{R}^2$. Then we have

$$\int_{\mathbb{D}_{\mathbf{t}}} K(\mathbf{t}-\eta) f(\eta)\, d\eta = 0,$$

for any $\mathbf{t} \in \mathbb{R}^2$. More generally, if $\mathbb{D}$ consists of sets contained in the euclidean spaces of dimension less than n, after the canonical embedding of this space into $\mathbb{R}^n$ we get:

$$\int_{\mathbb{D}_{\mathbf{t}}} K(\mathbf{t}-\eta) f(\eta)\, d\eta = 0,$$

for any $\mathbf{t} \in \mathbb{R}^n$. Therefore, in the formulation of previous theorem, it seems very reasonable to assume that there exists a point $\mathbf{t_0} \in \mathbb{R}^n$ such that $\operatorname{int}(\mathbb{D}_{\mathbf{t_0}}) \neq \emptyset$.

Remark 2.29 If $\mathbb{D} = [\alpha_1, +\infty) \times [\alpha_2, +\infty) \times \cdots \times [\alpha_n, +\infty)$, then $\mathbb{D}_{\mathbf{t}} = [\alpha_1, t_1] \times [\alpha_2, t_2] \times \cdots \times [\alpha_n, t_n] := [\boldsymbol{\alpha}, \mathbf{t}]$ and, under the hypothesis (E1), (E2) and (E3), assuming additionally the notation

$$\int_{\boldsymbol{\alpha}}^{\mathbf{t}} := \int_{\alpha_1}^{t_1} \cdots \int_{\alpha_n}^{t_n},$$

we have

$$\Gamma f(\mathbf{t}) = \int_{\boldsymbol{\alpha}}^{\mathbf{t}} K(\mathbf{t} - \eta) f(\eta)\, d\eta.$$

Example Let $\alpha,\ \beta$ be positive real numbers and consider the kernel function $K_e : \mathbb{R}^2 \to \mathbb{R}$ given by $K_e(x, y) := \exp(-\alpha x)\exp(-\beta y)$. Suppose that $\mathbb{D}$ is the first quadrant $[0, +\infty) \times [0, +\infty)$ and denote $\mathbf{t} = (x, y)$. Consider the integral operator

$$F(\mathbf{t}) = \iint\limits_{\mathbb{D}_{\mathbf{t}}} K_e(x - s, y - r) f(s, r)\, ds\, dr$$

with $f(\mathbf{t}) = 1 + e^{-(\alpha x + \beta y)}$ and R being any collection of sequences. Then

$$\begin{aligned} F(\mathbf{t}) &= \iint\limits_{\mathbb{D}_{\mathbf{t}}} K_e(x - s, y - r)(1 + e^{-(\alpha s + \beta r)})\, ds\, dr \\ &= \iint\limits_{\mathbb{D}_{\mathbf{t}}} K_e(x - s, y - r)\, ds\, dr + \iint\limits_{\mathbb{D}_{\mathbf{t}}} K_e(x - s, y - r) e^{-(\alpha s + \beta r)}\, ds\, dr \\ &= \iint\limits_{\mathcal{I}_{\mathbf{t}}} K_e(x - s, y - r)\, ds\, dr - \iint\limits_{\mathcal{I}_{\mathbf{t}} \cap \mathbb{D}^c} K_e(x - s, y - r)\, ds\, dr + \\ &\quad + \iint\limits_{\mathbb{D}_{\mathbf{t}}} K_e(x - s, y - r) e^{-(\alpha s + \beta r)}\, ds\, dr \\ &= F_1(\mathbf{t}) + F_2(\mathbf{t}), \end{aligned}$$

where

$$F_1(\mathbf{t}) := \iint\limits_{\mathcal{I}_{\mathbf{t}}} K_e(x - s, y - r)\, ds\, dr$$

and

$$F_2(\mathbf{t}) := \iint_{\mathbb{D}_{\mathbf{t}}} K_e(x-s, y-r)e^{-(\alpha s+\beta r)}\, ds\, dr - \iint_{\mathcal{I}_{\mathbf{t}} \cap \mathbb{D}^c} K_e(x-s, y-r)\, ds\, dr.$$

We see that $F_1(\cdot)$ is R-multi-almost periodic (note that $K_e(\cdot;\cdot)$ satisfies condition (E1)). On the other hand, $F_2(\cdot)$ is not R-multi-almost automorphic because for $(x_0, y) \in \mathbb{D}$, with fixed $x_0 \in (0, +\infty)$, we have

$$\lim_{|(x_0,y)| \to +\infty} F_2(x_0, y) \neq 0.$$

We close this section by providing the following illustrative application of Theorem 2.27:

Example (see also (Kostić 2019, Example 3.10.10)) Suppose that A, B and C are closed linear operators in X, $D(B) \subseteq D(A) \cap D(C)$, $B^{-1} \in L(X)$ and conditions (Favini and Yagi 1998, (6.4)–(6.5)) are satisifed with certain numbers $c > 0$ and $0 < \beta \leq \alpha = 1$. In Favini and Yagi (1998, Chapter VI), A. Favini and A. Yagi have analyzed the following second order differential equation

$$\frac{d}{dt}\big(Cu'(t)\big) + Bu'(t) + Au(t) = f(t),\ t > 0; \quad u(0) = u_0,\ Cu'(0) = Cu_1$$

by the usual rewriting into the first order matricial system

$$\frac{d}{dt}Mz(t) = Lz(t) + F(t),\ t > 0; \quad Mz(0) = Mz_0,$$

where

$$M = \begin{bmatrix} I & O \\ O & C \end{bmatrix},\ L = \begin{bmatrix} O & I \\ -A & -B \end{bmatrix},\ z_0 = \begin{bmatrix} u_0 \\ u_1 \end{bmatrix} \text{ and } F(t) = \begin{bmatrix} 0 \\ f(t) \end{bmatrix}\ (t > 0).$$

The multivalued linear operator $(L_{[D(B)]\times X} - \omega M_{[D(B)]\times X})(M_{[D(B)]\times X})^{-1}$ satisfies condition (P) used in the monograph (Kostić 2019), with a sufficiently large number $\omega > 0$, in the pivot space $[D(B)] \times X$. Hence, the operator $(L_{[D(B)]\times X} - \omega M_{[D(B)]\times X})(M_{[D(B)]\times X})^{-1}$ generates a degenerate semigroup $(T(t))_{t>0}$ in $[D(B)] \times X$, having an integrable singularity at zero and exponentially decaying growth rate at infinity, so that Favini and Yagi (1998, Theorem 3.8, Theorem 3.9) are applicable in the analysis of existence and uniqueness of solutions of the problem

$$\frac{d}{dt}Mz(t) = (L - \omega M)z(t) + F(t),\ t > 0; \quad Mz(0) = Mz_0. \tag{2.20}$$

Let us denote the components of $z(t)$ by $u(t)$ and $v(t)$; then a simple analysis of problem (2.20) enables us to consider the well-posedness of the following second-order differential equation:

$$\frac{d}{dt}\big(Cu'(t)\big) + (2\omega C + B)u'(t) + \big(A + \omega B + \omega^2 C\big)u(t) = f(t),\ t > 0;$$
$$u(0) = u_0,\ C\big[u'(0) + \omega u_0\big] = Cu_1. \tag{2.21}$$

