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On (ϕ, ψ)-Inframonogenic Functions in Clifford Analysis

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Abstract
Solutions of the sandwich equation ${}^{\phi}\underline{\partial}[f]^{\psi}\underline{\partial} = 0$, where ${}^{\phi}\underline{\partial}$ stands for the Dirac operator with respect to a structural set ϕ, are referred to as (ϕ, ψ)-inframonogenic functions and capture the standard inframonogenic ones as special case. We derive a new integral representation formula for such functions as well as for multidimensional Ahlfors–Beurling transforms closely connected to the use of two different orthogonal basis in $\mathbb{R}^m$. Moreover, we also establish sufficient conditions for the solvability of a jump problem for the system ${}^{\phi}\underline{\partial}[f]^{\psi}\underline{\partial} = 0$ in domains with fractal boundary.

Keywords Clifford analysis · Structural sets · Inframonogenic functions · Π-Operator

Mathematics Subject Classification 30G35

1 Introduction

Inframonogenic functions are the solutions of the the second order partial differential equation $\underline{\partial}[f]\underline{\partial} = 0$, where

$$\underline{\partial} = e_1 \frac{\partial}{\partial x_1} + e_2 \frac{\partial}{\partial x_2} + \cdots + e_m \frac{\partial}{\partial x_m}$$

stands for the orthogonal Dirac operator in $\mathbb{R}^m$ constructed with the generators $\{e_1, e_2, \ldots, e_m\}$ of the real Clifford algebra $\mathbb{R}_{0,m}$.

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Such functions were originally introduced in Malonek et al. (2010), where a Fischer decomposition for homogeneous polynomials in terms of inframonogenic polynomials was derived. In Malonek et al. (2011) the same authors proved a Cauchy-Kowalevski extension theorem for these functions. More recently in García et al. (2017, 2018, 2020) it became clear that inframonogenic functions have interesting connections and applications in some topics of linear elasticity theory and there are serious reasons why such functions should be of interest to mathematicians, engineers, and physicists.

On the other hand, as the results in Blaya et al. (2015b, 2016, 2017), Delanghe et al. (2001), Gürlebeck (1998), Gürlebeck et al. (1999), Gürlebeck and Nguyen (2015, 2014), Krausshar and Malonek (2001), Nguyen (2015) have demonstrated, significant progress in Clifford analysis has been achieved when instead of the standard basis $\{e_1, e_2, \ldots, e_m\}$, one considers an arbitrary orthonormal basis $\psi := \{\psi^1, \psi^2, ..., \psi^m\}$ and the corresponding Dirac operator

$$
{}^{\psi}\underline{\partial} := \psi^1 \frac{\partial}{\partial x_1} + \psi^2 \frac{\partial}{\partial x_2} + \cdots + \psi^m \frac{\partial}{\partial x_m}.
$$

This leads to the notion of ψ-hyperholomorphic functions, as the $\mathbb{R}_{0,m}$-valued solutions of the equation ${}^{\psi}\underline{\partial}[u] = 0$.

The application of ψ-hyperholomorphic functions is natural in establishing a new representation for the general solution of the Lamé–Navier system in linear elasticity theory, a fact that has already been noticed in previous studies, Gürlebeck and Nguyen (2014, 2015) for instance.

In connection with all this there arise reasonable motivations to study the generalized sandwich equation ${}^{\phi}\underline{\partial}[f]{}^{\psi}\underline{\partial} = 0$, obtained by the use of two orthogonal bases ϕ, ψ. The solutions of this equation are an extension of the inframonogenic functions and will be referred here to as (ϕ, ψ)-inframonogenic functions. Our aim is to obtain a Borel–Pompeiu formula, yielding a Cauchy integral representation for such functions. We also explore some interesting relations between our work and a generalized Π-operator (Ahlfors–Beurling transform) introduced in Blaya et al. (2016). We end with an application of a generalized Teodorescu operator in solving some boundary value problem for (ϕ, ψ)-inframonogenic functions in fractal domains.

2 Preliminaries

Let $\mathbb{R}_{0,m}$ be the 2^m-dimensional real Clifford algebra constructed over the orthonormal basis $\{e_1, ..., e_m\}$ of the Euclidean space $\mathbb{R}^m$ $(m > 2)$. The multiplication in $\mathbb{R}_{0,m}$ is determined by relations $e_j e_k + e_k e_j = -2\delta_{jk}$ and a general element of $\mathbb{R}_{0,m}$ is of the form $a = \sum_A a_A e_A$, $a_A \in \mathbb{R}$, where for $A = \{j_1, ..., j_k\} \subset \{1, ..., m\}$, $j_1 < ... < j_k$, $e_A = e_{j_1} \cdots e_{j_k}$. For the empty set $\emptyset$, we put $e_\emptyset = 1$, the latter being the identity element.

Notice that any $a \in \mathbb{R}_{0,m}$ may also be written as $a = \sum_{k=0}^m [a]_k$ where $[a]_k$ is the projection of a on $\mathbb{R}_{0,m}^{(k)}$. Here $\mathbb{R}_{0,m}^{(k)}$ denotes the subspace of k-vectores defined by

$$\mathbb{R}_{0,m}^{(k)} = \left\{ a \in \mathbb{R}_{0,m} : \ a = \sum_{|A|=k} a_A e_A, \ \ a_A \in \mathbb{R} \right\}.$$

The conjugation in $\mathbb{R}_{0,m}$ is defined as the anti-involution $a \mapsto \overline{a}$ for which $\overline{e_i} = -e_i$. A norm $\|.\|$ on $\mathbb{R}_{0,m}$ is defined by $\|a\|^2 = Sc[a\overline{a}]$ for $a \in \mathbb{R}_{0,m}$. We remark that for $\underline{x} \in \mathbb{R}^m$ we have $\|\underline{x}\| = |\underline{x}|$, the usual Euclidean norm.

We will consider functions defined on subsets of $\mathbb{R}^m$ and taking values in $\mathbb{R}_{0,m}$. Those functions might be written as $f = \sum_A f_A e_A$, where f_A are $\mathbb{R}$-valued functions. The notions of continuity, differentiability and integrability of a $\mathbb{R}_{0,m}$-valued function have the usual component-wise meaning. In particular, the spaces of all k-times continuous differentiable, k-times ν-Hölder continuously differentiable and p-integrable functions are denoted by $C^k(\mathbf{E})$, $C^{k,\nu}(\mathbf{E})$ and $L^p(\mathbf{E})$ respectively, where $\mathbf{E}$ is a given subset of $\mathbb{R}^m$.

The so-called Dirac operator $\underline{\partial}$ is defined to be

$$\underline{\partial} := e_1 \frac{\partial}{\partial x_1} + e_2 \frac{\partial}{\partial x_2} + \cdots + e_m \frac{\partial}{\partial x_m}.$$

An $\mathbb{R}_{0,m}$-valued function f, defined and differentiable in an open region Ω of $\mathbb{R}^m$, is called left monogenic (right monogenic) if $\underline{\partial}[f] = 0$ ($[f]\underline{\partial} = 0$) in Ω (see Brackx et al. 1982; Güerlebeck et al. 2008).