Suppose, for example, that $M[u_0\, u_1]^T$ belongs to the domain of continuity of $(T(t))_{t>0}$ and $f(\cdot)$ is Hölder continuous with an appropriate Hölder exponent. Then there exists a unique solution $z(t)$ of (2.20), continuous for $t \geq 0$, and

$$Mz(t) = M\begin{bmatrix} u(t) \\ v(t) \end{bmatrix} = T(t)M\begin{bmatrix} u_0 \\ u_1 \end{bmatrix} + \int_0^t T(t-s)\begin{bmatrix} 0 \\ f(s) \end{bmatrix} ds, \quad t \geq 0;$$

all this has been seen in Kostić (2019, Example 3.10.10). Assume now that there exist a compactly almost automorphic function $f_0 : \mathbb{R} \to X$ and a function $q \in C_0([0,\infty) : X)$ such that $f(t) = f_0(t) + q(t)$ for all $t \geq 0$. Then Theorem 2.27 implies that a unique solution $u(\cdot)$ of (2.21) satisfies that $Mz(\cdot) = [u(\cdot) \quad C(u'(\cdot) + \omega u(\cdot))]^T$ is a sum of a compactly $([D(B)] \times X)$-valued almost automorphic function and a function belonging to the space $C_0([0,\infty) : [D(B)] \times X)$, which can be simply applied in the analysis of the existence and uniqueness of asymptotically compactly almost automorphic solutions of the damped Poisson-wave type equations in the spaces $X := H^{-1}(\Omega)$ or $X := L^p(\Omega)$; see also Favini and Yagi (1998, Example 6.1) for further information.

3 Applications to the Abstract Volterra Integro-differential Equations

In this section, we present some applications of our abstract results in the qualitative analysis of solutions for various classes of the abstract Volterra integro-differential equations.

3.1 Applications to the Semilinear Integral Equations

In this subsection, we present some applications of established composition results and the results about the invariance of $(\mathrm{R}, \mathcal{B})$-multi-almost automorphy under the actions of multi-dimensional convolution products. We start by stating the following result:

Theorem 3.1 *Let $F, G : \mathbb{R}^n \times X \to X$ be two $(\mathrm{R}, \mathcal{B})$-multi-almost automorphic functions, where $\mathcal{B}$ is the collection of all bounded subsets of X, R is any collection of sequences in $\mathbb{R}^n$ satisfying that for each sequence $(\mathbf{b}_k)$ in R any its subsequence also belongs to R. Suppose that, for every bounded subset B of X, we have*

$$\sup_{\mathbf{t}\in\mathbb{R}^n; x\in B} \Big[\|F(\mathbf{t}; x)\| + \|G(\mathbf{t}; x)\|\Big] < \infty. \tag{3.1}$$

If [E1] *holds with $Y = X$, then there exists a unique bounded* R*-multi-almost automorphic solution of the integral equation*

$$u(\mathbf{t}) = F(\mathbf{t}; u(\mathbf{t})) + \int_{\mathcal{I}_{\mathbf{t}}} K(\mathbf{t} - \eta) G(\eta, u(\eta))\, d\eta, \quad \mathbf{t} \in \mathbb{R}^n, \tag{3.2}$$

provided that the function $G(\cdot; \cdot)$ satisfies the estimate (2.13) *with some finite real constant $L > 0$, the function $F(\cdot; \cdot)$ satisfies the estimate* (2.13) *with some finite real constant $L_F > 0$ and the meaning clear, and*

$$L_F + L \int_{(0,\infty)^n} \|K(\eta)\|_{L(X)}\, d\eta < 1. \tag{3.3}$$

Proof Due to Proposition 2.11(ii), the vector space $\mathcal{X}$ of all bounded R-multi-almost automorphic functions $u : \mathbb{R}^n \to X$ endowed with the sup-norm is a Banach space. Furthermore, Theorem 2.20 in combination with the estimate (3.1) implies that, for every function $u : \mathbb{R}^n \to X$ which belongs to $\mathcal{X}$, the functions $\mathbf{t} \mapsto F(\mathbf{t}; u(\mathbf{t}))$, $\mathbf{t} \in \mathbb{R}^n$ and $\mathbf{t} \mapsto G(\mathbf{t}; u(\mathbf{t}))$, $\mathbf{t} \in \mathbb{R}^n$ are bounded R-multi-almost automorphic. Applying after that Theorem 2.25, we get that the integral operator

$$\mathbf{t} \mapsto (\Gamma u)(\mathbf{t}) := F(\mathbf{t}; u(\mathbf{t})) + \int_{\mathcal{I}_{\mathbf{t}}} K(\mathbf{t} - \eta) G(\eta, u(\eta))\, d\eta, \quad \mathbf{t} \in \mathbb{R}^n,$$

is well defined and maps the space $\mathcal{X}$ into itself. The final conclusion simply follows from the Banach contraction principle and a simple calculation involving the estimate (3.3). □

Without any substantial difficulties, we can similarly consider the existence and uniqueness of bounded compactly R-multi-almost automorphic solutions of the integral equation (3.2), provided that the functions $F(\cdot; \cdot)$ and $G(\cdot; \cdot)$ satisfy conditions sufficient for applying Corollary 2.21. Furthermore, we can similarly consider the existence and uniqueness of bounded (R, W_{R})-multi-almost automorphic solutions of the equation (and its semilinear analogues)

$$u(\mathbf{t}) = f(\mathbf{t}) + \int_{\mathcal{I}_{\mathbf{t}}} K(\mathbf{t} - \eta) u(\eta)\, d\eta, \quad \mathbf{t} \in \mathbb{R}^n,$$

where $f(\cdot)$ is bounded (R, W_{R})-multi-almost automorphic, (E1) holds and, for every set $D \in W_{\mathrm{R}}$ and for every compact set $C \subseteq [0, \infty)^n$, we have that $D - C \subseteq D'$ for some set $D' \in W_{\mathrm{R}}$; cf. also the formulation of Theorem 2.25.

It is worth noting that the equation (3.2) can be used for modeling of some two-dimensional nonlinear Volterra integral equations of convolution type of the second kind with infinite delay; see Aziz et al. (2014) for some examples in the absence of delay and Courant and Hilbert (1989, Chapter 10) for some other results in this direction. In actual fact, we can consider the well-posedness of equation

$$u(x, y) = g(x, y) + \int_{-\infty}^{x} \int_{-\infty}^{y} K(x, y, s, t, u(s, t))\, ds\, dt, \quad (x, y) \in \mathbb{R}^2,$$

provided that $K(x, y, s, t, u(s, t))$ has the form:

$$K(x, y, s, t, u(s, t)) = k(s - x, t - y)h(s, t, u(s, t));$$

our results about the invariance of $\mathbb{D}$-asymptotical $(\mathrm{R}, \mathcal{B})$-multi-almost automorphy can be applied in the qualitative analysis of solutions to the following two-dimensional nonlinear Volterra integral equation ($\mathbf{t} = (x, y)$):

$$f(\mathbf{t}) = g(\mathbf{t}; f(\mathbf{t})) + \int_{0}^{x} \int_{0}^{y} K(\mathbf{t} - \eta)h(\eta, f(\eta))\, d\eta,$$

as well.

We close this subsection by observing that, in the fourth part of Chávez et al. (2020, Section 3), we have considered the existence and uniqueness of Bohr almost periodic solution of the following Hammerstein integral equation of convolution type on $\mathbb{R}^n$ (cf. (Corduneanu 1991, Section 4.3, pp. 170-180) for more details on the subject):

$$y(\mathbf{t}) = g(\mathbf{t}) + \int_{\mathbb{R}^n} k(\mathbf{t} - \mathbf{s})F(\mathbf{s}, y(\mathbf{s}))\, d\mathbf{s}, \quad \mathbf{t} \in \mathbb{R}^n. \tag{3.4}$$

Keeping in mind the deduced composition principles and the results about the convolution invariance of $(\mathrm{R}, \mathcal{B})$-multi-almost automorphy, we can easily transfer the established results to the multi-dimensional almost automorphic functions. For example, assume that $g : \mathbb{R}^n \to X$ is (compactly) almost automorphic, R is the collection of all sequences in $\mathbb{R}^n$, $\mathcal{B}$ is the collection of all compact subsets of X, $F(\cdot; \cdot)$ is $(\mathrm{R}, \mathcal{B}, \mathrm{P}_{\mathrm{R},\mathcal{B}})$-multi-almost automorphic, where for each $B \in \mathcal{B}$ we have that $\mathrm{P}_{\mathrm{R},\mathcal{B}}(B)$ is the collection of all compact subsets of $\mathbb{R}^n \times X$, and there exists a finite constant $L > 0$ such that (2.13) holds with the function $G(\cdot; \cdot)$ replaced therein with the function $F(\cdot; \cdot)$. If $k \in L^1(\mathbb{R}^n)$ and $L\|k\|_{L^1(\mathbb{R}^n)} < 1$, then (3.4) has a unique (compactly) almost automorphic solution (see Proposition 2.12 and Corollary 2.21).

3.2 Spatially Almost Automorphic Solutions of the Multidimensional Heat Equation and the Multidimensional Wave Equation

In this subsection, we will first study the initial value problem for the homogeneous heat equation with nonlocal diffusion

$$\begin{aligned} u_t - \Delta u &= 0 \quad \text{in} \quad [0, +\infty) \times \mathbb{R}^n, \\ u(0, x) &= F(x) \quad \text{in} \quad \mathbb{R}^n \times \{0\}. \end{aligned} \tag{3.5}$$

Suppose for simplicity that $X = BUC(\mathbb{R}^n : \mathbb{C})$, the Banach space of bounded uniformly continuous functions on $\mathbb{R}^n$ equipped with the sup-norm. Then it is well known that the Gaussian semigroup

$$(G(t)F)(x) := \big(4\pi t\big)^{-(n/2)} \int_{\mathbb{R}^n} F(x-y)e^{-\frac{|y|^2}{4t}}\, dy, \quad t > 0,\ F \in X,\ x \in \mathbb{R}^n,$$

can be extended to a bounded analytic C_0-semigroup of angle $\pi/2$, generated by the Laplacian Δ_x as well as that the unique solution of (3.5) is given by $(t, x) \mapsto (G(t)F)(x)$, $t \geq 0$, $x \in \mathbb{R}^n$. Suppose now that a number $t_0 > 0$ is fixed. Then Proposition 2.12 shows that the function $\mathbb{R}^n \ni x \mapsto u(x, t_0) \equiv (G(t_0)F)(x) \in \mathbb{C}$ is bounded, R-multi-almost automorphic provided that R is any non-empty collection of sequences in $\mathbb{R}^n$ and the function $F(\cdot)$ is bounded, R-multi-almost automorphic. We can similarly apply Proposition 2.12 to the Poisson semigroup in $\mathbb{R}^n$; see Chávez et al. (2020) for more details.

In the remainder of this subsection, we shall revisit the classical theory of partial differential equations of second order and provide some new applications in the qualitative analysis of solutions of the wave equations in $\mathbb{R}^3$ (the obtain conclusions are new even for spatially almost periodic solutions):

$$u_{tt}(t, x) = d^2\Delta_x u(t, x), \quad x \in \mathbb{R}^3,\ t > 0;\ \ u(0, x) = g(x),\ \ u_t(0, x) = h(x), \tag{3.6}$$

where $d > 0$, $g \in C^3(\mathbb{R}^3 : \mathbb{R})$ and $h \in C^2(\mathbb{R}^3 : \mathbb{R})$. By the famous Kirchhoff formula (see e.g., (Salsa 2008, Theorem 5.4, pp. 277-278) for the notion used and more details about the spherical means; we will use the same notion and notation), the function

$$\begin{aligned} u(t, x) &:= \frac{\partial}{\partial t}\Bigg[\frac{1}{4\pi d^2 t}\int_{\partial B_{dt}(x)} g(\boldsymbol{\sigma})\, d\sigma\Bigg] + \frac{1}{4\pi d^2 t}\int_{\partial B_{dt}(x)} h(\boldsymbol{\oe})\, d\sigma \\ &= \frac{1}{4\pi}\int_{\partial B_1(0)} g(x + dt\boldsymbol{\omega})\, d\omega + \frac{dt}{4\pi}\int_{\partial B_1(0)} \nabla g(x + dt\boldsymbol{\omega})\cdot\boldsymbol{\omega}\, d\omega \\ &\quad + \frac{t}{4\pi}\int_{\partial B_1(0)} h(x + dt\boldsymbol{\omega})\, d\omega, \quad t \geq 0,\ x \in \mathbb{R}^3, \end{aligned} \tag{3.7}$$

is a unique solution of problem (3.6) which belongs to the class $C^2([0,\infty) \times \mathbb{R}^3)$. Fix now a number $t_0 > 0$. Then the function $x \mapsto u(t_0, x)$, $x \in \mathbb{R}^3$ is Bohr c-almost periodic (c-uniformly recurrent) provided that the functions $g(\cdot)$, $\nabla g(\cdot)$ and $h(\cdot)$ are c-almost periodic (c-uniformly recurrent), where $c \in \mathbb{C}\setminus\{0\}$. Similarly, let us assume that the functions $g(\cdot)$, $\nabla g(\cdot)$ and $h(\cdot)$ are bounded R-multi-almost automorphic, where R is any collection of sequences in $\mathbb{R}^3$ such that, for every sequence $(\mathbf{b}_k) \in$ R, any subsequence $(\mathbf{b}_{k_l})$ of $(\mathbf{b}_k)$ also belongs to R (the last condition is superfluous in the case that $g \equiv 0$). If we replace the functions $g(\cdot)$ and $\nabla g(\cdot)$ in (3.7) with the corresponding limit functions $g^*(\cdot)$ and $\nabla g^*(\cdot)$ for the sequence $(\mathbf{b}_k)$ from the definition of R-multi-almost automorphy, then the use of dominated convergence theorem shows that the function $x \mapsto u(t_0, x)$, $x \in \mathbb{R}^3$ is likewise bounded R-multi-almost automorphic;

furthermore, the same statement holds for the notion of bounded $(\mathrm{R}, \mathrm{P}_{\mathrm{R}})$-multi-almost automorphy provided that the following holds:

(i) For every sequence $(\mathbf{b}_k) \in \mathrm{R}$ and for every subsequence $(\mathbf{b}_{k_l})$ of $(\mathbf{b}_k)$, we have $P_{(\mathbf{b}_k)} \subseteq P_{(\mathbf{b}_{k_l})}$;
(ii) For every sequence $(\mathbf{b}_k) \in \mathrm{R}$, for every set $D \in P_{(\mathbf{b}_k)}$ and for every compact set $C \subseteq \mathbb{R}^3$, we have the existence of a set $D' \in P_{(\mathbf{b}_k)}$ such that $D + C \subseteq D'$.