More generally, for fixed orthonormal base $\psi := \{\psi^1, \psi^2, \ldots, \psi^m\}$ in $\mathbb{R}^m$ (structural set) we introduce the so-called ψ-hyperholomorphic functions (left or right respectively), which belong to $\ker[{}^{\psi}\underline{\partial}(\cdot)]$ or $\ker[(\cdot){}^{\psi}\underline{\partial}]$, where

$${}^{\psi}\underline{\partial} := \psi^1 \frac{\partial}{\partial x_1} + \psi^2 \frac{\partial}{\partial x_2} + \cdots + \psi^m \frac{\partial}{\partial x_m}.$$

As has been already mentioned in the introduction, we are rather interested in a second order partial differential equation involving two different orthogonal bases at once. So we define for an open set $\Omega \subset \mathbb{R}^m$, the following subclass of $\mathbb{R}_{0,m}$-valued functions:

$$\mathcal{I}_{\phi,\psi}(\Omega) = \{u \in C^2(\Omega) : \ {}^{\phi}\underline{\partial}[u]{}^{\psi}\underline{\partial} = 0\}.$$

Observe that for $\phi = \psi = \{e_1, e_2, \ldots, e_m\}$, the class $\mathcal{I}_{\phi,\psi}(\Omega)$ becomes the space of inframonogenic functions $\mathcal{I}(\Omega)$, introduced in Malonek et al. (2010, 2011). The above is reason enough for referring the elements of $\mathcal{I}_{\phi,\psi}$ to as (ϕ, ψ)-inframonogenic functions.

It is easy to see that such functions violate the maximum principle. Indeed, for $\phi = \psi$ or $\phi = -\psi$ in $\mathbb{R}^3$ consider the function given by

$$g(\underline{x}) = (ax_1^2 + bx_2^2 + cx_3^2 - 1)\psi^1,$$

where $a, b, c \in \mathbb{R}_+ \backslash \{0\}$, $a - b - c = 0$. Clearly, g vanishes on the boundary of the ellipsoid

$$\mathcal{E} = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : \ ax_1^2 + bx_2^2 + cx_3^2 < 1\}.$$

On the other hand, a direct calculation gives ${}^{\psi}\underline{\partial}[g]^{\psi}\underline{\partial} = 0$ in $\mathcal{E}$ with g being non identically zero there.

Now let us ask the question: can we find two different structural sets ϕ, ψ (with $\phi \neq \pm\psi$) for which the (ϕ, ψ)-inframonogenic functions satisfy the maximum principle? This question has a negative answer based in the following reasoning, which for the sake of simplicity, will be confined exclusively to $\mathbb{R}^3$.

Let us consider the function given by

$$g_1(\underline{x}) := (2x_1^2 + x_2^2 + x_3^2 - 1)(\phi^1\psi^2 + \phi^2\psi^1 + \phi^3\psi^3\psi^2\psi^1 + \phi^1\phi^2\phi^3\psi^3).$$

Obviously g_1 does vanish on the boundary of the ellipsoid

$$\mathcal{E}_1 := \{(x_1, x_2, x_3) \in \mathbb{R}^3 : \ 2x_1^2 + x_2^2 + x_3^2 < 1\}.$$

Moreover, we have

$$\begin{aligned}
{}^{\phi}\underline{\partial}[g_1]^{\psi}\underline{\partial} &= 4\phi^1\phi^1\psi^2\psi^1 + 2\phi^2\phi^1\psi^2\psi^2 + 2\phi^3\phi^1\psi^2\psi^3 \\
&\quad + 4\phi^1\phi^2\psi^1\psi^1 + 2\phi^2\phi^2\psi^1\psi^2 + 2\phi^3\phi^2\psi^1\psi^3 \\
&\quad + 4\phi^1\phi^3\psi^3\psi^2\psi^1\psi^1 + 2\phi^2\phi^3\psi^3\psi^2\psi^1\psi^2 + 2\phi^3\phi^3\psi^3\psi^2\psi^1\psi^3 \\
&\quad + 4\phi^1\phi^1\phi^2\phi^3\psi^3\psi^1 + 2\phi^2\phi^1\phi^2\phi^3\psi^3\psi^2 + 2\phi^3\phi^1\phi^2\phi^3\psi^3\psi^3 \\
&= 4\psi^1\psi^2 + 2\phi^1\phi^2 + 2\phi^3\phi^1\psi^2\psi^3 \\
&\quad - 4\phi^1\phi^2 - 2\psi^1\psi^2 + 2\phi^3\phi^2\psi^1\psi^3 \\
&\quad - 4\phi^3\phi^1\psi^2\psi^3 + 2\phi^3\phi^2\psi^1\psi^3 - 2\psi^1\psi^2 \\
&\quad - 4\phi^3\phi^2\psi^1\psi^3 + 2\phi^3\phi^1\psi^2\psi^3 + 2\phi^1\phi^2 = 0.
\end{aligned}$$

Therefore, g_1 is a (ϕ, ψ)-inframonogenic $\mathbb{R}_{0,3}$-valued polynomial with vanishing trace on $\partial\mathcal{E}_1$.

Similarly the functions

$$g_2(\underline{x}) = (x_1^2 + x_2^2 + 2x_3^2 - 1)(\phi^1\psi^3 + \phi^3\psi^1 + \phi^2\psi^3\psi^2\psi^1 + \phi^1\phi^2\phi^3\psi^2)$$

and

$$g_3(\underline{x}) = (x_1^2 + 2x_2^2 + x_3^2 - 1)(\phi^2\psi^3 + \phi^3\psi^2 + \phi^1\psi^3\psi^2\psi^1 + \phi^1\phi^2\phi^3\psi^1)$$

are (ϕ, ψ)-inframonogenic polynomials with vanishing traces on the boundaries of the ellipsoids

$$\mathcal{E}_2 = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : \ x_1^2 + x_2^2 + 2x_3^2 < 1\}$$

and

$$\mathcal{E}_3 = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : \ x_1^2 + 2x_2^2 + x_3^2 < 1\},$$

respectively.

At least one of the polynomials g_1, g_2, g_3 must not be identically zero as otherwise the system

$$\phi^1\psi^2 + \phi^2\psi^1 + \phi^3\psi^3\psi^2\psi^1 + \phi^1\phi^2\phi^3\psi^3 = 0 \tag{1}$$
$$\phi^1\psi^3 + \phi^3\psi^1 + \phi^2\psi^3\psi^2\psi^1 + \phi^1\phi^2\phi^3\psi^2 = 0 \tag{2}$$
$$\phi^2\psi^3 + \phi^3\psi^2 + \phi^1\psi^3\psi^2\psi^1 + \phi^1\phi^2\phi^3\psi^1 = 0 \tag{3}$$

is satisfied, implying ϕ to be ψ or $-\psi$, which leads to a contradiction.

To see that this implication has been properly made we proceed as follows.

Multiplying from the left (1) by ϕ^3, (2) by ϕ^2 and (3) by ϕ^1, we obtain

$$-\phi^1\phi^3\psi^2 - \phi^2\phi^3\psi^1 - \psi^3\psi^2\psi^1 - \phi^1\phi^2\psi^3 = 0 \tag{4}$$
$$-\phi^1\phi^2\psi^3 + \phi^2\phi^3\psi^1 - \psi^3\psi^2\psi^1 + \phi^1\phi^3\psi^2 = 0 \tag{5}$$
$$\phi^1\phi^2\psi^3 + \phi^1\phi^3\psi^2 - \psi^3\psi^2\psi^1 - \phi^2\phi^3\psi^1 = 0, \tag{6}$$

which after the elementary operations "(5)–(4)" and "(6)–(4)" leads to

$$\phi^1\phi^3\psi^2 + \phi^2\phi^3\psi^1 = 0 \tag{7}$$
$$\phi^1\phi^2\psi^3 + \phi^1\phi^3\psi^2 = 0. \tag{8}$$

Again, multiplying from the left (7) by ϕ^3 and (8) by $-\phi^1$, one obtains

$$\phi^1\psi^2 + \phi^2\psi^1 = 0 \tag{9}$$
$$\phi^2\psi^3 + \phi^3\psi^2 = 0. \tag{10}$$

By expanding ψ in terms of the basis ϕ, we have $\psi^1 = k_{11}\phi^1 + k_{12}\phi^2 + k_{13}\phi^3$, $\psi^2 = k_{21}\phi^1 + k_{22}\phi^2 + k_{23}\phi^3$ y $\psi^3 = k_{31}\phi^1 + k_{32}\phi^2 + k_{33}\phi^3$. Then (9) can be written in the form

$$-k_{21} - k_{12} + (k_{22} - k_{11})\phi^1\phi^2 + k_{23}\phi^1\phi^3 + k_{13}\phi^2\phi^3 = 0,$$

from which we get the equalities

$$\psi^1 = k_{11}\phi^1 + k_{12}\phi^2, \ \ \psi^2 = -k_{12}\phi^1 + k_{11}\phi^2.$$

Similarly, (10) may be rewritten as

$$-k_{31}\phi^1\phi^2 - k_{32} + (k_{33} - k_{11})\phi^2\phi^3 + k_{12}\phi^1\phi^3 = 0,$$

from which we conclude that $k_{31} = 0$, $k_{32} = 0$, $k_{12} = 0$, $k_{33} - k_{11} = 0$ and hence that $\psi = k_{11}\phi$. The normality of both structural sets leads to $\psi = \phi$ or $\psi = -\phi$, as claimed.

Summarizing the above examples we can conclude that there does not exist a maximum principle for (ϕ, ψ)-inframonogenic functions, and this happens for any structural sets ϕ and ψ.

Remark 1 The following fine property of inframonogenic functions was revealed in García et al. (2017): a function $f = \sum_{k=0}^{m}[f]_k$ belongs to $\mathcal{I}(\Omega)$ if and only if $[f]_k \in \mathcal{I}(\Omega)$ for $k = 0, 1, \ldots, m$. In exactly the same way it is obvious that this property is satisfied by (ϕ, ϕ)-inframonogenic functions, i.e.,

$$(f \in \mathcal{I}_{\phi,\phi}(\Omega)) \Longleftrightarrow ([f]_k \in \mathcal{I}_{\phi,\phi}(\Omega),\ k = \overline{0, m}).$$

The situation with two distinct structural sets is, however, totally different. The following simple example illustrates this phenomenon.

Let be $\phi = \{e_1, e_3, e_2\}$ and $\psi = \{\frac{\sqrt{2}}{2}(e_1 + e_3), \frac{\sqrt{2}}{2}(e_1 - e_3), e_2\}$ two structural sets and the function given by

$$f(\underline{x}) = \frac{1}{2}x_1^2 + \frac{1}{2}x_2^2 + x_2x_3e_1 - \frac{\sqrt{2}}{2}x_3^2e_2 + x_1x_3e_3 + \frac{\sqrt{2}}{2}x_3^2e_1e_2e_3.$$

On the one hand we have

$$\begin{aligned}
{}^{\phi}\underline{\partial}[f]^{\psi}\underline{\partial} &= \frac{\sqrt{2}}{2}e_1(e_1 + e_3) + \frac{\sqrt{2}}{2}e_3(e_1 - e_3) + \sqrt{2}e_2(-e_2 + e_1e_2e_3)e_2 + e_1e_3e_2 \\
&\quad + \frac{\sqrt{2}}{2}e_2e_3(e_1 + e_3) + e_3e_1e_2 + \frac{\sqrt{2}}{2}e_2e_1(e_1 - e_3) \\
&= -\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}e_1e_3 + \frac{\sqrt{2}}{2}e_3e_1 + \frac{\sqrt{2}}{2} + \sqrt{2}e_2 - \sqrt{2}e_1e_2e_3 - e_1e_2e_3 \\
&\quad + \frac{\sqrt{2}}{2}e_1e_2e_3 - \frac{\sqrt{2}}{2}e_2 + e_1e_2e_3 - \frac{\sqrt{2}}{2}e_2 + \frac{\sqrt{2}}{2}e_1e_2e_3 \\
&= 0.
\end{aligned}$$

but on the other

$$\begin{aligned}
{}^{\phi}\underline{\partial}[f]_0{}^{\psi}\underline{\partial} &= {}^{\phi}\underline{\partial}\left[\frac{1}{2}x_1^2 + \frac{1}{2}x_2^2\right]{}^{\psi}\underline{\partial} = -\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}e_1e_3 + \frac{\sqrt{2}}{2}e_3e_1 + \frac{\sqrt{2}}{2} = 0 \\
{}^{\phi}\underline{\partial}[f]_1{}^{\psi}\underline{\partial} &= {}^{\phi}\underline{\partial}\left[x_2x_3e_1 - \frac{\sqrt{2}}{2}x_3^2e_2 + x_1x_3e_3\right]{}^{\psi}\underline{\partial} \\
&= \sqrt{2}e_2 - e_1e_2e_3 + \frac{\sqrt{2}}{2}e_1e_2e_3 - \frac{\sqrt{2}}{2}e_2 + e_1e_2e_3 - \frac{\sqrt{2}}{2}e_2 + \frac{\sqrt{2}}{2}e_1e_2e_3 \\
&= \sqrt{2}e_1e_2e_3 \neq 0
\end{aligned}$$

$$ {}^{\phi}\underline{\partial}[f]_2{}^{\psi}\underline{\partial} = 0 $$

$$ {}^{\phi}\underline{\partial}[f]_3{}^{\psi}\underline{\partial} = {}^{\phi}\underline{\partial}\left[\frac{\sqrt{2}}{2}x_3^2 e_1e_2e_3\right]{}^{\psi}\underline{\partial} = -\sqrt{2}e_1e_2e_3 \neq 0. $$

It is worth to point out that ${}^{\psi}\underline{\partial}$ factorizes the Laplace operator in $\mathbb{R}^m$ in the sense that ${}^{\psi}\underline{\partial}^2 = -\Delta_m$. The fundamental solution of ${}^{\psi}\underline{\partial}$ is thus given by

$$ K_{\psi}(\underline{x}) := {}^{\psi}\underline{\partial}[E_1(\underline{x})], $$

where

$$ E_1(\underline{x}) = \frac{1}{(m-2)\sigma_m|\underline{x}|^{m-2}}, \quad \underline{x} \neq 0, $$

is the fundamental solution of the Laplacian Δ_m and σ_m stands for the surface area of the unit sphere in $\mathbb{R}^m$. Thus, the function

$$ K_{\psi}(\underline{x}) = -\frac{\underline{x}_{\psi}}{\sigma_m|\underline{x}|^m}, $$

where $\underline{x}_{\psi} = \sum_{i=1}^{m} x_i\psi^i$ if $\underline{x} = \sum_{i=1}^{m} x_ie_i$, called Cauchy kernel, satisfies in $\mathbb{R}^m\setminus\{0\}$ the equation ${}^{\psi}\underline{\partial}[K_{\psi}(\underline{x})] = [K_{\psi}(\underline{x})]{}^{\psi}\underline{\partial} = 0$.