We can similarly provide some applications in the qualitative analysis of solutions of the wave equations in $\mathbb{R}^2$:

$$u_{tt}(t,x) = d^2 \Delta_x u(t,x), \quad x \in \mathbb{R}^2,\ t > 0; \quad u(0,x) = g(x), \quad u_t(0,x) = h(x), \tag{3.8}$$

where $d > 0$, $g \in C^3(\mathbb{R}^2 : \mathbb{R})$ and $h \in C^2(\mathbb{R}^2 : \mathbb{R})$. By the Poisson formula (see e.g., (Salsa 2008, Theorem 5.5, pp. 280–281)), we have that the function

$$\begin{aligned} u(t,x) &:= \frac{\partial}{\partial t}\left[\frac{1}{2\pi d}\int_{\partial B_{dt}(x)} \frac{g(\boldsymbol{\sigma})}{\sqrt{d^2t^2 - |x-y|^2}}\, d\sigma\right] \\ &+ \frac{1}{2\pi d}\int_{\partial B_{dt}(x)} \frac{h(\boldsymbol{\sigma})}{\sqrt{d^2t^2 - |x-y|^2}}\, d\sigma \\ &= d\int_{B_1(0)} \frac{g(x+dt\sigma)}{\sqrt{1-|\sigma|^2}}\, d\sigma + d^2 t\int_{B_1(0)} \frac{\nabla g(x+dt\sigma)\cdot\sigma}{\sqrt{1-|\sigma|^2}}\, d\sigma \\ &+ dt\int_{B_1(0)} \frac{h(x+dt\sigma)}{\sqrt{1-|\sigma|^2}}\, d\sigma, \quad t \geq 0,\ x \in \mathbb{R}^2, \end{aligned}$$

is a unique solution of problem (3.8) which belongs to the class $C^2([0,\infty) \times \mathbb{R}^3)$. Then we can argue as in the three-dimensional case.

Concerning the one-dimensional case, it should be recalled that the unique regular solution of wave equation

$$u_{tt}(t,x) = d^2 \Delta_x u(t,x), \quad x \in \mathbb{R},\ t > 0; \quad u(0,x) = g(x), \quad u_t(0,x) = h(x),$$

where $d > 0$, $g \in C^2(\mathbb{R} : \mathbb{R})$ and $h \in C^1(\mathbb{R} : \mathbb{R})$, is given by the d'Alembert formula

$$u(x,t) = \frac{1}{2}\big[g(x-at) + g(x+at)\big] + \frac{1}{2a}\int_{x-at}^{x+at} h(s)\, ds, \quad x \in \mathbb{R},\ t > 0.$$

In Chávez et al. (2020, Example 1.2), we have assumed that the functions $g(\cdot)$ and $h^{[1]}(\cdot) \equiv \int_0^{\cdot} g(s)\, ds$ are almost periodic; then the solution $u(x,t)$ can be extended to the whole real line in time variable and it is almost periodic in $(x,t) \in \mathbb{R}^2$. Arguing similarly, we may conclude the assumptions that the functions $g(\cdot)$ and $h^{[1]}(\cdot) \equiv \int_0^{\cdot} h(s)\, ds$ are almost automorphic imply that the solution $u(x,t)$ is almost automorphic in $(x,t) \in \mathbb{R}^2$. Details can be left to the interested readers.

Consider now the inhomogeneous wave equation

$$\begin{aligned} &u_{tt}(t,x) - d^2\Delta_x u(t,x) = f(t,x), \quad x \in \mathbb{R},\ t > 0; \\ &u(0,x) = g(x), \quad u_t(0,x) = h(x), \end{aligned} \tag{3.9}$$

where $d > 0$, $f(t, x)$ is continuously differentiable in the variable $t \in \mathbb{R}$ and continuous in the variable $x \in \mathbb{R}$, $g \in C^2(\mathbb{R} : \mathbb{R})$ and $h \in C^1(\mathbb{R} : \mathbb{R})$. Using the d'Alembert formula and the Duhamel principle (we will not consider the higher dimensions here for simplicity), the unique solution of (3.9) is given by

$$
\begin{aligned}
u(x,t) &= \frac{1}{2}\big[g(x-at)+g(x+at)\big]+\frac{1}{2a}\int_{x-at}^{x+at} h(s)\,ds \\
&\quad + \frac{1}{2d}\int_0^t \Bigg[\int_{x-d(t-s)}^{x+d(t-s)} f(r,s)\,dr\Bigg]\,ds \\
&:= u_h(x,t)+\frac{1}{2d}\int_0^t \Bigg[\int_{x-d(t-s)}^{x+d(t-s)} f(r,s)\,dr\Bigg]\,ds, \quad x\in\mathbb{R},\ t>0.
\end{aligned}
$$

If we assume that the functions $g(\cdot)$ and $h^{[1]}(\cdot) \equiv \int_0^{\cdot} h(s)\,ds$ are almost automorphic, then we get from the above that the solution $u_h(x, t)$ is almost automorphic in $(x, t) \in \mathbb{R}^2$. It is clear that the function

$$
(x,t) \mapsto u_p(x,t) \equiv \frac{1}{2d}\int_0^t \Bigg[\int_{x-d(t-s)}^{x+d(t-s)} f(r,s)\,dr\Bigg]\,ds
$$

can be defined for all $(x, t) \in \mathbb{R}^2$. Suppose now that $L > 0$ and the function $f(\cdot, \cdot)$ has the property that $\lim_{|x|\to+\infty} f(x, t) = 0$, uniformly in $t \in [0, L]$. Set $\mathbb{D} := \{(x, t) \in \mathbb{R}^2 : t \in [0, L]\}$. Then $u_p \in C_{0,\mathbb{D}}(\mathbb{R}^2 : \mathbb{R})$ since

$$
u_p(x,t) = \frac{1}{2d}\int_0^t \Bigg[\int_{x-ds}^{x+ds} f(r,t-s)\,dr\Bigg]\,ds, \quad x\in\mathbb{R},\ t\in\mathbb{R}
$$

and there exists a sufficiently large real number $x_0 > 0$ such that, for every $x \in \mathbb{R}$ with $|x| \geq x_0$, for every $t \in [0, L]$ and for every $s \in [0, t]$, we have $|f(r, t - s)| \leq \epsilon$ for all $r \in [x - dL, x + dL]$ and therefore

$$
\big|u_p(x,t)\big| \leq \epsilon \cdot L^2, \quad (x,t)\in\mathbb{D},\ |x|\geq x_0.
$$

Hence, the solution $u(x, t)$ obtained by a combination of the d'Alembert formula and the Duhamel principle will be $\mathbb{D}$-asymptotically R-multi-almost automorphic with R being the collection of all sequences in $\mathbb{R}^2$.