In the sequel, when speaking of a domain Ω, it will always be assumed to be open and simply connected set of $\mathbb{R}^m$ with sufficiently smooth boundary Γ. In Sect. 5 we will consider domains with boundary satisfying conditions of a more general character. If necessary, we shall use the temporary notation $\Omega_+ = \Omega$, $\Omega_- = \mathbb{R}^m\setminus(\Omega\cup\Gamma)$.

The Cauchy kernel generates the following two important integral operators:

$$ (\mathcal{T}_{\psi}^{l}g)(\underline{x}) = -\int_{\Omega} K_{\psi}(\underline{y}-\underline{x})g(\underline{y})dV(\underline{y}) \tag{11} $$

and

$$ (\mathcal{C}_{\phi,\psi}^{l}g)(\underline{x}) = \int_{\Gamma} K_{\phi}(\underline{y}-\underline{x})n_{\psi}(\underline{y})g(\underline{y})dS(\underline{y}), \quad \underline{x} \notin \Gamma. \tag{12} $$

When $\phi = \psi$, $\mathcal{C}_{\phi,\psi}^{l}$ reduces to the usual Cauchy type integral

$$ (\mathcal{C}_{\psi}^{l}g)(\underline{x}) = \int_{\Gamma} K_{\psi}(\underline{y}-\underline{x})n_{\psi}(\underline{y})g(\underline{y})dS(\underline{y}), \quad \underline{x} \notin \Gamma. $$

The Stokes theorem, conveniently used, leads after some calculations to a formula connecting the integrals $\mathcal{C}_{\phi,\psi}^{l}$, $\mathcal{T}_{\psi}^{l}$ and a multidimensional Ahlfors-Beurling transform given by $\Pi_{\phi,\psi}^{l} := {}^{\phi}\underline{\partial}[\mathcal{T}_{\psi}^{l}]$. More precisely,

Theorem 1 (Blaya et al. (2016) Generalized Borel–Pompeiu formula) *Let* $f \in C^1(\overline{\Omega}, \mathbb{R}_{0,m})$. *Then*

$$(\mathcal{C}^l_{\phi,\psi} f)(\underline{x}) + (\mathcal{T}^l_\phi {}^\psi\underline{\partial}[f])(\underline{x}) = \Pi^l_{\phi,\psi}[f](\underline{x}), \quad \underline{x} \notin \Gamma. \tag{13}$$

Of course, we have the right version of formula (13) given by

$$(\mathcal{C}^r_{\phi,\psi} f)(\underline{x}) + (\mathcal{T}^r_\phi [f]^\psi\underline{\partial})(\underline{x}) = \Pi^r_{\phi,\psi}[f](\underline{x}), \quad \underline{x} \notin \Gamma, \tag{14}$$

where $\mathcal{T}^r_\psi g$ (resp. $\mathcal{C}^r_{\phi,\psi} g$) is obtained from (11) (resp. from (12)) by interchanging the Cauchy kernel with $g(\underline{y})$ and $\Pi^r_{\phi,\psi} := [\mathcal{T}^r_\psi]^\phi\underline{\partial}$.

For further use and more in the direction of this paper, let us introduce the new integral operators

$$(\mathcal{C}^0_{\phi,\psi} g)(\underline{x}) = \int_\Gamma K_\phi(\underline{y}-\underline{x}) n_\phi(\underline{y}) g(\underline{y})(\underline{y}_\psi - \underline{x}_\psi) dS(\underline{y}),$$
$$(\mathcal{C}^1_{\phi,\psi} g)(\underline{x}) = \sum_{i=1}^m \phi^i \left[\int_\Gamma E_1(\underline{y}-\underline{x}) n_\phi(\underline{y}) g(\underline{y}) dS(\underline{y})\right] \psi^i,$$
$$(\mathcal{T}^0_{\phi,\psi} g)(\underline{x}) = -\int_\Omega K_\phi(\underline{y}-\underline{x}) g(\underline{y})(\underline{y}_\psi - \underline{x}_\psi) dV(\underline{y}),$$
$$(\mathcal{T}^1_{\phi,\psi} g)(\underline{x}) = -\sum_{i=1}^m \phi^i \left[\int_\Omega E_1(\underline{y}-\underline{x}) g(\underline{y}) dV(\underline{y})\right] \psi^i,$$

each one of them connecting the two arbitrary structural sets ϕ and ψ in its own particular way.

For the sake of brevity, we still introduce two more operators:

$$\mathcal{C}^{\mathbf{i},r}_{\phi,\psi} g = \frac{1}{2}(\mathcal{C}^0_{\phi,\psi} g + \mathcal{C}^1_{\phi,\psi} g),$$
$$\mathcal{T}^{\mathbf{i},r}_{\phi,\psi} g = \frac{1}{2}(\mathcal{T}^0_{\phi,\psi} g + \mathcal{T}^1_{\phi,\psi} g).$$

The first of them can be thought as a sort of (ϕ, ψ)-inframonogenic Cauchy transform while the second represents a generalized Teodorescu operator. Note the following formulas which will be used in the sequel

$$\left[\mathcal{C}^{\mathbf{i},r}_{\phi,\psi} g\right]^\psi\underline{\partial} = \mathcal{C}^l_\phi g, \quad \left[\mathcal{T}^{\mathbf{i},r}_{\phi,\psi} g\right]^\psi\underline{\partial} = \mathcal{T}^l_\phi g. \tag{15}$$

That this is so, is easily checked directly. Indeed,

$$\int_\Gamma K_\phi(\underline{y}-\underline{x}) n_\phi(\underline{y}) g(\underline{y})(\underline{y}_\psi - \underline{x}_\psi) dS(\underline{y}) = \int_\Gamma (\underline{y}_\phi - \underline{x}_\phi) n_\phi(\underline{y}) g(\underline{y}) K_\psi(\underline{y}-\underline{x}) dS(\underline{y}).$$

Then,

$$
\begin{aligned}
[\mathcal{C}^0_{\phi,\psi} g(\underline{x})]^{\psi}\underline{\partial}_{\underline{x}} &= \left[\int_\Gamma (\underline{y}_\phi - \underline{x}_\phi) n_\phi(\underline{y}) g(\underline{y}) K_\psi(\underline{y} - \underline{x}) dS(\underline{y})\right]^{\psi}\underline{\partial}_{\underline{x}} \\
&= -\int_\Gamma [\underline{x}_\phi n_\phi(\underline{y}) g(\underline{y}) K_\psi(\underline{y} - \underline{x})]^{\psi}\underline{\partial}_{\underline{x}} dS(\underline{y}) \\
&= -\sum_{i=1}^m \phi^i \int_\Gamma n_\phi(\underline{y}) g(\underline{y}) K_\psi(\underline{y} - \underline{x}) dS(\underline{y}) \psi^i.
\end{aligned}
$$