3.3 Applications to the Abstract Ill-posed Cauchy Problems

In this subsection, we will revisit once more the theory of integrated solution operator families, C-regularized solution operator families and their applications to the ill-posed abstract Cauchy problems. For more details about the notion used, we refer the reader to the monographs by Arendt et al. (2001) and Kostić (2011, 2015) by the third named author.

Without going into full details, which is almost impossible to be done, we will only present two illustrative examples which strongly justify the introduction of function spaces analyzed in this paper (similar conclusions hold for the corresponding classes of multi-dimensional almost periodic type functions). In order to achieve our aims, we mainly apply Proposition 2.12 concerning the convolution invariance of introduced function spaces (the use of symbol D is clear from the context).

1. Suppose that $k \in \mathbb{N}$, $a_\alpha \in \mathbb{C}$, $0 \leq |\alpha| \leq k$, $a_\alpha \neq 0$ for some α with $|\alpha| = k$, $P(x) = \sum_{|\alpha|\leq k} a_\alpha i^{|\alpha|} x^\alpha$, $x \in \mathbb{R}^n$, $P(\cdot)$ is an elliptic polynomial, i.e., there exist $C > 0$ and $L > 0$ such that $|P(x)| \geq C|x|^k$, $|x| \geq L$, $\omega := \sup_{x\in\mathbb{R}^n} \Re(P(x)) < \infty$, and X is one of the spaces $L^p(\mathbb{R}^n)$ $(1 \leq p \leq \infty)$, $C_0(\mathbb{R}^n)$, $C_b(\mathbb{R}^n)$ [the space of bounded continuous functions $f : \mathbb{R}^n \rightarrow \mathbb{C}$ equipped with the sup-norm], $BUC(\mathbb{R}^n)$ [the space of bounded uniformly continuous functions $f : \mathbb{R}^n \rightarrow \mathbb{C}$ equipped with the sup-norm]. Define

$$P(D) := \sum_{|\alpha|\leq k} a_\alpha f^{(\alpha)} \text{ and } D(P(D)) := \{f \in E : P(D)f \in E \text{ distributionally}\},$$

$n_X := n|(1/2) - (1/p)|$, if $X = L^p(\mathbb{R}^n)$ for some $p \in (1, \infty)$ and $n_X > n/2$, otherwise. Then we know that the operator $P(D)$ generates an exponentially bounded r-times integrated semigroup $(S_r(t))_{t\geq 0}$ in X for any $r > n_X$ as well as that the operator $P(D)$ generates an exponentially bounded n_X-times integrated semigroup $(S_{n_X}(t))_{t\geq 0}$ in $L^p(\mathbb{R}^n)$ provided $p \in (1, \infty)$; see e.g., (Kostić 2011, Example 2.8.6) and references quoted therein. We will consider the general case $r > n/2$ and the spaces $C_b(\mathbb{R}^n)$, $BUC(\mathbb{R}^n)$ below; in the setting of L^p-spaces, certain applications can be given for the multi-dimensional Weyl almost periodic functions and the multi-dimensional Weyl almost automorphic solutions (see Kostić 2021 for more details). It is well known that for each $t \geq 0$ there exists a function $f_t \in L^1(\mathbb{R}^n)$ such that

$$\big[S_r(t)f\big](x) := \big(f_t * f\big)(x), \quad x \in \mathbb{R}^n,\ f \in X.$$

Let us fix a number $t_0 \geq 0$, and let us assume that the function $X \ni f$ is R-multi-almost automorphic, where R is any non-empty collection of sequences in $\mathbb{R}^n$. Applying Proposition 2.12, we get that the function $x \mapsto [S_r(t_0)f](x)$, $x \in \mathbb{R}^n$ is R-multi-almost automorphic and belongs to X. In terms of the corresponding abstract first-order Cauchy problem, this means that there exists a unique X-valued continuous function $t \mapsto u(t)$, $t \geq 0$ such that $\int_0^t u(s)\, ds \in D(P(D))$ for every $t \geq 0$ and

$$u(t) = P(D) \int_0^t u(s)\, ds - \frac{t^r}{\Gamma(r+1)} f, \quad t \geq 0;$$

furthermore, the solution $t \mapsto u(t)$, $t \geq 0$ of this abstract Cauchy problem has the property that its orbit consists solely of R-multi-almost automorphic functions. Suppose now that the collection R additionally satisfies that for each sequence $\mathbf{b} \in$ R any its subsequence also belongs to R and consider, for simplicity, case in which $r \in \mathbb{N}$. If we assume that $f \in D(P(D)^r)$ and all functions

$$f,\ P(D)f, \ldots, P(D)^r f$$

are R-multi-almost automorphic, then it is well known that the function

$$u(t) := S_r(t)P(D)^r f + \frac{t^{r-1}}{(r-1)!}P(D)^{r-1} f + \cdots + tP(D)f + f, \quad t \geq 0 \quad (3.10)$$

is a unique continuous X-valued function which satisfies that $\int_0^t u(s)\,ds \in D(P(D))$ for every $t \geq 0$ and

$$u(t) = P(D)\int_0^t u(s)\,ds - f, \quad t \geq 0;$$

due to the representation formula (3.10) and our assumptions, the solution $t \mapsto u(t)$, $t \geq 0$ of this abstract Cauchy problem has the property that its orbit consists solely of R-multi-almost automorphic functions; see Kostić (2015, Subsection 2.9.7) for more details regarding the existence and growth of mild solutions of operators generating fractionally integrated C-semigroups and fractionally integrated C-cosine functions in locally convex spaces.

2. Suppose now that X is $C_b(\mathbb{R}^n)$ or $BUC(\mathbb{R}^n)$, $m \in \mathbb{N}$, $a_\alpha \in \mathbb{C}$ for $0 \leq |\alpha| \leq k$ and $a_\alpha \neq 0$ for some α with $|\alpha| = k$. Consider the operator $P(D)$ with its maximal distributional domain and its associated polynomial $P(x)$ defined as above. Set

$$h_{t,\beta}(x) := \bigl(1+|x|^2\bigr)^{-\beta/2}\sum_{j=0}^{\infty}\frac{t^{2j}P(x)^j}{(2j)!}, \quad x \in \mathbb{R}^n,\ t \geq 0,\ \beta \geq 0,$$

$\Omega(\omega) := \{\lambda^2 : \Re\lambda > \omega\}$, if $\omega > 0$ and $\Omega(\omega) := \mathbb{C} \setminus (-\infty, \omega^2]$, if $\omega \leq 0$. Assume $r \in [0, k]$, $\omega \in \mathbb{R}$ and condition (Kostić 2015, (W); Example 2.2.14) holds. Then, for every $\beta > (k - \frac{r}{2})\frac{n}{4}$, $P(D)$ generates an exponentially bounded $C_\beta(0)$-regularized cosine function $(C_\beta(t))_{t\geq 0}$ in X satisfying

$$C_\beta(t)f = \mathcal{F}^{-1}h_{t,\beta} * f, \quad t \geq 0,\ f \in X,$$

where $\mathcal{F}^{-1}$ denotes the inverse Fourier transform in $\mathbb{R}^n$. Since $\mathcal{F}^{-1}h_{t,\beta} \in L^1(\mathbb{R}^n)$ for every $t \geq 0$, we can repeat verbatim the arguments from the first application. For example, suppose that the function $X \ni f$ is R-multi-almost automorphic. Then the function $t \mapsto C_\beta(t)f$, $t \geq 0$ is a unique continuous X-valued function which satisfies that $\int_0^t (t-s)u(s)\,ds \in D(P(D))$ for every $t \geq 0$ and

$$u(t) = P(D)\int_0^t (t-s)u(s)\,ds - C_\beta(0)f, \quad t \geq 0;$$

as above, for each fixed number $t \geq 0$ we have that $u(t)$ is a spatially R-multi-almost automorphic function which belongs to X. See also Kostić (2015, Section 2.5), where we have analyzed the generation of fractional resolvent families by (non-)coercive differential operators; the obtained results can be applied with the obvious choice of operators $A_j \equiv -i\partial/\partial x_j$ $(1 \leq j \leq n)$.

Before proceeding to the final section of paper, let us note that Ding et al. (2008) investigated that the asymptotically almost automorphic solutions of the following integro-differential equation (with nonlocal initial data), which models the heat conduction in materials with memory:

$$u'(t) = Au(t) + \int_0^t B(t-s)u(s)\,ds + f(t, u(t)),\ t \geq 0, \tag{3.11}$$

$$u(0) = u_0 + g(u); \tag{3.12}$$

here, $u_0 \in X$, A and $(B(t))_{t\geq 0}$ are linear, closed and densely defined operators on X. Some results about the existence and uniqueness of the asymptotically almost automorphic solutions to the integro-differential equation (3.11)–(3.12) have been established in Chávez et al. (2021), as well. It could be of some importance to reconsider the statement of Ding et al. (2008, Theorem 2.7), given in the one-dimensional setting, for asymptotically R-almost automorphic type functions, where R denotes a certain collection of sequences in R which has the property that, for every sequence $(b_k) \in$ R, any subsequence (b_{k_l}) of (b_k) also belongs to R. It seems that this can be done with some obvious modifications, not only in the case of consideration (Ding et al. 2008, Theorem 2.7), but also in the case of consideration of many other structural results obtained so far regarding the time almost automorphic solutions of the abstract PDEs (see Kostić 2019 and references cited therein); details can be left to the interested readers.

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4 Appendix: Almost Automorphic Functions on Semi-topological Groups

The first systematic study of almost automorphic functions on topological groups was conducted by Veech (1965, 1967) (see also the papers by Reich (1970) and Terras (1972)). Following Milnes (1977), who considered only the scalar-valued case, we say that a continuous function $f : G \to Y$, where G is a (semi-)topological group, is almost automorphic if and only if for any sequence (n_i') in G there exists a subsequence (n_i) of (n_i') such that the joint limit $\lim_{i,j} f(n_i n_j^{-1} t) = f(t)$ exists for all $t \in G$. It is clear that $\mathbb{R}^n \times X$ is a semi-topological group as well as that the notion of $(\mathrm{R}_{\mathrm{X}}, \mathcal{B})$-multi-almost automorphy can be extended in this rather general framework. For more details about almost periodic functions on topological groups, see the research monograph by Levitan (1953) and the reference list given in the forthcoming monograph (Kostić 2021).

In this section, we will briefly explain the main ideas and results about almost automorphic functions on semi-topological groups established by Milnes (1977); we will also remind the readers of some known results about almost automorphic functions on topological groups obtained by other authors (there is a vast literature about topological groups and their generalizations; we will only refer the reader to the recent book (Morris 2019) edited by S. A. Morris and references cited therein).

Let G be a topological space which is also a multiplicative group. Then we say that G is a semi-topological space if and only if the mappings $s \mapsto st$ and $s \mapsto ts$ from G into G are continuous for all $t \in G$; furthermore, G is called a topological group if, in addition to the above, we have that the mapping $(s, t) \mapsto st^{-1}$ from $G \times G$ into G is continuous. By $\mathcal{J}$ we denote the topology on G and by $C_b(G : Y)$ we denote the space of all bounded continuous functions $f : G \to Y$ equipped with the sup-norm $\| \cdot \|_\infty$. We say that:

(i) a subset D of a semi-topological group G is left relatively dense if and only if there exists a finite set of elements $\{s_i : 1 \le i \le N\}$ in G such that $G \subseteq \bigcup_{i=1}^{N}(s_i D)$;
(ii) a topological group G is totally bounded if and only if for every non-empty neighbourhood V in G we have the existence of a finite set of elements $\{s_i : 1 \le i \le N\}$ in G such that $G \subseteq \bigcup_{i=1}^{N}(s_i V)$.

For any $s \in G$, the left (right) translate f_s (f^s) of f is defined through $f_s(\cdot) := f(s\cdot)$ ($f_s(\cdot) := f(\cdot s)$). A subspace C of $C_b(G : Y)$ is called translation invariant if and only if f_s and f^s belong to C for every $f \in C$. If $f : G \to Y$ and $g : G \to Y$ are given functions and $(\alpha_i)_{i\in I}$, resp. $(n_i)_{i\in\mathbb{N}}$, is a net in G, resp. a sequence in G, then we write $T_\alpha f = g$ if and only if the net of left translations f_{α_i}, resp. f_{n_i}, converges pointwise on G. The right uniformly continuous subspace $RUC_b(G : Y)$ of $C_b(G : Y)$ is defined as the set of all functions $f \in C_b(G : Y)$ such that $\| f^{\alpha_i} - f^s \|$ tends to zero whenever $(\alpha_i)_{i\in I}$ is a net in G converging to $s \in G$; the left continuous subspace $LUC_b(G : Y)$ of $C_b(G : Y)$ is defined similarly.

Definition 4.1 Let G be a semi-topological group. Then we say that a continuous function $f : G \to Y$ is left almost automorphic if and only if every net $\alpha' \subseteq G$ has a subnet $\alpha \subseteq G$ such that $T_\alpha f = g$ and $T_{\alpha^{-1}} g = f$, where $\alpha^{-1} = (\alpha_i^{-1})$; the notion of right almost automorphy is introduced similarly, with the analogous conditions involving right translates. By $LAA(G : Y)$ and $RAA(G : Y)$ we denote the family of all left almost automorphic functions on G and the right almost automorphic functions on G, respectively.