On the other hand, we have

$$
\begin{aligned}
\mathcal{C}^1_{\phi,\psi} g(\underline{x}) t^{\psi}\underline{\partial}_{\underline{x}} &= \left[\sum_{i=1}^m \phi^i \int_\Gamma E_1(\underline{y} - \underline{x}) n_\phi(\underline{y}) g(\underline{y}) dS(\underline{y}) \psi^i\right]^{\psi}\underline{\partial}_{\underline{x}} \\
&= \sum_{i=1}^m \int_\Gamma [\phi^i E_1(\underline{y} - \underline{x}) n_\phi(\underline{y}) g(\underline{y}) \psi^i]^{\psi}\underline{\partial}_{\underline{x}} dS(\underline{y}) \\
&= 2\int_\Gamma K_\phi(\underline{y} - \underline{x}) n_\phi(\underline{y}) g(\underline{y}) dS(\underline{y}) \\
&\quad + \sum_{i=1}^m \phi^i \left[\int_\Gamma n_\phi(\underline{y}) g(\underline{y}) K_\psi(\underline{y} - \underline{x}) dS(\underline{y})\right] \psi^i
\end{aligned}
$$

and hence

$$
\left[\mathcal{C}^{\mathbf{i},r}_{\phi,\psi} g(\underline{x})\right]^{\psi}\underline{\partial}_{\underline{x}} = \frac{1}{2}([\mathcal{C}^0_\phi g(\underline{x})]^{\psi}\underline{\partial}_{\underline{x}} + [\mathcal{C}^1_{\phi,\psi} g(\underline{x})]^{\psi}\underline{\partial}_{\underline{x}}) = (\mathcal{C}^l_\phi g)(\underline{x}).
$$

The second identity in (15) may be proved in a quite analogous way.

3 Cauchy Formula for (ϕ, ψ)-Inframonogenic Functions

In this section we will prove a new Borel–Pompeiu formula involving the sandwich operator ${}^{\phi}\underline{\partial}(.)^{\psi}\underline{\partial}$. The Cauchy representation for (ϕ, ψ)-inframonogenic functions is thus derived as a simple corollary.

We begin with an auxiliary lemma, in which the differentiability of f is tacitly assumed to legitimate the single or iterative action of the Dirac operators.

Lemma 1 *The following formulas hold*

(1) $K_\phi(\underline{y} - \underline{x})\phi^i = -\phi^i K_\phi(\underline{y} - \underline{x}) + 2[K_\phi(\underline{y} - \underline{x})\phi^i]_0$, $i = 1, \ldots, m$,
(2) ${}^{\phi}\underline{\partial}_{\underline{y}}[([f(\underline{y})]^{\psi}\underline{\partial}_{\underline{y}})(\underline{y}_\psi - \underline{x}_\psi)] = ({}^{\phi}\underline{\partial}_{\underline{y}}[f(\underline{y})]^{\psi}\underline{\partial}_{\underline{y}})(\underline{y}_\psi - \underline{x}_\psi) + \sum_{i=1}^m \phi^i ([f(\underline{y})]^{\psi}\underline{\partial}_{\underline{y}})\psi^i$.

Proof The proof of (1) is a matter of direct calculation. To prove (2) we proceed as follows:

$$\begin{aligned}
{}^{\phi}\partial_{\underline{y}}[([f(\underline{y})]^{\psi}\partial_{\underline{y}})(\underline{y}_{\psi}-\underline{x}_{\psi})] &= \sum_{i=1}^{m}\phi^i\frac{\partial}{\partial y_i}[([f(\underline{y})]^{\psi}\partial_{\underline{y}})(\underline{y}_{\psi}-\underline{x}_{\psi})]\\
&= \sum_{i=1}^{m}\phi^i\left\{\frac{\partial[f(\underline{y})]^{\psi}\partial_{\underline{y}}}{\partial y_i}(\underline{y}_{\psi}-\underline{x}_{\psi})+([f(\underline{y})]^{\psi}\partial_{\underline{y}})\frac{\partial(\underline{y}_{\psi}-\underline{x}_{\psi})}{\partial y_i}\right\}\\
&= ({}^{\phi}\partial_{\underline{y}}[f(\underline{y})]^{\psi}\partial_{\underline{y}})(\underline{y}_{\psi}-\underline{x}_{\psi})+\sum_{i=1}^{m}\phi^i([f(\underline{y})]^{\psi}\partial_{\underline{y}})\psi^i.
\end{aligned}$$

Theorem 2 *Let $f \in C^2(\Omega \cup \Gamma)$. Then we have in Ω*

$$f(\underline{x}) = (\mathcal{C}^r_{\psi} f)(\underline{x}) + (\mathcal{C}^{\mathbf{i},r}_{\phi,\psi}[f]^{\psi}\underline{\partial})(\underline{x}) + (\mathcal{T}^{\mathbf{i},r}_{\phi,\psi}\, {}^{\phi}\underline{\partial}[f]^{\psi}\underline{\partial})(\underline{x}). \tag{16}$$

Proof The harder part of the proof was already done when we realized what the appropriate structure of the integrals involved in formula (16) should be. For the rest of the proof we use the same type of calculations used in (García et al. (2017), Theorem 3.1).

Removing from Ω a sufficiently small open ball $B_\epsilon(\underline{x})$ gives the domain $\Omega_\epsilon := \Omega\setminus\overline{B_\epsilon(\underline{x})}$ with boundary $\partial\Omega_\epsilon := \Gamma \cup (-\partial B_\epsilon(\underline{x}))$.

Now let

$$\begin{aligned}
I^\epsilon &= \int_{\Omega_\epsilon} K_\phi(\underline{y}-\underline{x})\,{}^{\phi}\partial_{\underline{y}}[([f(\underline{y})]^{\psi}\partial_{\underline{y}})(\underline{y}_{\psi}-\underline{x}_{\psi})]dV(\underline{y}),\\
J^\epsilon &= \int_{\Omega_\epsilon}\sum_{i=1}^{m} K_\phi(\underline{y}-\underline{x})\phi^i([f(\underline{y})]^{\psi}\partial_{\underline{y}})\psi^i dV(\underline{y}).
\end{aligned}$$

After applying the Stokes formula and having in mind the ϕ-hyperholomorphicity of K_ϕ we have

$$I^\epsilon = \int_{\partial\Omega_\epsilon} K_\phi(\underline{y}-\underline{x})n_\phi(\underline{y})([f(\underline{y})]^{\psi}\partial_{\underline{y}})(\underline{y}_{\psi}-\underline{x}_{\psi})dS(\underline{y}). \tag{17}$$

With the help of Lemma 1-(1) we obtain

$$\begin{aligned}
J^\epsilon &= \int_{\Omega_\epsilon}\sum_{i=1}^{m}\{-\phi^i K_\phi(\underline{y}-\underline{x})([f(\underline{y})]^{\psi}\partial_{\underline{y}})\psi^i + 2[K_\phi(\underline{y}-\underline{x})\phi^i]_0([f(\underline{y})]^{\psi}\partial_{\underline{y}})\psi^i\}dV(\underline{y})\\
&= -\sum_{i=1}^{m}\phi^i\left[\int_{\Omega_\epsilon} K_\phi(\underline{y}-\underline{x})([f(\underline{y})]^{\psi}\partial_{\underline{y}})dV(\underline{y})\right]\psi^i - 2\int_{\Omega_\epsilon}([f(\underline{y})]^{\psi}\partial_{\underline{y}})K_\psi(\underline{y}-\underline{x})dV(\underline{y}).
\end{aligned}$$

On the other hand, we have

$$\int_{\Omega_\epsilon} E_1(\underline{y}-\underline{x})({}^{\phi}\underline{\partial}_{\underline{y}}[f(\underline{y})]^{\psi}\underline{\partial}_{\underline{y}})dV(\underline{y}) + \int_{\Omega_\epsilon} K_\phi(\underline{y}-\underline{x})([f(\underline{y})]^{\psi}\underline{\partial}_{\underline{y}})dV(\underline{y})$$
$$= \int_{\partial\Omega_\epsilon} E_1(\underline{y}-\underline{x})\underline{n}_\phi(\underline{y})([f(\underline{y})]^{\psi}\underline{\partial}_{\underline{y}})dS(\underline{y}),$$

where again use has been made of the Stokes formula.

Consequently,

$$\begin{aligned} J^\epsilon &= \sum_{i=1}^{m} \phi^i \left[\int_{\Omega_\epsilon} E_1(\underline{y}-\underline{x})({}^{\phi}\underline{\partial}_{\underline{y}}[f(y)]^{\psi}\underline{\partial}_{\underline{y}})dV(\underline{y})\right]\psi^i \\ &\quad - \sum_{i=1}^{m} \phi^i \left[\int_{\partial\Omega_\epsilon} E_1(\underline{y}-\underline{x})\underline{n}_\phi(\underline{y})([f(\underline{y})]^{\psi}\underline{\partial}_{\underline{y}})dS(\underline{y})\right]\psi^i \\ &\quad - 2\int_{\Omega_\epsilon} ([f(\underline{y})]^{\psi}\underline{\partial}_{\underline{y}})K_\psi(\underline{y}-\underline{x})dV(\underline{y}). \end{aligned} \tag{18}$$

By the Lemma 1-(2) one has

$$I^\epsilon - J^\epsilon = \int_{\Omega_\epsilon} K_\phi(\underline{y}-\underline{x})({}^{\phi}\underline{\partial}_{\underline{y}}[f(y)]^{\psi}\underline{\partial}_{\underline{y}})(\underline{y}_\psi - \underline{x}_\psi)dV(\underline{y}).$$

Hence, by (17) and (18) we obtain

$$\begin{aligned} &\int_{\Omega_\epsilon} K_\phi(\underline{y}-\underline{x})({}^{\phi}\underline{\partial}_{\underline{y}}[f(y)]^{\psi}\underline{\partial}_{\underline{y}})(\underline{y}_\psi - \underline{x}_\psi)dV(\underline{y}) \\ &= \int_{\partial\Omega_\epsilon} K_\phi(\underline{y}-\underline{x})n_\phi(\underline{y})([f(\underline{y})]^{\psi}\underline{\partial}_{\underline{y}})(\underline{y}_\psi - \underline{x}_\psi)dS(\underline{y}) \\ &\quad + \sum_{i=1}^{m} \phi^i \left[\int_{\partial\Omega_\epsilon} E_1(\underline{y}-\underline{x})\underline{n}_\phi(\underline{y})([f(\underline{y})]^{\psi}\underline{\partial}_{\underline{y}})dS(\underline{y})\right]\psi^i \\ &\quad - \sum_{i=1}^{m} \phi^i \left[\int_{\Omega_\epsilon} [E_1(\underline{y}-\underline{x})]({}^{\phi}\underline{\partial}_{\underline{y}}[f(\underline{y})]^{\psi}\underline{\partial}_{\underline{y}})dV(\underline{y})\right]\psi^i \\ &\quad + 2\int_{\Omega_\epsilon} ([f(\underline{y})]^{\psi}\underline{\partial}_{\underline{y}})K_\psi(\underline{y}-\underline{x})dV(\underline{y}). \end{aligned}$$

At this stage, the same analysis as in (García et al. (2017), Theorem 3.1) leads to (16), after letting ϵ tend to 0. □

Remark 2 Looking at the above proof it is easy to check that the Borel-Pompeiu formula (16) will remain valid if the condition $f \in C^2(\Omega \cup \Gamma)$ is replaced by the weaker assumption

$$f \in C^2(\Omega) \cap C^1(\Omega \cup \Gamma), \quad \int_\Omega |{}^\phi\underline{\partial}[f(\underline{y})]^\psi\underline{\partial}| d\underline{y} < +\infty.$$

As we mentioned before, the above theorem yields a Cauchy representation formula for (ϕ, ψ)-inframonogenic functions.

Corollary 1 *Let $f \in C^2(\Omega) \cap C^1(\Omega \cup \Gamma)$ be (ϕ, ψ)-inframonogenic in Ω. Then it follows that*

$$f(\underline{x}) = (\mathcal{C}^r_\psi f)(\underline{x}) + (\mathcal{C}^{\mathbf{i},r}_{\phi,\psi}[f]^\psi\underline{\partial})(\underline{x}), \ \ \underline{x} \in \Omega. \tag{19}$$

Remark 3 For $\phi = \psi$ being the standard orthonormal basis, this formula becomes the Cauchy integral formula for inframonogenic functions which was proved in (García et al. (2017), Theorem 3.2).

Remark 4 A left analogue of the formula (16) arises if one uses the corresponding operators

$$\mathcal{C}^{\mathbf{i},l}_{\phi,\psi} g(\underline{x}) := \frac{1}{2}\Bigg[\int_\Gamma (\underline{y}_\phi - \underline{x}_\phi) g(\underline{y}) n_\psi(\underline{y}) K_\psi(\underline{y} - \underline{x}) dS(\underline{y}) + \sum_{i=1}^m \phi^i \Big[\int_\Gamma g(\underline{y}) n_\psi(\underline{y}) E_1(\underline{y} - \underline{x}) dS(\underline{y})\Big]\psi^i\Bigg]$$

and

$$\mathcal{T}^{\mathbf{i},l}_{\phi,\psi} g(\underline{x}) := -\frac{1}{2}\Bigg[\int_\Omega (\underline{y}_\phi - \underline{x}_\phi) g(\underline{y}) K_\psi(\underline{y} - \underline{x}) dV(\underline{y}) + \sum_{i=1}^m \phi^i \Big[\int_\Omega g(\underline{y}) E_1(\underline{y} - \underline{x}) dV(\underline{y})\Big]\psi^i\Bigg].$$

Indeed,

$$f(\underline{x}) = (\mathcal{C}^l_\phi f)(\underline{x}) + (\mathcal{C}^{\mathbf{i},l}_{\phi,\psi}\, {}^\phi\underline{\partial}[f])(\underline{x}) + (\mathcal{T}^{\mathbf{i},l}_{\phi,\psi}\, {}^\phi\underline{\partial}[f]^\psi\underline{\partial})(\underline{x}). \tag{20}$$

4 Π-Operators and Inframonogenicity

The complex Π-operator (Ahlfos–Beurling transform)

$$\Pi_\Omega[h(z)] = -\frac{1}{\pi}\int_\Omega \frac{h(\epsilon)}{(\epsilon - z)^2} d\epsilon_1 d\epsilon_2$$

plays an essential role in Complex Analysis and is particularly useful in the theory of quasiconformal mappings in the plane. This strongly singular operator behaves isometrically on $L^2(\Omega)$ and in the sense of distribution one has

$$\Pi_\Omega[\partial_{\overline{z}} h(z)] = \partial_z h.$$

The operator $\Pi^r_{\phi,\psi}$ may be seen as a multidimensional generalization of Π_Ω and many of its mapping and invertibility properties were described in Blaya et al. (2016).