A function $f \in C_b(G : Y)$ is called almost periodic if and only if the set of all left translations $\{f_s : s \in G\}$ is relatively compact in $C_b(G : Y)$. Any almost periodic function $f \in C_b(G : Y)$ is left almost automorphic and satisfies that the convergence in $T_\alpha f = g$ is uniform on G, along with the convergence in $T_{\alpha^{-1}} g = f$. We know that $LAA(G : Y)$ and $RAA(G : Y)$ are translation invariant spaces as well as that the limit $T_\alpha f = g$ need not be continuous on G.

Suppose, for the time being, that $Y = \mathbb{C}$. Then we know that, if G is a Hausdorff topological group that is complete in a left invariant metric or locally compact and $f \in C_b(G : \mathbb{C})$, then we can always find a net $(\alpha_i)_{i\in I}$ such that $T_\alpha f = g$ is discontinuous on G if and only if $f \notin RUC_b(G : \mathbb{C})$. In what follows, it will be said that the Bohr topology B on a semi-topological group G is that topology which has the property that a subbase of B-neighbourhoods of a point $s \in G$ forms the sets $\{t \in G : |f(t) - f(s)| < \epsilon\}$, where $f : G \to \mathbb{C}$ is almost periodic and $\epsilon > 0$; a function $f : G \to \mathbb{C}$ is said to be Bohr continuous if and only if the function $f(\cdot)$ is continuous for the Bohr topology. Due to Milnes (1977, Theorem 8), a necessary and sufficient condition for a topological group G to be totally bounded is that every continuous complex-valued function on G is Bohr continuous.

For the scalar-valued functions, Milnes (1977, Theorem 13) states that for any continuous function $f : G \to \mathbb{C}$, where G is a semi-topological group, the following conditions are mutually equivalent:

1. (2.) $f(\cdot)$ is left (right) almost automorphic.
3. $f(\cdot)$ is Bohr continuous.
4. For every $\epsilon > 0$ and for every finite set $N \subseteq G$, there exists a left relatively dense subset $D \subseteq G \ni D^{-1}D \subseteq \{s \in G : \sup_{r,t\in N} |f(rst) - f(rt)| < \epsilon\}$.
5. (6.) For every $\epsilon > 0$ and $t \in G$, there exists a left relatively dense subset $D \subseteq G \ni D^{-1}D \subseteq \{s \in G : \sup_{r,t\in N} |f(ts) - f(t)| < \epsilon\}$ ($D \subseteq G \ni D^{-1}D \subseteq \{s \in G : \sup_{r,t\in N} |f(st) - f(t)| < \epsilon\}$).
7. For every net $\alpha \subseteq G$, there exists a subnet $\alpha \subseteq G$ such that the joint limit $\lim_{i,j} f(s\alpha_i\alpha_j^{-1}t) = f(st)$ for all $s,\ t \in G$.
8. (9.) For every net $\alpha \subseteq G$, there exists a subnet $\alpha \subseteq G$ such that the joint limit $\lim_{i,j} f(\alpha_i\alpha_j^{-1}t) = f(t)$ for all $t \in G$ ($\lim_{i,j} f(t\alpha_i\alpha_j^{-1}) = f(t)$ for all $t \in G$).
10. For every sequence $\mathbf{n}' \subseteq G$, there exists a subnet $\mathbf{n} \subseteq G$ such that the joint limit $\lim_{i,j} f(sn_in_j^{-1}t) = f(st)$ for all $s,\ t \in G$
11. (12.) For every sequence $\mathbf{n}' \subseteq G$, there exists a subnet $\mathbf{n} \subseteq G$ such that the joint limit $\lim_{i,j} f(n_in_j^{-1}t) = f(t)$ for all $t \in G$ ($\lim_{i,j} f(tn_in_j^{-1}) = f(t)$ for all $t \in G$).

Although it would be very unpleasant to clarify the validity or non-validity of above conditions for the vector-valued functions $f : G \to Y$, especially for those Banach spaces Y which are not separable (see e.g., the proof of (Milnes 1977, Theorem 10)), we would like to note that some equivalence relations clarified above hold for the vector-valued functions $f : G \to Y$ on topological groups G. For example, B. Basit has proved, in Basit (1974, Theorem 1.2), that a bounded continuous function $f : G \to Y$ is almost automorphic if and only if $f(\cdot)$ is Levitan almost periodic (see (Basit 1974, Definition 1.1)), which immediately implies the equivalence of [1. (2.)] and [8. (9.)] in this framework. Keeping this in mind, it seems reasonable to further explore the following notion (more details will appear somewhere else; see also the research study of Weyl multi-dimensional almost automorphic functions carried out in Kostić (2021), where this approach has been essentially followed):

Definition 4.2 Suppose that $F : \mathbb{R}^n \times X \to Y$ is a continuous function as well as that for each $B \in \mathcal{B}$ and $(\mathbf{b}_k = (b_k^1, b_k^2, \ldots, b_k^n)) \in \mathrm{R}$ we have $W_{B,(\mathbf{b}_k)} : B \to P(P(\mathbb{R}^n))$ and $\mathrm{P}_{B,(\mathbf{b}_k)} \in P(P(\mathbb{R}^n \times B))$. Then we say that $F(\cdot;\cdot)$ is:

(i) jointly $(\mathrm{R}, \mathcal{B})$-multi-almost automorphic if and only if for every $B \in \mathcal{B}$ and for every sequence $(\mathbf{b}_k = (b_k^1, b_k^2, \ldots, b_k^n)) \in \mathrm{R}$ there exists a subsequence $(\mathbf{b}_{k_l} = (b_{k_l}^1, b_{k_l}^2, \ldots, b_{k_l}^n))$ of $(\mathbf{b}_k)$ such that

$$\lim_{(l,m)\to+\infty} F\Big(\mathbf{t} - \big(b_{k_l}^1, \ldots, b_{k_l}^n\big) + \big(b_{k_m}^1, \ldots, b_{k_m}^n\big); x\Big) = F(\mathbf{t}; x), \tag{4.1}$$

pointwisely for all $x \in B$ and $\mathbf{t} \in \mathbb{R}^n$;

(ii) jointly $(\mathrm{R}, \mathcal{B}, W_{\mathcal{B},\mathrm{R}})$-multi-almost automorphic if and only if for every $B \in \mathcal{B}$ and for every sequence $(\mathbf{b}_k = (b_k^1, b_k^2, \ldots, b_k^n)) \in \mathrm{R}$ there exists a subsequence $(\mathbf{b}_{k_l} = (b_{k_l}^1, b_{k_l}^2, \ldots, b_{k_l}^n))$ of $(\mathbf{b}_k)$ such that (4.1) holds pointwisely for all $x \in B$ and $\mathbf{t} \in \mathbb{R}^n$ as well as that for each $x \in B$ the convergence in (4.1) is uniform in $\mathbf{t}$ for any set of the collection $W_{B,(\mathbf{b}_k)}(x)$;

(iii) jointly $(\mathrm{R}, \mathcal{B}, \mathrm{P}_{\mathcal{B},\mathrm{R}})$-multi-almost automorphic if and only if for every $B \in \mathcal{B}$ and for every sequence $(\mathbf{b}_k = (b_k^1, b_k^2, \ldots, b_k^n)) \in \mathrm{R}$ there exists a subsequence $(\mathbf{b}_{k_l} = (b_{k_l}^1, b_{k_l}^2, \ldots, b_{k_l}^n))$ of $(\mathbf{b}_k)$ such that (4.1) holds pointwisely for all $x \in B$ and $\mathbf{t} \in \mathbb{R}^n$ as well as that the convergence in (4.1) is uniform in $(\mathbf{t}; x)$ for any set of the collection $\mathrm{P}_{B,(\mathbf{b}_k)}$.