In our context this operator has the important property that it maps $\mathcal{I}_{\phi,\psi}(\Omega)$ into $\mathcal{I}_{\phi,\phi}(\Omega)$, a remarkable fact if one takes into account Remark 1. This conclusion follows directly from

$$^{\phi}\underline{\partial}[\Pi^r_{\phi,\psi}[f]]^{\phi}\underline{\partial} = {}^{\phi}\underline{\partial}[[\mathcal{T}^r_\psi f]^{\phi}\underline{\partial}]^{\phi}\underline{\partial} = {}^{\phi}\underline{\partial}[\mathcal{T}^r_\psi f]^{\psi}\underline{\partial}^{\psi}\underline{\partial} = {}^{\phi}\underline{\partial}[f]^{\psi}\underline{\partial}, \tag{21}$$

where use has been made of the identity $[\mathcal{T}^r_\psi f]^{\psi}\underline{\partial} = f$.

Similarly, we obtain

$$^{\phi}\underline{\partial}[\Pi^l_{\phi,\psi}[f]]^{\phi}\underline{\partial} = {}^{\phi}\underline{\partial}[{}^{\phi}\underline{\partial}[\mathcal{T}^l_\psi f]]^{\phi}\underline{\partial} = {}^{\psi}\underline{\partial}\,{}^{\psi}\underline{\partial}[\mathcal{T}^l_\psi f]^{\phi}\underline{\partial} = {}^{\psi}\underline{\partial}[f]^{\phi}\underline{\partial}, \tag{22}$$

which one can think of as a left version of (21).

Formula (22) means that $\Pi^l_{\phi,\psi}$ does this time map $\mathcal{I}_{\psi,\phi}(\Omega)$ into $\mathcal{I}_{\phi,\phi}(\Omega)$, and we note the subtle change in the positions of ϕ and ψ.

A generalized Borel–Pompeiu formula for $\Pi^r_{\phi,\psi}[f]$ and $\Pi^l_{\phi,\psi}[f]$ may be obtained in a similar manner as in the proof of Theorem 2.

Theorem 3 *Let $f \in C^2(\Omega \cup \Gamma)$. Then it holds that*

$$\Pi^r_{\phi,\psi}[f](\underline{x}) = (\mathcal{C}^r_{\phi,\psi} f)(\underline{x}) + (\mathcal{C}^{\mathbf{i},r}_{\phi,\phi}[f]^{\psi}\underline{\partial})(\underline{x}) + (\mathcal{T}^{\mathbf{i},r}_{\phi,\phi}\,{}^{\phi}\underline{\partial}[f]^{\psi}\underline{\partial})(\underline{x}), \tag{23}$$

$$\Pi^l_{\phi,\psi}[f](\underline{x}) = (\mathcal{C}^l_{\phi,\psi} f)(\underline{x}) + (\mathcal{C}^{\mathbf{i},l}_{\phi,\phi}\,{}^{\psi}\underline{\partial}[f])(\underline{x}) + (\mathcal{T}^{\mathbf{i},l}_{\phi,\phi}\,{}^{\psi}\underline{\partial}[f]^{\phi}\underline{\partial})(\underline{x}). \tag{24}$$

Remark 5 For $\phi = \psi$ being the standard orthonormal basis, each one of the above formulas reduces to the Borel-Pompeiu formula derived in (García et al. (2017), Theorem 3.1). There is an essential difference between formulas (13) and (24). Indeed, if f is (ψ, ϕ)-inframonogenic, then (24) yields a representation of $\Pi^l_{\phi,\psi}[f]$ in terms of surface integral operators. On the other hand, a similar integral representation is derived from (13) only if one requires that f satisfies the much more restrictive assumption of ψ-hyperholomorphicity. The same difference between (14) and (23) arises in analogous manner.

We note that formula (16) for $\psi = \phi$ and identity (21) yields

$$\Pi^r_{\phi,\psi}[f](\underline{x}) = (\mathcal{C}^r_{\phi}\Pi^r_{\phi,\psi}[f])(\underline{x}) + (\mathcal{C}^{\mathbf{i},r}_{\phi,\phi}[f]^{\psi}\underline{\partial})(\underline{x}) + (\mathcal{T}^{\mathbf{i},r}_{\phi,\phi}\,{}^{\phi}\underline{\partial}[f]^{\psi}\underline{\partial})(\underline{x}).$$

As an easy consequence of these observations and formula (23) we have the following interesting relations, which have already been derived in Blaya et al. (2016) by a different method.

Proposition 1 *Let $f \in C^2(\Omega \cup \Gamma)$. Then it follows that*

$$\mathcal{C}^r_{\phi,\psi} f = \mathcal{C}^r_{\phi}\Pi^r_{\phi,\psi}[f].$$

Similarly,

$$\mathcal{C}^l_{\phi,\psi} f = \mathcal{C}^l_\phi \Pi^l_{\phi,\psi}[f].$$

5 Jump Problem in Fractal Domains

This section is devoted to a jump problem for (ϕ, ψ)-inframonogenic functions in which the smoothness assumption on Γ is replaced by another one, of much more general character. The previously introduced Teodorescu type operator $\mathcal{T}^{\mathbf{i},r}_{\phi,\psi}$ will prove extremely useful for such general investigations.

The Cauchy formula (19) says that if a solution of the Dirichlet problem

$$\begin{cases} {}^{\phi}\underline{\partial}[F]^{\psi}\underline{\partial} = 0 \text{ in } \Omega \\ F = f \text{ on } \Gamma, \end{cases} \tag{25}$$

exists in the class $C^2(\Omega) \cap C^1(\overline{\Omega})$, then it can be represented by

$$F(\underline{x}) = (\mathcal{C}^r_\psi f)(\underline{x}) + (\mathcal{C}^{\mathbf{i},r}_{\phi,\psi}[f]^{\psi}\underline{\partial})(\underline{x}). \tag{26}$$

However, because of the absence of a maximum principle for (ϕ, ψ)-inframonogenic functions, the usual uniqueness proof in the case of the Dirichlet problem (25) loses its validity and other solutions in the class $C^2(\Omega) \cap C(\overline{\Omega})$ may exist.

In addition to the Dirichlet problem there arises the following jump problem for (ϕ, ψ)-inframonogenic functions:

$$\begin{cases} {}^{\phi}\underline{\partial}[F]^{\psi}\underline{\partial} = 0, & \underline{x} \in \Omega_+ \cup \Omega_-, \\ F^+(\underline{x}) - F^-(\underline{x}) = f(\underline{x}), & \underline{x} \in \Gamma, \\ [[F]^{\psi}\underline{\partial}]^+(\underline{x}) - [[F]^{\psi}\underline{\partial}]^-(\underline{x}) = f_1(\underline{x}), & \underline{x} \in \Gamma, \\ F(\infty) = ([F]^{\psi}\underline{\partial})(\infty) = 0, \end{cases} \tag{27}$$

where f, f_1 are assumed in $C^{0,\nu}(\Gamma)$.

It follows directly from (15) and the Plemelj-Sokhotski formula that the function

$$F(\underline{x}) = (\mathcal{C}^r_\psi f)(\underline{x}) + (\mathcal{C}^{\mathbf{i},r}_{\phi,\psi} f_1)(\underline{x}) \tag{28}$$

is a solution of (27). The uniqueness of such a solution is ensured by a combination of Painleve and Liouville theorems in Clifford analysis (Brackx et al. 1982).

When one omits the smoothness conditions on Γ, the function (28) does not represent in general a solution of (27) or, which is even more inconvenient, it loses its usual sense. The natural question arises whether it is possible to construct a solution of (27), analogous to (28), where this time Ω is assumed to be a domain with fractal boundary Γ. Fractals are not only relevant from a mathematical point of view, but also have important applications in engineering and are widely used in modern telecommunication systems (Bellido et al. 2017; Karim et al. 2010; Tumakov et al. 2020). It is for these reasons that it is not unreasonable to consider the above problem under such a general geometric conditions.

In the case of fractal domains, the merely Hölder continuity of the boundary traces f, f_1 does not, in general, offer the possibility of constructing the solution of (27). The method we will mainly sketch below has its roots in the seminal work of (Kats 1983), (see also Blaya et al. 2015a).

Firstly, we define an appropriate way of measuring the fractality of Γ. In this direction we prefer the concept of d-summable boundary, introduced by Harrison and Norton in (1992). We say that Γ is d-summable for some $m-1<d<m$, if the improper integral

$$\int_0^1 N_\Gamma(\tau)\,\tau^{d-1}\,d\tau$$

converges, where $N_\Gamma(\tau)$ stands for the minimal number of balls of radius τ needed to cover Γ. The relationship between the above notion and fractality is clarified by checking that any d-summable boundary Γ has fractal dimension $\mathrm{Dim}(\Gamma)\le d$. Conversely, whenever $\mathrm{Dim}(\Gamma)<d$, then Γ is d-summable.

Secondly, we assume that f belongs (component-wisely) to the so-called higher order Lipschitz class $\mathrm{Lip}(1+\nu,\Gamma)$ introduced by Whitney (1934) and more extensively studied in Stein (1970). Roughly speaking, this means that there exist a jet $\{f, f^{\mathbf{j}}, |\mathbf{j}|=1\}$ such that the field of polynomials (in the usual multi-index notation)

$$f(\underline{y})+\sum_{|\mathbf{j}|=1} f^{\mathbf{j}}(\underline{y})(\underline{x}-\underline{y})^{\mathbf{j}},\ \ \underline{y}\in\Gamma$$

is the field of Taylor polynomials of a $C^{1,\nu}$-function in $\mathbb{R}^m$. A classical theorem of Whitney (1934) shows that such a function f may be extended to a $C^{1,\alpha}$-smooth function $\tilde{f}$ in $\mathbb{R}^m$, satisfying moreover that

$$|\partial_{\underline{x}}^{\mathbf{j}}\tilde{f}(\underline{x})|\le c\,\mathrm{dist}(\underline{x},\Gamma)^{\nu-1},\ \ |\mathbf{j}|=2,\ \ \underline{x}\in\mathbb{R}^m\backslash\Gamma.$$

Finally, if Γ is d-summable and f is assumed to be in $\mathrm{Lip}(1+\nu,\Gamma)$ with $\nu>\frac{d}{m}$, then it follows from (Blaya et al. 2015a, Lemma 4.1) that ${}^{\phi}\underline{\partial}[\tilde{f}]^{\psi}\underline{\partial}$ belongs to $L^p(\Omega)$ with $p=\frac{m-d}{1-\nu}>m$. Consequently, the functions $\mathcal{T}^{\mathbf{i},r}_{\phi,\psi}[{}^{\phi}\underline{\partial}[f]^{\psi}\underline{\partial}]$ and $\mathcal{T}^{l}_{\phi}[{}^{\phi}\underline{\partial}[\tilde{f}]^{\psi}\underline{\partial}]$ are continuous in the whole space $\mathbb{R}^m$ (see (Blaya et al. 2015a, Lemma 4.2)).

Summarizing, we are led to the following

Theorem 4 *Let $f\in Lip(1+\nu,\Gamma)$ and let Γ be d-summable with $\nu>\dfrac{d}{m}$. Then the jump problem*

$$\begin{cases} {}^{\phi}\underline{\partial}[F]^{\psi}\underline{\partial}=0, & \underline{x}\in\Omega_+\cup\Omega_-,\\ F^+(\underline{x})-F^-(\underline{x})=f(\underline{x}), & \underline{x}\in\Gamma,\\ [[F]^{\psi}\underline{\partial}]^+(\underline{x})-[[F]^{\psi}\underline{\partial}]^-(\underline{x})=([\tilde{f}]^{\psi}\underline{\partial})(\underline{x}), & \underline{x}\in\Gamma,\\ F(\infty)=([F]^{\psi}\underline{\partial})(\infty)=0 \end{cases}\qquad(29)$$

has a solution given by

$$F(\underline{x}) = \tilde{f}(\underline{x})\chi_\Omega(\underline{x}) - \mathcal{T}^{\mathbf{i},r}_{\phi,\psi}[{}^{\phi}\underline{\partial}[\tilde{f}]^{\psi}\underline{\partial}](\underline{x}), \tag{30}$$

where χ_Ω *stands for the characteristic function of* Ω.

Proof That F satisfies ${}^{\phi}\underline{\partial}[F]^{\psi}\underline{\partial} = 0$ may be verified directly on the basis of the formulas

$$\left[\mathcal{T}^{\mathbf{i},r}_{\phi,\psi}\, {}^{\phi}\underline{\partial}[\tilde{f}]^{\psi}\underline{\partial}\right]^{\psi}\underline{\partial} = \mathcal{T}^{l}_{\phi}[{}^{\phi}\underline{\partial}[\tilde{f}]^{\psi}\underline{\partial}],\ \ {}^{\phi}\underline{\partial}[\mathcal{T}^{l}_{\phi}[{}^{\phi}\underline{\partial}[\tilde{f}]^{\psi}\underline{\partial}]] = \begin{cases} {}^{\phi}\underline{\partial}[\tilde{f}]^{\psi}\underline{\partial} & \text{in } \Omega_+ \\ 0 & \text{in } \Omega_-. \end{cases}$$

The next thing we have to do is to prove that F given by (30) satisfies the jump conditions in (29), which is simply deduced from the previously mentioned continuity of $\mathcal{T}^{\mathbf{i},r}_{\phi,\psi}[{}^{\phi}\underline{\partial}[\tilde{f}]^{\psi}\underline{\partial}]$ and $\mathcal{T}^{l}_{\phi}[{}^{\phi}\underline{\partial}[\tilde{f}]^{\psi}\underline{\partial}]$. On the other hand, it is a matter of routine to check the vanishing conditions $F(\infty) = ([F]^{\psi}\underline{\partial})(\infty) = 0$. □

Finally, it should be mentioned that completely different situation arises when we consider the question of uniqueness of (29) in the fractal setting for the reason that the Painleve theorem does not hold in such high level of generality.

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