Arguing as above, it can be simply verified that any $(\mathrm{R}, \mathcal{B})$-multi-almost periodic function $F : \mathbb{R}^n \times X \to Y$ is jointly $(\mathrm{R}, \mathcal{B}, \mathrm{P}_{\mathcal{B},\mathrm{R}})$-multi-almost automorphic with $\mathrm{P}_{\mathcal{B},\mathrm{R}} = \{\mathbb{R}^n \times B\}$. We also have the following:

Proposition 4.3 *Suppose that $F : \mathbb{R}^n \to Y$ is a c-uniformly recurrent function (see Definition 1.2-(ii)), where the sequence $(\boldsymbol{\omega}_k)$ satisfies $\lim_{k\to+\infty} |\boldsymbol{\omega}_k| = +\infty$ and (1.3). Let* R *denote the collection consisting of the sequence $(\boldsymbol{\tau}_k)$ and all its subsequences. Then the function $F(\cdot)$ is jointly $(\mathrm{R}, \mathrm{P}_\mathrm{R})$-multi-almost automorphic with P_R being the singleton $\{\mathbb{R}^n\}$.*

Proof Let $(\boldsymbol{\tau}'_k)$ be any subsequence of $(\boldsymbol{\tau}_k)$. Then we have (1.3) with the sequence $(\boldsymbol{\omega}_k)$ replaced with the sequence $(\boldsymbol{\tau}'_k)$ therein; therefore, we also have

$$\lim_{k\to+\infty} \sup_{\mathbf{t}\in\mathbb{R}^n} \left\| F(\mathbf{t} - \boldsymbol{\tau}'_k; x) - c^{-1} F(\mathbf{t}; x) \right\|_Y = 0. \tag{4.2}$$

The final conclusion simply follows from the above estimates, the corresponding definition of joint $(\mathrm{R}, \mathrm{P}_\mathrm{R})$-multi-almost automorphy and the decomposition:

$$\begin{aligned}
&\sup_{\mathbf{t}\in\mathbb{R}^n} \left\| F\big(\mathbf{t} - \boldsymbol{\omega}'_l + \boldsymbol{\tau}'_m; x\big) - F(\mathbf{t}; x) \right\|_Y \\
&\le \sup_{\mathbf{t}\in\mathbb{R}^n} \left\| F\big(\mathbf{t} - \boldsymbol{\tau}'_l + \boldsymbol{\tau}'_m; x\big) - cF\big(\mathbf{t} - \boldsymbol{\tau}'_l; x\big) \right\|_Y + \sup_{\mathbf{t}\in\mathbb{R}^n} \left\| cF\big(\mathbf{t} - \boldsymbol{\tau}'_l; x\big) - F(\mathbf{t}; x) \right\|_Y \\
&= \sup_{\mathbf{t}\in\mathbb{R}^n} \left\| F\big(\mathbf{t} - \boldsymbol{\tau}'_l + \boldsymbol{\tau}'_m; x\big) - cF\big(\mathbf{t} - \boldsymbol{\tau}'_l; x\big) \right\|_Y + |c| \sup_{\mathbf{t}\in\mathbb{R}^n} \left\| F\big(\mathbf{t} - \boldsymbol{\tau}'_l; x\big) - c^{-1} F(\mathbf{t}; x) \right\|_Y.
\end{aligned}$$

□

Furthermore, we can similarly introduce and analyze the notions of joint $(\mathrm{R}_X, \mathcal{B})$-multi-almost automorphy, joint $(\mathrm{R}_X, \mathcal{B}, W_{\mathcal{B},\mathrm{R}_X})$-multi-almost automorphy and joint $(\mathrm{R}_X, \mathcal{B}, \mathrm{P}_{\mathcal{B},\mathrm{R}_X})$-multi-almost automorphy (see Definition 4.2).

The results on approximations of almost automorphic functions, proved by Veech (1965, 1967) on topological groups, continue to hold on semi-topological groups without any essential changes. For example, by Milnes (1977, Theorem 18), we know that a continuous function $f : G \to Y$ is almost automorphic if and only if there exists a uniformly bounded sequence (f_k) of almost periodic functions $f_k : G \to \mathbb{C}$ $(k \in \mathbb{N})$ such that, for every $s \in G$ and $\epsilon > 0$, we have the existence of a Bohr neighbourhood V of s and an integer $k_0 \in \mathbb{N}$ such that, for very integer $k \geq k_0$, we have $|f_k(t) - f(t)| < \epsilon$ for all $t \in V$. See also Veech (1965, Subsection 6.2) for some elementary facts regarding analytic almost automorphic functions defined on the additive group of integers $\mathbb{Z}$.

The complete characterization of those semi-topological groups for which the equality $AP(G : \mathbb{C}) = AA(G : \mathbb{C})$ holds is given in Milnes (1977, Theorem 23). In Milnes (1977, Theorem 25), P. Milnes has shown that, if G is arbitrary semi-topological group and $f : G \to Y$ is almost automorphic, then $f(\cdot)$ is almost periodic if and only if $T_\alpha f \in AA(G : \mathbb{C})$ whenever it exists, extending thus a result of W. A. Veech known on topological groups before that. It is also worth noting that Terras (1972) has constructed an almost automorphic function $f : \mathbb{Z} \to \mathbb{R}$ for which the limit $\lim_{N\to+\infty} \frac{1}{2N+1} \sum_{i=-N}^{N} f(i)$ does not exist. It is well known that this example can be transferred to the continuous setting as well as that there exists an almost automorphic function $f : \mathbb{R} \to \mathbb{R}$ such that the limit

$$\mathcal{M}(f) := \lim_{t\to+\infty} \frac{1}{2t} \int_{-t}^{t} f(s)\, ds$$

does not exist.

Concerning the notion of almost automorphy and the notion of almost periodicity for functions defined on (semi-)topological groups, it should be noted that some definitions for introducing these notions do not require a priori the continuity or measurability of function $f : G \to Y$ under consideration; see the research articles by Davies (1967) and Veech (1969) for some results obtained in this direction.

Concerning differences of almost periodic and almost automorphic functions defined on topological groups, with values in general locally convex spaces, we refer the reader to the research articles by Basit and Emam (1983) and Dimitrova and Dimitrov (2003). Mention should be made of paper by Péraire (1993), as well.

We close the paper with the observation that the Stepanov multi-dimensional almost automorphic type functions and their applications have recently been considered in Kostić et al. (2021).

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